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(I.C.)

AN

INTRODUCTION

TO

THE THEORY AND PRACTICE

OF

PLANE AND SPHERICAL

TRIGONOMETRY,

AND

THE STEREOGRAPHIC PROJECTION OF THE SPHERE;

INCLUDING

THE THEORY OF NAVIGATION:

COMPREHENDING

A Variety of Rules, Formulæ, &c. with their Practical Applications to the Mensuration of Heights and Distances; to determining the Latitude by two Altitudes of the Sun, the Longitude by the Lunar Observations, and to other important Problems on the Sphere, and on Nautical Astronomy.

BY THOMAS KEITH.

THE FIFTH EDITION, CORRECTED AND IMPROVED.

La Trigonométrie est sans contredit une des plus utiles applications de la Géométrie élémentaire : elle est la base de la Géodésie, de la Géographie, de l'Astronomie et de la Navigation. Discours sur les Progrès des Sciences, rendu par l'Institut de France, 1809, page 6.

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PREFACE.

TRIGONOMETRY is an important branch of the mathematical sciences; the speculative parts, like the Elements of Euclid, habituate the mind to close and demonstrative reasoning; and the practical parts are of extensive use in the common concerns of life.

By Trigonometry we determine the magnitudes of the earth and planets, and the positions of the fixed stars with respect to each other, by which we are enabled to depict the appearance of the heavens in a small compass.

The distances of the planets from the sun, their motions, eclipses, and other phænomena, are calculated by Trigonometry; as are likewise the distances and positions of places on the earth, with their latitudes and longitudes; it may, therefore, justly be considered as the basis of Astronomy and Geography.

Navigation, with all its modern improvements, depends entirely on Trigonometry, which is likewise the foundation of maritime surveying, and of almost every branch of practical mathematics; accordingly we find this subject has been studied in the earliest ages of mathematical learning. Among the ancients were *Hipparchus*, *Theodosius*, *Menelaus*, *Ptolemy*, &c. who contributed to the advancement of this science.

The improvements in Astronomy, Navigation, and Trigonometry, nearly kept pace with each other. The invention of Logarithms by *Baron Napier* was an invaluable acquisition to these sciences; and the improvements made by this illustrious person, in Spherical Trigonometry, will be a lasting monument of his penetration and judgment. From this period the history of Logarithms and that of Trigonometry are closely connected, and there is scarcely a writer on one of these subjects, who has not likewise introduced the other.

The first authors of note, after the invention of Logarithms, were *Briggs* and *Gellibrand.*—Dr. *Charles Hutton,* in the Introduction to his Logarithms, has given a very complete account of the different writers on that subject, which likewise includes the principal authors on Trigonometry; to this valuable work the Reader is, therefore, referred for further information in the history of the science.

The authors on Trigonometry may be divided into two classes, theoretical and practical; for none of them have combined the theory with the practice, in such a manner as to render the subject plain and intelligible to a learner: the most valuable and scientifical are, in general, too abstruse, and the practical scarcely furnish the student with the rationale of a single rule or operation.

The object of the ensuing treatise is to simplify the theory, yet to retain a methodical and accurate mode of investigation, and to exemplify this theory by a variety of important and useful examples.

The demonstrations are frequently founded on principles strictly Geometrical, especially where those principles can be rendered very plain and perspicuous by the assistance of simple diagrams; and sometimes the process of demonstration is conducted by algebraical signs, particularly where the Geometrical method would require a complicated figure, or a long and tedious process.

In the construction of various formulæ the algebraical mode of deduction tends greatly to simplify the subject; yet the definitions and the elementary parts of the science must be acquired from Geometrical principles illustrated by diagrams; otherwise a student will never obtain a clear and satisfactory knowledge of the subject. Should any person attempt to teach the elementary principles of the science by the assistance of algebraical characters, and algebraical formulæ alone, without the aid of Geometry, he would most assuredly deceive both himself and his pupils.

It appears then, if the above observations be correct, that Trigonometry cannot be learnt expeditiously and completely, either from principles purely geometrical, or purely algebraical, without sacrificing utility to the uniformity of system.

By way of illustration, let us compare some parts of the Trigonometry at the end of Dr. *Simson*'s Euclid, with similar parts of the Trigonometry at the end of *Legendre*'s Geometry.

FIRST, *Where the three sides of a spherical triangle are given, to find an angle.*

In Prop. XXVIII. of Dr. *Simson*'s work, the process is conducted from principles purely Geometrical, and the demonstration is long and tedious; whereas in *Legendre*'s Geometry, article LXXVIII, page 389, 6th edition, (where the demonstration is similar to that given at page 185 of the ensuing treatise,) it is short and perspicuous,

SECONDLY, in article LXXVII, page 387 of *Legendre*'s Geometry, where he has shown that *the sines of the sides of a spherical triangle are directly proportional to the sines of their opposite angles;* the demonstration, which is purely algebraical, is complicated, unnatural, and almost incomprehensible to a young student; whilst that given by Dr. *Simson* in Prop. XXIII, from Geometrical principles, is simple and obvious.* See likewise page 176 of the ensuing work, where the rule is deduced from *Baron Napier.*

These remarks are not introduced with a design to criticise the works of either of these eminent authors, but to show the insufficiency of the Geometrical and of the Algebraical mode of demonstration independent of each other.

The following work may perhaps, at first view, appear to be too much extended for an introductory treatise, but a little attention to the subject will soon correct this supposition. The student is here supplied with THREE DISTINCT TREATISES, which are all essential towards his future progress in the science. *Firstly,* A treatise on Logarithms and the necessary tables. *Secondly,* A treatise on practical trigonometry; and *thirdly,* a theoretical treatise on the subject. A work which is deficient in any one of these three dependent parts, will oblige the student to have recourse to different authors, whereby the want of uniformity of reference, and of regularity of composition, will considerably retard his progress at the commencement of his studies.

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* It may be proper to remark that *Legendre* has given a Geometrical demonstration, at page 385 of his treatise, which is equally as plain as that given by Dr. *Simson*.

"An Introduction" (says Lord *Bacon*) "ought to have two properties; the one, that of a perspicuous and clear method; the other, that of an universal latitude and comprehension; where the student may have a little pre-notion of every thing, like a model towards a great building."

Convinced of the propriety of these remarks, the author has employed much time and attention, in the arrangement and execution of the subject; bearing constantly in mind the purpose for which he was writing, viz. *the Instruction of young Students.*

The Work is divided into FOUR BOOKS.

BOOK I. Contains the nature and use of Logarithms, the explanation of the Tables, the construction and use of *Gunter's* Scale, and some preparatory Geometrical Problems.

BOOK II. Comprises the principles and practice of Plane Trigonometry, illustrated by a variety of useful examples in the mensuration of distances, heights, &c.

The substance of this book is the same as that of Book I. of the first edition, with the exception of some additions, and an alteration in the arrangement of the several chapters. The easy parts are here placed together, and the demonstrations of the rules are separated from their practical applications; so that the student who wishes to acquire the practical parts only, may omit the demonstrations and proceed regularly with those parts which are the sole object of his pursuit; whilst the theoretical reader may, if he thinks proper, omit the practical parts, and continue the theory without interruption.

The greater part of the *fourth Chapter* was added to the second edition, and will be found to contain information which will be useful on several occasions.

The *fifth Chapter* is nearly the same as *Chap. I.* BOOK I. of the first edition, except that the formulæ have been somewhat extended. The substance of the two general Scholia from pages 6th and 11th of the second edition of Mr. *Emerson's* Trigonometry, (which were noticed in the 11th page of the first edition of this work,) has been introduced, and commodiously arranged according to the notation of *Mauduit*, *Legendre*, &c. This chapter likewise contains a variety of formulæ, exclusive of those already mentioned, together with series for finding the sines, cosines, &c. of arcs, and the construction of a table of sines by the continual bisection of an arc. This method of construction was made use of on account of its simple and obvious principles, and is similar to that given in the Trigonometry of *Nikitin* and *Souvoroff*, inspectors of the Imperial Academy at Cronstadt, and to the method used by the late Dr. *Horsley*, Bishop of St. Asaph, at page 137 of his treatise.

BOOK III. Comprehends the Theory and Practice of Spherical Trigonometry, and the Stereographic Projection of the Sphere.

The *first Chapter* contains Spherical Geometry, including various properties of Spherical Angles, Arcs, and Triangles, demonstrated in a plain and easy manner.

The *second Chapter* is wholly occupied with the Stereographical Projection of the Sphere, which was placed at the end of Spherical Trigonometry in the first edition; it has been brought forward, in this edition, in order that the constructions of the different practical cases of spherical triangles may appear with the calculations. The only alteration here is in the arrangement.

The *third Chapter* contains the investigation of general rules for calculating the sides and angles of right-angled Spherical Triangles. The XXIIId proposition, page 167, which is partly derived from the general figure page 151, is very extensive in its application. The original construction of this figure is ascribed, by Dr. *Horsely*, to *Copernicus*, the celebrated astronomer. The demonstration of *Baron Napier's* rule is derived from the XXIIId proposition. This rule is the most simple and comprehensive that ever was invented, for solving the different cases of right-angled spherical triangles, but has generally been explained in a confused and unintelligible manner. Valuable as this rule is, it has been censured as "Too artificial and restricted to be generally employed in the present advanced state of the science." It is, notwithstanding this aspersion, susceptible of general application, and the species of the different cases may be as accurately ascertained from its equations, as from any rules, or formulæ, hitherto invented, in consequence of which, it will, undoubtedly, continue to be used, by those who know how to appreciate its merit, when the works of the author who has presumed to censure it shall be forgotten.

The *fourth Chapter* shows the manner of applying *Baron Napier's* rule to the solutions of the different cases of oblique-angled spherical triangles, from which several useful rules are derived.

The *fifth Chapter* is, in substance, the same as the IVth Chapter of BOOK II. in the first edition, but the mode of demonstration has been varied. In this edition the method used by *Lagrange* and *Legendre* has been adopted, as being more simple than that used in the first edition. This chapter contains the whole of the formulæ given by *Legendre*, from page 386 to 399 of his Geometry, besides several others. These formulæ, however, are not copied from *Legendre*, each succeeding step being regularly deduced from the preceding, or from different propositions, to which references are given. These particulars, so essential to a learner, are entirely disregarded by *Legendre*: for illustration, compare article LXXXVIII. page 398 of *Legendre's* Geometry, with U. 192. and W. 192. of the succeeding work, &c. At page 193, &c. the various formulæ relating to Spherical Trigonometry are concentrated, and classed under their appropriate heads. Some of these formulæ are the same with those given by *Delambre* in his preface to *Borda's* trigonometrical tables.

The formulæ, where auxiliary angles are introduced (Q. 190, &c.) are, in effect, analogous to those which may be deduced from *Baron Napier's* rule, by drawing a perpendicular from the vertical angle of a spherical triangle upon the base, as in proposition xxv, page 173, &c.

The *sixth*, *seventh*, *eighth*, and *ninth Chapters* contain the logarithmical solutions of the various cases of Spherical Trigonometry.

The *tenth*, *eleventh*, and *twelfth Chapters* include the substance of BOOK III. of the first edition, though the matter has undergone a considerable change; the original examples were all adapted to the year 1796, but in this edition those which depend on the Nautical Almanac are carried forward to the year 1822. The Nautical Almanac for that year can easily be procured. This necessary alteration has been attended with considerable labour, and, it is presumed, will be of much advantage to the young student. The method of finding the correct distance between the sun and the moon, or between the moon and a star, by having the apparent distance and the apparent altitudes of these objects given, is shorter than any that has hitherto appeared, where the common tables only are used, besides having the advantage of being perfectly correct; for it is not liable to those inaccuracies which frequently arise from taking the logarithmical, or natural, sines and cosines out of the tables; as the observed distance is never very small, nor ever near 180 degrees.

The Nautical Almanac was first published in the year 1767; from which time, no treatise on Trigonometry had appeared, containing astronomical examples adapted to that work, until the first edition of the following treatise; wherein the author, with great labour and attention, suited all his examples to the year 1796, and corrected the situations of all the stars to the same year; the *whole* of these examples, with the rules and observations upon them, were *copied*[*] and republished, without a single acknowledgment, in less than five years after their first appearance.

The *thirteenth Chapter* contains the fluxional analogies of spherical triangles, with a few examples showing their use. This subject was first considered by Mr. *Cotes*, of Trinity College, Cambridge, in a treatise entitled De Æstimatione Errorum in Mixtâ Mathesi, which was published by Dr. *Robert Smith*, in the year 1722, six years after the author's decease. This branch of the science has hitherto received very little improvement, either from the English or foreign mathematicians, although a century has nearly elapsed since the first publication of Mr. *Cotes's* Treatise.

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[*] In a *second edition* of the work into which these subjects were copied, it is stated that " nearly all the practical questions before given, *many of which had* " *passed, in the same state, from one work to another, for more than a century past,* " *are here reproposed with new data,*" &c. This insinuation will not have the effect intended, with those who are capable of examining the matter. However, in consequence thereof, I think it proper to state, that several of the rules, and likewise examples, stand in *that* second edition without alteration; they are, of course, precisely the same as those in the *first* edition of *my* work, and some of them are *incorrect*. The data will show how long they have been in use.

The substance of all the *differential equations* given by *Cagnoli*, *La Lande*, *Delambre*, &c. is included in the XIIIth Chapter; these differential equations may readily be deduced from *Simpson's Fluxions*.

The *fourteenth Chapter* consists of Miscellaneous Propositions, which could not easily be classed under different heads.

The first comprehends the French division of the circle. The sexagesimal division of the circle, which is ascribed to the Egyptians, has often been objected to by mathematicians, as by *Oughtred*, *Wallis*, &c. who recommend a decimal, or centesimal division. Several tables have been constructed centesimally, particularly one by Dr. *John Newton*. Dr. *Charles Hutton*, in a memoir published in the Philosophical Transactions for 1783, proposed to divide the quadrant into equal decimal parts of the radius; by which means the degrees, or the divisions of the arc, would be the real lengths of the arcs, in terms of the radius.

The French have lately adopted the centesimal division of the circle in their calculations, and tables of greater extent than those by Dr. *Newton* have been published, as the tables of *Hobert* and *Idelcr*, printed at Berlin, in the year 1799, the tables of *Borda*, printed at Paris, an. IX, with a preface by *Delambre*, &c.

The advantages of this new division of the circle, should it be generally adopted in practical calculations, are few and trifling, when compared with the confusion and perplexity it would occasion. It is true that degrees, &c. would be more readily turned into minutes or seconds, *et vice versâ*, and some other advantages of minor importance would be obtained, were the new division to be *universally* adopted; at the same time all our valuable tables would be rendered useless; the many well-established trigonometrical and astronomical works, which from time to time have been published, would be little better than waste paper; the most valuable mathematical instruments, which have been constructed by celebrated artists, must be considered as lumber in the different observatories of Europe; the latitudes and longitudes of places must be changed, which change would render all the different works on Geography useless; or otherwise the Astronomers, and those in the habit of making trigonometrical calculations, must be perpetually turning the old division of the circle into the new, or the new into the old.

The best instruments in the observatories of Paris, Palermo, and of the different capitals of Europe, have been constructed by British artists, viz. by *Ramsden*, *Troughton*, &c. and are graduated according to the sexagesimal division of the circle; it is plain, therefore, that all observations made with these instruments must be reduced to the centesimal division of the circle before they can be used in calculation. Again, The logarithmical tables of sines, tangents, &c. which were originally constructed by the British mathematicians, have passed through so many hands, and have been so often examined, that they may be depended upon as correct; whilst the new tables would require great caution in using them.

In devising short-methods of constructing the new tables, foreign mathematicians have, however, discovered several formulæ, which may be useful in examining or extending the old tables, and in some astronomical calculations. These formulæ comprehend the most beneficial parts of their labours, as their new division of the circle will doubtless, ere long, share the same fate as their new calendar; unless the works, which may hereafter be framed according to that model, should possess extraordinary and continued merit.

From an attentive investigation of this subject, it will appear, that, if any merit can be attached to the new division of the circle, the invention cannot, with any degree of justice, be ascribed to the French mathematicians, and that their anxious desire for its universal adoption is the natural result of the French revolution, the general tendency of which has invariably been to depreciate the merit of every subject which has not the glory of the French nation or the praise of Frenchmen for its basis.

The improvements made by the French in the various branches of mathematics, though highly extolled, will be found, in many cases, to be more specious than real; the modern analysis so *universally* adopted (and which in some instances is certainly illogical, if not unscientifical) has contributed greatly to vitiate the taste and explode that solid and accurate method of reasoning which is so conspicuous in the writings of the ancients. A flimsy mode of demonstration, grounded on a dextermous management of algebraical characters, has frequently been substituted for perspicuity and logical exactness. By making these remarks, the author does not mean to insinuate that the French have made no improvements in the mathematical sciences, he is convinced that they have made several, and that in a variety of instances their method of demonstration is neat and perspicuous; but his opinion is, that the value of those improvements has been greately over-rated.

After considering the French division of the circle, the remainder of the *fourteenth Chapter* is occupied with problems which have been frequently applied in extensive trigonometrical surveys.

BOOK IV. treats of the Theory of Navigation, a subject intimately connected with Trigonometry, and immediately derived from the sphere.

The work CONCLUDES with a table of the logarithms of all numbers from an unit to ten thousand; a table of natural sines; a table of logarithmical sines and tangents to every degree and minute of the quadrant; with some other tables useful in the solutions of the different astronomical problems. These tables occupy only fify pages, and will be found a valuable acquisition both to the teacher and the learner, as they will save the expence of a large set of tables, and answer evey purpose of instruction with equal advantage.

THE FOLLOWING OBSERVATIONS ARE CONTINUED FROM THE SECOND EDITION.

After the foregoing developement of the nature and design of the work, and of the improvements which this edition contains, I feel myself compelled, however reluctantly, to notice the injustice done me by a contemporary author.

Mr. Bonnycastle, a gentleman well known from his various mathematical works, published a treatise on Plane and Spherical Trigonometry in the year 1806, into which he copied, without the least acknowledgment, a very considerable portion of my work.

In consequence of this I waited upon his bookseller, Mr. Johnson, and expressed my displeasure with the transaction. He assured me verbally (and by letters which are now in my possession) that no steps whatever should be taken, by advertisements or otherwise, to dispose of Mr. Bonnycastle's publication until he (Mr. Johnson the proprietor of the work) had indemnified me for the injury which I might sustain.

I stated that I should be satisfied by his agreeing to cancel such parts of Mr. Bonnycastle's book as I could prove to be copied from mine. This demand, so just and reasonable in itself, did not appear to meet with that approbation which might have been expected, and several letters passed between Mr. Johnston and myself on the subject; the last (excepting one) was dated the 30th of November 1806, which repeated the assurance that the work should not be advertised either directly or indirectly.

Notwithstanding these repeated assurances, a *partial* review of Mr. Bonnycastle's work appeared in the Philosophical Magazine for November 1806, evidently drawn up with the design of counteracting the effects of any unfavourable impression that might have gone abroad respecting it, and of paralizing the efforts I might make to enforce my demand. The parts copied could not be cancelled without destroying nearly half of the work.

In the same Magazine for January 1807, I made some remarks upon this review, wherein I stated that "Exclusive of detached matter (which is very considerable) there are upwards of *seventy pages* (of Mr. Bonnycastle's treatise) in which there are scarcely ten lines in any one page which are not directly copied from my work; and in many pages not a single line nor figure but what is copied." The truth of which I illustrated by pointing out some errors in my book, which Mr. Bonnycastle had copied most minutely. My remarks produced a reply, in February, which clearly convinced me that the feelings of Mr. Bonnycastle and those of his reviewer, were so strictly in unison, that the slighest irritation of the nerves of the one, immediately caused those of the other to vibrate.

There was not a sentence, or a word, in my observations, at which this reviewer (unconnected with Mr. Bonnycastle) could possibly be offended, yet his reply was not only intemperate but illiberal. I made some observations upon this reply, in a letter to the Editor of the Philosophical Magazine (dated 9th March 1807); part of which letter was inserted in the succeeding Magazine, but several of the most essential passages were suppressed.

The reviewers says, "The charges which Mr. Keith has adduced against Mr. Bonnycastle it is not *my* immediate business to refute." I would ask, for what purpose then did he make his reply? He makes a most ridiculous comment on the words *copying*, and *collecting* from the Nautical Almanac. Notwithstanding which, my observations were perfectly correct, for the tables of the sun's right ascension and declination inserted in my Book, were *collected* from the different pages of the Nautical Almanac, with the alteration of the last figures, and the arrangement was my own; whereas Mr. Bonnycastle's tables were exactly *copied* from my work. Now, why did Mr. Bonnycastle copy these tables from my treatise? The answer is obvious, they were copied for the express purpose of solving the various astronomical problems which he had taken from my work, for they were of no other use whatever.

The reviewer next makes some remarks on *two* diagrams noticed in my *first* letter to the Editor of the Magazine; and here, it appeaas, he has carefully measured the figures to the greatest nicety, and exults at the (*supposed*) advantage which he has obtained over me. In consequence of which, like a conqueror of true courage and humanity, he assumes a milder tone in the next sentence, and says, "There is in general only *one* way of constructing these figures; so that the cry of plagiarism on such an occasion is perfectly ridiculous, as I have no doubt Mr. Keith very well knows."

I feel highly flattered that this learned reviewer will allow me to know any thing, and in return for this mark of his condescension, I beg leave to refer him to the 120th page of the second volume of *Hodgeson's* System of Mathematics, printed in the year 1723, wherein he will find that a right-angled or quadrantal spherical triangle may be stereographically constructed, not only *one* way, but *one hundred and twenty* different ways!

"How far Mr. Bonnycastle may judge it right" (says the reviewer) "to enter into any altercation with Mr. Keith, in consequence of the liberties he has taken with his character, is not for me to decide."

If what I have charged Mr. Bonnycastle (in the Philosophical Magazine) with copying from my work, has really *not* been copied, and *can* be found in any other treatise, prior to the first publication of mine, then, indeed, I have accused him unjustly, and shall, undoubtedly, meet with that contempt, which so base a conduct would merit. But if, on the contrary, my charges are well founded, and the matter in question *cannot* be found in any other publication (however insignificant or unimportant the subjects which are copied may be); I stand upon a foundation not to be shaken by the false assertions and illiberal insinuations of this, or of any other critic.

The reviewer concludes his letter, by paying a handsome compliment to his own superior talents and acquirements, with which he seems to be highly pleased; and, as it is my fixed determination never to notice him again, I shall take my leave of him whilst he remains in that happy state of mind.

With regard to Mr. Bonnycastle, who now holds one of the first mathematical situations in the kingdom, if he can conscientiously acquit his conduct of reprehension, I envy not the happiness of his disposition, but leave him in quiet possession of the honour and emolument which he may have acquired.

THOMAS KEITH.

ADVERTISEMENT

TO

THE FOURTH EDITION.

THIS edition of the ensuing work has been carefully revised. All the astronomical examples which depend on the Nautical Almanac, are adapted to the year 1822, and they may be solved either with or without the Almanac; because the several articles necessary to be taken from that work are given at the end of the different examples. By these means the students will readily learn the use of the Nautical Almanac, and in a school where several are studying the same subject, their progress will not be retarded by waiting for the Almanac. The examples will be found sufficiently numerous and appropriate for the purpose of instruction. They might easily have been extended, *with different dates*, by a selection from various authors; but such examples are perfectly useless to a student, who has not in his possession a Nautical Almanac corresponding with each date.

No. 1. York-buildings, New-road,
St. Mary-le-bone, London, October, 1820.

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EXPLANATION OF THE CHARACTERS OR MARKS USED IN THE FOLLOWING WORK.

$+$, *Plus* or *more*, the sign of addition; as $AD + DC$, signifies that the line AD is to be increased by the line DC; and $4 + 3$ signifies that the number 4 is to be increased by the number 3.

$-$, *Minus* or *less*, the sign of subtraction, and shows that the second quantity is to be taken from the first; as $CB - GB$ shows that the line CB is to be diminished by the line GB.

$\times$, *Into* or *by*, the sign of multiplication; as $ED \times DC$ signifies the rectangle formed by the lines ED and DC, and $a \times b$ expresses the product of the quantity a by the quantity b. Also $a \cdot b$ or ab signifies the same thing.

$\div$, *Divide by*, as $PB \div CS$, or $\frac{PB}{CS}$ signifies that PB is to be divided by CS.

AB^2, AB^3, signify the square and the cube of AB; also $\overline{14}|^{3}$ signifies that 14 is to be involved to the third power, and then the fourth root is to be extracted.

$\sqrt{A}$ or $A^{\frac{1}{2}}$, $\sqrt[3]{A}$ or $A^{\frac{1}{3}}$, express the square and cube root of A.

$=$, *Equal to*, as $AB = CD$, shews that AB is equal to CD.

$\sim$, *Difference*, as $A \sim B$, shews that the difference between A and B is to be taken.

A vinculum or *parenthesis*, serves to link two or more quantities together, as $\overline{A + B} \times m$, or $(A + B) \cdot m$, signifies that A and B are first to be added together, and then to be multiplied by the quantity *m*.

$\left\{ \begin{matrix} : & :: & : \\ \textit{is to} & & \textit{as is to} \end{matrix} \right\}$ *Proportion*, A : B :: C : D signifies that A has to B the same ratio which C has to D, and is usually read A is to B as C is to D.

∵ *Therefore*,

∠ *Angle*, as ∠ A, signifies the angle A.

⊐ *Greater than*, as A ⊐ B, shows that A is greater than B.

⊏ *Less than*, as A ⊏ B, shows A to be less than B.

The other characters are explained among the definitions in the work.

N. B. The letters within the parentheses, at the beginning of the different paragraphs of the work, are for references. Thus, (C. 2.) refers to the article marked (C) at page 2.; (H. 25.) refers to the article marked (H) at page 25, and so on.

ERRATA.

Page 93 and 94, in the note, *for* Chap. XI. *read* Chap. XIV.
Page 305, line 10. *for* 5th of October, *read* 5th of August.

AN
INTRODUCTION
TO
PLANE AND SPHERICAL
TRIGONOMETRY.

BOOK I.

CHAPTER I.

THE NATURE AND PROPERTIES OF LOGARITHMS.

(A) *Definition.* LOGARITHMS are a series of numbers contrived to facilitate arithmetical calculations; so that by them the work of multiplication is performed by addition, division by subtraction, involution by multiplication, and the extraction of roots by division.

They may therefore be considered as *indices* to a series of numbers in geometrical progression, where the first term is an unit. Let

$1 . r^1 . r^2 . r^3 . r^4 . r^5 . r^6$, &c. be such a series, increasing from 1; or $1 . \frac{1}{r} . \frac{1}{r^2} . \frac{1}{r^3} . \frac{1}{r^4} . \frac{1}{r^5} . \frac{1}{r^6}$, &c. decreasing from 1; which last series, agreeably to the established notation in algebra, may be thus expressed, $1 . r^{-1} . r^{-2} . r^{-3} . r^{-4} . r^{-5} . r^{-6}$, &c. Here the common ratio is r, and the indices $1 . 2 . 3$, &c. or $-1 . -2 . -3$, &c. are logarithms. Hence it is obvious, that if a series of numbers be in geometrical progression, their logarithms will constitute a series in arithmetical progresssion. And, where the series is increasing, the terms of the geometrical progression are obtained by multiplication, and those of the arithmetical progression, or logarithms, by addition; on the contrary, if the series be decreasing, the terms of the geometrical progression are obtained by division, and those of the arithmetical progression by subtraction.

The same observations apply to logarithms when they are fractions, thus if $r^{\frac{1}{n}}$ denote any number, then will $1 . r^{\frac{1}{n}} . r^{\frac{2}{n}} . r^{\frac{3}{n}} . r^{\frac{4}{n}} . r^{\frac{5}{n}} . r^{\frac{6}{n}}$, &c. constitute an increasing series of numbers in geometrical progression, of which the indices $0 . \frac{1}{n} . \frac{2}{n} . \frac{3}{n} . \frac{4}{n} . \frac{5}{n} . \frac{6}{n}$, &c. are the logarithms, or $1 . \frac{1}{r^{\frac{1}{n}}} . \frac{1}{r^{\frac{2}{n}}} . \frac{1}{r^{\frac{3}{n}}} . \frac{1}{r^{\frac{4}{n}}} . \frac{1}{r^{\frac{5}{n}}} . \frac{1}{r^{\frac{6}{n}}}$, will constitute a decreasing series which may be thus expressed, $r^{-\frac{1}{n}} . r^{-\frac{2}{n}} . r^{-\frac{3}{n}} . r^{-\frac{4}{n}} . r^{-\frac{5}{n}} . r^{-\frac{6}{n}}$, &c. and of which $0 . -\frac{1}{n} . -\frac{2}{n} . -\frac{3}{n} . -\frac{4}{n} . -\frac{5}{n} . -\frac{6}{n}$, &c. are the logarithms.

(B) Considering logarithms as *indices* to a series of numbers in geometrical progression, where r and n may represent any numbers whatever, it follows that there may be as many different kinds of logarithms as there can be taken different sorts of geometrical series: numbers in very different progressions may likewise have the *same* logarithms, and on the contrary, the same geometrical series may have different series of logarithms corresponding to them, but in every case the logarithm of 1 is 0.

(C) The tables of logarithms in common use are constructed upon a supposition that $r = 10$; hence it appears that the logarithm of any number whatever is the index of some power of 10.

Thus, the logarithm of 10 is 1, being the index of 10^1; the logarithm of 100 is 2, being the index of 10^2; the logarithm of 1000 is 3, being the index of 10^3; the logarithm of 10000 is 4, being the index of 10^4, &c. Hence the logarithms of all numbers between 1 and 10 will be greater than 0 and less than 1, that is, they will be decimals; between 10 and 100 they will be greater than 1 and less than 2, that is, they will be expressed by 1 with decimals annexed; between 100 and

1000 the logarithms are expressed by 2 with decimals annexed; between 1000 and 10000 they are expressed by 3 with decimals annexed.

Again, the logarithm of $\frac{1}{10}=\frac{1}{10^1}$ is -1, being the index of 10^{-1}; the logarithm of $\frac{1}{100}=\frac{1}{10^2}$ is -2, being the index of 10^{-2}; the logarithm of $\frac{1}{1000}=\frac{1}{10^3}$ is -3, being the index of 10^{-3}, &c. Hence the logarithms of all numbers decreasing from 1 to ·1 will be expressed by -1, with decimals annexed; decreasing from ·1 to ·01, they will be expressed by -2, with decimals annexed; decreasing from ·01 to ·001, by -3, with decimals annexed, &c.

And, *universally, if* $10^x = a$, *then* x *is the logarithm of* a; where a may be any number from 1 to 101,000, the extent of our best modern tables.

If 10^{x}[*] $=2$, *then* $x=0\cdot30103$ *the log. of* 2.
If $10^x=3$, *then* $x=0\cdot47712$ *the log. of* 3.
If $10^x=4$, *then* $x=0\cdot60206$ *the log. of* 4.
If $10^x=50$, *then* $x=1\cdot69897$ *the log. of* 50.
If $10^x=94261$, *then* $x=4\cdot97433$ *the log. of* 94261.

The logarithm of 9·4261 = ·97433 = x, *viz.* $10^x = 10^{\cdot97433}=9\cdot4261$, multiply by 10.

Then $10^{x+1}=10^{1\cdot97433}=94\cdot261$.

Multiply again by 10,

And $10^{x+2}=10^{2\cdot97433}=942\cdot61$.

Multiply a third time by 10,

Then $10^{x+3}=10^{3\cdot97433}=9426\cdot1$.

Multiply once more by 10,

Then $10^{x+4}=10^{4\cdot97433}=94261$ and so on. If instead of multipying by 10, we divide by 10 in the equation $10^x=9\cdot4261$, then will

$$10^{x-1}=10^{\cdot97433-1}=10^{-1\cdot97433}=\cdot94261;$$

---

[*] The design of this chapter is to show the nature and properties of logarithms, and not the different methods of constructing them.

Divide again by 10,

Then $10^{x-2} = 10^{-1{\cdot}97433-1} = 10^{-2{\cdot}97433} = {\cdot}094261$.

Divide a third time by 10,

Then $10^{x-3} = 10^{-2{\cdot}97433-1} = 10^{-3{\cdot}97433} = {\cdot}0094261$, &c. as far as you please.

(D) Hence it appears that all numbers whatever, whether they be whole numbers, mixed numbers, or decimals, have the decimal part of their logarithms the same, the only difference being in the index or whole number; and that the index to the logarithm of a whole number is *affirmative*, and the index to the logarithm of a pure decimal is *negative;* but, in every case, the *decimal* part of the logarithm is affirmative.

CHAP. II.

THE USE OF THE TABLE OF LOGARITHMS.

PROPOSITION I.

(E) *To find the logarithm of any whole number, or mixed decimal, consisting of one, two, three, or four figures.*

This proposition will appear plain from the following examples, observing that the decimal parts only of the logarithms are put down in the tables; because the index, or whole number, of the logarithm is always an unit less than the number of figures contained in the natural whole number to which it belongs. (C. 2.)

Required the logarithm of 5.

Look in the table in the column marked *No.* for 5, and against it in the column *log.* stands ·69897. The natural number containing but one figure, the index is therefore 0.

Required the logarithm of 74.

Look for 74 in the column marked *No.* and against it stands ·86923. The natural number containing two figures, the index is 1, therefore the logarithm of 74 is 1·86923.

Required the logarithm of 743.

Look for 743 in the column marked *No.* and against it stands ·87099. The natural number containing three figures, the index is 2, therefore the logarithm of 743 is 2·87099.

Required the logarithm of 7438.

Look for 743 in the column marked *No.* as before, and directing your eye to the numbers in the top column of the table, find the remaining figure 8, guide your hand down that column till you come in the same line with 743, against which you will find ·87146. The natural number containing four figures, the index is 3. Hence the logarithm of 7438 is 3·87146.

	No.	Log.	No.	Log.	No.	Log.
Examp.	37	=1·56820	3754	=3·57449	4980	=3·69723
	4	=0·60206	375·4	=2·57449	7986	=3·90233
	91·6	=1·96190	37·54	=1·57449	3700	=3·56820
	9·16	=0·96190	3·754	=0·57449	4000	=3·60206

PROPOSITION II.

(F) *To find the logarithm of any whole number, or mixed decimal, to five or six places of figures.*

Rule. Find the logarithm to the first four figures as above, then take the difference between this logarithm and the next greater in the table. Multiply this difference by the figures you have above four, and cut off so many figures from the right hand of the product as you multiply by, the rest must be added to the logarithm first taken out of the table.

Required the logarithm of 59684.

Logarithm of 5968 is 77583	7 diff.
The next greater is 77590	4 remaining figure
Diff. 7	2·8 product=3 nearly.

Then 77583+3=77586. The natural number consisting of five figures, the index is 4, therefore the logarithm of 59684 is 4·77586.

Examples. No.	Log.	No.	Log.
59·684	= 1·77586	65·439	= 1·81584
94268	= 4·97436	27863	= 4·44503
785920	= 5·89538	131755	= 5·11977

PROPOSITION III.

(G) *To find the logarithm of a pure decimal.*

Rule. Find the logarithm corresponding to the significant figures, as if they were whole numbers. Then if the first significant figure be in the place of *tenths*, the index will be —1; if in the place of hundredths it will be —2; if in the place of thousandths it will be —3, and so on.

Thus, the logarithm of ·3754 is — 1·57449
The logarithm of ·03754 = — 2·57449
The logarithm of ·003754 = — 3·57449
The logarithm of ·0003754 = — 4·57449, &c.

PROPOSITION IV.

(H) *To find the logarithm of a vulgar fraction.*

Rule. Reduce compound fractions, mixed fractions, &c. to simple fractions. Then subtract the logarithm of the denominator from the logarithm of the numerator; if you carry 1, add it to the index of the logarithm of the denominator; then take the difference of the indices for the index to the logarithm of the fraction. For a proper fraction this index will be −, for an improper fraction it will be +.

Required the logarithm of $\frac{5}{18}$.

The logarithm of 5 = 0·69897
The logarithm of 18 = 1·25527

The logarithm of $\frac{5}{18}$ = − 1.44370

In the same manner the logarithm of $\frac{93}{94}$ = − 1·99535; the log. of $\frac{55}{72}$ = − 1·88303; the log. of $\frac{3}{5}$ = − 1·77815; the log. of $\frac{5349}{999}$ = + 0·72870, &c.

PROPOSITION V.

(I) *To find the number answering to any logarithm to four places of figures.*

Rule. Look in the column of logarithms marked 0 at the top of the table, if you do not find your logarithm there, take the nearest less to it which you can find in that column; the first three figures in the natural number will stand in the column marked *No.*, and looking in the same line from the left hand towards the right, you will either find your logarithm or the nearest less to it in the table, the fourth figure of the natural number will stand at the top of the table.

Required the natural number answering to the logarithm 3·90233.

Rejecting the index 3, I look in the column 0, where the first three figures of the logarithm are 902, and against 90200 I find the natural number 798, and looking along that line, under 6, I find 90233 the given logarithm, therefore the natural number is 7986, and the index being three, the natural number is a whole number.

Examples.	Log.	No.	Log.	No.
	1·76343 =	58	2·85775 =	720·7
	2·77056 =	589·6	3·86682 =	7359

PROPOSITION VI.

(K) *To find the number answering to any logarithm, to five or six places of figures.*

RULE. Find the nearest less logarithm in the tables to the given one, and take out the four figures answering to it as before. Take the difference between this logarithm and the next greater in the tables, and also between this logarithm and the given one; divide the latter difference, with ciphers annexed to it, by the former, and place the quotient to the right hand of the natural number already found.

Required the natural number answering to the log. 4·59859.

Given log. 59859	nearest less log. 59857
Next less 59857 nat. numb. 3968	next greater 59868
diff. 2 annex ciphers.	diff. 11

then 200, &c. ÷ 11 = 18, &c. the natural number is therefore 39681·8, the index being 4, there are five whole numbers.

Examples. Log.	No.	Log.	No.
4·97436 =	94267	2·50181 =	317·55
2·97436 =	942·67	4·81584 =	65440
4·89538 =	78592	5·37081 =	234861

PROPOSITION VII.

(L) *To find the product of two whole or mixed numbers.*[*]

RULE. Add the logarithms of the numbers together, the natural number answering to the sum will be the product required.

When several numbers are to be multiplied together, some of which are less than an unit, add the logarithms of the numbers together; when you come to the indices, add the affirmative indices and what you carry into one sum, and the negative indices into another. Take the difference between these sums for the index of the product, prefixing the sign of the greater sum.

Required the product of 84 *and* 56.

Logarithm of 84	=	1·92428
Logarithm of 56	=	1·74819
Product 4704 Log.	=	3·67247

Required the product of 76·5 *by* 5·5.

Logarithm of 76·5	=	1·88366
Logarithm of 5·5	=	0·74036
Product 420·75 Log.	=	2·62402

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[*] Let $10^v = a$; $10^{-x} = b$; $10^{-y} = c$; $10^z = d$, &c. Then $10^{v-x-y+z} = a\,b\,c\,d$, &c. Where v is the logarithm of a; $-x$ the logarithm of b; $-y$ the logarithm of c; z the logarithm of d, &c. And $v-x-y+z$ is the logarithm of $a\,b\,c\,d$. (A. 1. *and* C. 2.)

Required the product of ·84 × ·056 × 37.

The logarithm of ·84 = − 1·92428
of ·056 = − 2·74819
of 37 = + 1·56820

Sum of the decimals and affirmative index + 3·24067
Sum of the negative indices − 3.

Logarithm of the product = 0·24067

The number answering to which is 1·7405, the product required.

Required the product of ·37 × 426 × ·5 × ·004 × ·275 × 336.
Answer 29·128.

PROPOSITION VIII.

(M) *To divide one number by another.*[*]

Subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient.

If any of the indices be negative, or if the divisor be greater than the dividend, change the index of the divisor: then if the indices have unlike signs take their difference, and prefix the sign of the greater; if they have like signs take their sum, and prefix the common sign. When there is an unit to carry from the decimal part of the logarithm of the divisor, subtract it from the index of that logarithm if it be negative, otherwise add it, before the signs are changed.

Divide 3450 *by* 23.

Logarithm of 3450 = 3·53782
Logarithm of 23 = 1·36173

Quotient 150 Log. = 2·17609

Divide 420·75 *by* 76·5.

Logarithm of 420·75 = 2·62402
Logarithm of 5·5 = 0·74036

Quotient 76·5 Log. = 1·88366

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[*] Let $10^x = a$, and $10^y = c$, then x is the logarithm of a, and y is the logarithm of c. Now $\frac{10^x}{10^y} = 10^{x-y} = \frac{a}{c}$, where $x - y$ is the logarithm of $\frac{a}{c}$.

Divide ·00375 *by* ·0678.

Logarithm of ·00375 = − 3·57403
Logarithm of .0678 = − 2·83123

Quotient ·05531 log. = − 2·74280

Divide $\frac{5}{9}$ *by* 937½

Logarithm of $\frac{5}{9}$ = − 1·74473
Logarithm of 937·5 = + 2·97197

Quotient ·0005926 log. = − 4·77276

PROPOSITION IX.

(N) *To involve a number to any power; that is to square, cube, and number, &c.**

RULE. Multiply the logarithm of the number by the number expressing the power, viz. by 2 for the square, 3 for the cube, 4 for the biquadrate, &c.; and the product will be the logarithm of the power required.

If the index to the given logarithm be negative; multiply this index and the decimal part of the logarithm separately by the index of the given power, and subtract the former product from the latter; when you come to the indices, or whole numbers of these products, their difference must be taken, but if there be an unit to carry from the decimal part of the lower product, it must be added to the index of that product before you take the difference. The result will in every case be negative.

Required the cube of 1·2.

Logarithm of 1·2 = 0·07918
3

The cube is 1·728 Log. = 0·23754

Required the square of 25.

Logarithm of 25 = 1.39794
2

The square is 625 Log. = 2·79588

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* Let $10^x = a$, and n = the index of the power, then $10^{nx} = a^n$, where x is the logarithm of a, and nx the logarithm of a^n.

Required the third power of ·0725.

The logarithm of ·0725 is — 2·86034

Then ·86034 × 3 = 2·58102 product of the decimal.

and 2 × 3 = 6· - product of the indices.

Power ·000381 log. = — 4·58102

Required the 6·25 *power of* ·0032.

The logarithm of ·0032 is — 3·50515.

Then ·50515 × 6·25 = 3·15719 { product of the dec.

and 3 × 6·25 = 18·75 { product of the ind.

Power·0000000000000000 25538log.— =16·40719

PROPOSITION X.

(O) *To extract the square or cube root, &c. of any number.*[*]

Divide the logarithm of the number by 2 for the square-root, 3 for the cube-root, &c. and the quotient will be the logarithm of the root.

If the index to the logarithm be negative, and does not exactly contain the divisor, increase it by such a number as will make it exactly divisible, and increase the logarithm also by the same number before you begin to divide.

What is the square-root of 3·24?

Logarithm of 3·24 is 0·51054, which divided by 2 gives 0·25527, the number answering to which is 1·8.

What is the cube-root of 10648?

Logarithm of 10648 is 4·02726, which divided by 3 gives 1·34242, the number answering to which is 22.

What is the cube-root of ·0003811?

The logarithm of ·0003811 is — 4·58104 = — 4 + 0·58104; and by adding 2 to each part, it is = — 6 + 2·58104, divide by 3, then — 2·86034 is the logarithm of the root, the number answering to which is ·0725, the root sought.

What is the ·72 *root of* ·096?

The logarithm of ·096 is — 2·98227 = — 2 + 0·98227, and by adding ·16 to each part (in order that the negative index may divide even by ·72) it becomes — 2·16 + 1·14227, divide by ·72 then − 3· + 1·58648 = − 2·58648 is the logarithm of the root; hence the root sought is ·03859.

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[*] Let $10^x = a$, and n = the index of the root. Then $10^{\frac{x}{n}} = a^{\frac{1}{n}}$, where x is the logarithm of a, and $\frac{x}{n}$ is the logarithm of $a^{\frac{1}{n}}$.

PROPOSITION XI.

(P) *To find the value of a quantity having a vulgar fraction for its exponent.*

RULE. Multiply the logarithm of the given number by the numerator of the exponent, and divide the product by the denominator; the quotient will be the logarithm of the quantity required.

The multiplication must be performed as directed in the 9th Proposition, and the division according to the directions given in the 10th Proposition: for, the numerator denotes the power to which the given number is to be raised, and the denominator shows what root of that power is to be extracted.

What is the value of $\overline{\cdot 096}\rvert^{\frac{25}{18}}$?

The logarithm of ·096 = −2·98227 which multiplied by 25 (Prop. IX.) produces −26·55675; this divided by 18 (Prop. X.) gives −2·58648; the number answering to which is ·03859. *Answer.*

PROPOSITION XII.

(Q) *To find a fourth proportional to three given numbers; or to work a question in the rule of three by logarithms.*

RULE. Add the logarithms of the second and third terms together, and from the sum subtract the logarithm of the first term, the remainder will be the logarithm of the fourth term.

What is the fourth proportional to ·75; 36; *and* ·008?

Logarithm of 36 = 1·55630
Logarithm of ·008 = − 3·90309

Sum = − 1·45939
Logarithm of ·75 = − 1·87506 subtract.

− 1·58433 the number

answering to which is ·384 the fourth proportional required. For, ·75 : 36 :: ·008 : ·384.

PROMISCUOUS EXAMPLES, EXERCISING ALL THE PROPOSITIONS.

(1.) Find the logarithm of $\frac{36\frac{2}{3}}{48}$. *Ans.* − 1·88303.

(2.) Required the logarithm of ·563′ or ·56⅓. *Ans.* − 1·75076

(3.) Required the logarithm of ·06′28803′. *Ans.* − 2·79751

(4.) Find the logarithm of ·0084′97133′. *Ans.* —3·92927.

(5.) Required the product of $56\frac{1}{3} \times \cdot 7 \times \frac{19}{20} \times \frac{44\frac{2}{3}}{47} \times \frac{43}{68\frac{4}{5}}$. *Ans.* Logarithm of the product is 1·34735, and the product = 22·251.

(6.) Required the product of ·05′94405′ × ·583′ × ·0322916′ × ·4′28571′ × ·1′8′. *Ans.* Log. of the product is — 5·94075, and the product = ·000087247.

(7.) Divide ·0565 by ·25. *Quotient* = ·226.

(8.) Divide ·00375 by ·0678. *Quotient* = ·05531.

(9.) Divide 54498 by ·093. *Quotient* = 586000.

(10.) Divide $\frac{182}{369}$ by $\frac{1528}{13}$. *Quotient* = ·0041963.

(11.) Involve 1·05 to the 40th power. *Ans.* 7·0399.

(12.) Required the 3·75 power of 14·79; or find the value of $\overline{14\cdot 79}|^{\frac{15}{4}}$. *Ans.* 24399·5.

(13.) Required the ·34′54′ power of 94·75′; or find the value of $\overline{94\cdot 7\frac{5}{9}}|^{\frac{3451}{9990}}$. *Ans.* 4·81736.

(14.) Involve ·09475′ to the ·34′54′ power. *Ans.* ·44307.

(15.) Find the cube-root of ·000381078. *Ans.* ·0725.

(16.) What is the ·625 root of ·027588 ? *Ans.* ·0032.

(17.) Find a fourth proportional to 58′ . 13″; 11″·75; and 24 hours. *Ans.* 4′ . 50″·6.

(18.) Find a fourth proportional to 23^h . 12′ . 37″; 24 hours; and 7^h . 59′ . 34″. *Ans.* 8^h . 15′ . 53″.

CHAP. III.

THE USE OF THE TABLES OF SINES AND TANGENTS.*

PROPOSITION I.

(R.) *To find the natural sine or cosine of an arc, also the logarithmical sine, tangent, secant, &c.*

Rule. If the degrees in the arc be less than 45, look for them at the top of the table, and for the minutes (if any) in the left hand column marked M, against which, in the column signed at the top of the table with the proposed name, viz. *sine*, *cosine*, &c. stands the sine, cosine, &c. required. If the degrees are more than 45, they must be found at the bottom of the table, and the minutes (if any) must be found in the right hand column. The name in this case, viz. sine, tangent, &c. must be taken at the bottom of the table. To find the secants see the first page of Table III.

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* The construction of these tables will be found at the end of Book II. Chap. V. Before the student reads this and the following chapter, it will be proper for him to read the definitions, &c. in Book II. Chap. I.

The natural sines must be looked for in the table entitled *natural sines*; and the logarithmical sines in the table entitled *logarithmical sines and tangents.*.

Required the natural and logarithm. sine and cosine of 39°.42′.

Natural sine of 39°. 42′ = ·63877, cosine = ·76940.

Logarithmical sine of 39°. 42′ = 9·80534, cosine = 9·88615.

Required the natural and logarithm. sine and cosine of 73°.27′.

Natural sine of 73° . 27′ = ·95857, cosine = ·28485.

Logarithmical sine of 73° . 27′ = 9·98162, cosine = 9·45462.

If the sine, tangent, &c. be wanted to any number of degrees above 90; subtract those degrees from 180° and find the sine, tangent, &c. of the remainder: or subtract 90° from the given number of degrees, and find the *cosine*, *co-tangent*, &c. of the remainder, which is the same thing.

Required the logarithmical sine, tangent, secant, cosine, co-tangent, and co-secant of 137°.29′.

180°
137° . 29′

rem. 42° . 31′ sine = 9·82982, cosine = 9·86752, tangent = 9·96231, co-tangent = 10·03769, secant = 10·13248, co-secant = 10·17018, and these are respectively equal to the cosine, sine, co-tangent, tangent, co-secant, and secant of 47° . 29′ = 137° . 29′ – 90°.

PROPOSITION II.

(S) *To find the logarith. sine, cosine, &c. of an arc to seconds.*

Find the logarithm to the degrees and minutes as in Proposition I. take the difference between this logarithm and the next greater or less in the same column, according as you want a sine or cosine, tangent or co-tangent, &c. multiply this difference by the number of seconds given, and divide the product by 60; add the quotient to the given logarithm if it be a sine, tangent, or secant, but subtract the quotient from the given logarithm if it be a cosine, co-tangent, or co-secant, and the sum, or remainder, will be the logarithm required.

Required the logarith. sine, tangent, and secant of 35°.44′.24″.

Log. sine 35°.44′ = 9·76642 tang. = 9·85700 sec. = 10·09058
next *greater* sine = 9·76660 tang. = 9·85727 sec. = 10·09067

	diff. 18	diff. 27	diff. 9
	24	24	24
	60 | 432	60 | 648	60 | 216
Add the propor. part	7	10	3

Then, sine 35°.44′.24″ = 9·76649. tan. = 9·85710, sec.= 10·09061.

In the same manner the *natural* sine is found, being ·58410.

Required the logarithmical cosine, co-tangent, and co-secant of 35°. 44′. 24″.

Log.cos.35°.44′	=9·90942	co-tan.=10·14300	co-sec.=10·23358
next *less* cosine	=9·90933	co-tan.=10·14273	co-sec.=10·23340
	diff. 9	diff. 27	diff. 18
	24	24	24
	60 | 216	60 | 648	60 | 432
Subtract prop. part	3	10	7

Then cosine 35°. 44′. 24″ = 9·90939, co-tan. = 10·14290, co-secant = 10·23351.

In a similar manner the *natural* cosine is found, being ·81167.

PROPOSITION III.

(T) *To find the degrees, minutes, or degrees, minutes, and seconds, corresponding to any given logarithm, sine, tangent, &c.*

Rule. Find the nearest logarithm to the given one in the table, and the degrees answering to it will be found at the top of the column if the name be there, and the minutes on the left hand; but if the name be at the bottom of the table, the degrees must be found at the bottom of the table, and the minutes on the right hand. To find the arc to seconds, take the difference between the two nearest logarithms to the given one which you can find in the table, also the difference between the given logarithm and the nearest less. Multiply the second difference by 60, and divide the product by the first difference, the quotient will give a number of seconds, which must be added to the degrees and minutes corresponding to the nearest less number in the tables, if your given logarithm be a sine, tangent, or secant; but if your given logarithm be a cosine, co-tangent, or co-secant, the number of seconds must be subtracted from the degrees and minutes corresponding to the nearest less number in the tables.

Find the degrees, minutes, and seconds, corresponding to the logarithmical sine 9·43299.

Nearest sine less than the given one 9·43278 21
Nearest sine greater than the given one 9·43323 60
First difference 45 45 | 1260
Given sine 9·43299
Nearest less 9·43278 answering to 15°.43′ 28quot.
Second diff. 21

Therefore the required arc is 15°. 43′. 28″.

The same manner of proceeding must be observed in finding a tangent, secant, or natural sine.

Find the degrees, minutes, and seconds, corresponding to the logarithmical cosine 9·43297.

Nearest cosine less than the given one 9·43278
Nearest cosine greater than the given one 9·43323
First difference 45

Given cosine 9·43297
Nearest less 9·43278 answering to 74°.17′.
Second difference 19

19
60
45 | 1140
25quot.

Therefore the required arc is 74°.16′.34″.

PROPOSITION IV.

(U) *To find the natural or logarithmical versed sine of an arc, by the help of a table of natural or logarithmical sines.*

To find the natural versed sine; subtract the natural cosine from an unit if the arc be less than 90°, but if greater than 90, add it to an unit.

To find the logarithmical versed sine; find the logarithmical sine of half the arc, double it, and subtract 9·69897 from the product.

Required the natural versed sine of 65°.45′.
Radius = 1·
Nat. cosine 65°.45′ = ·41072
versed sine 65°.45′. = ·58928

Required the logarithm. versed sine of 72°.14′.
Logarithmical sine of 36°.7′ half arc = 9·77043
2
19·54086
9·69897
Log.vers.sine 72°.14″ = 9·84189

Required the natural versed sine of 115°.35′.
Natural cosine 115°.35′ or cosine 64°.25′ = ·43182
To which add 1·
vers.sine of 115°.35′ = 1·43182

Required the log. versed sine of 37°.53′.
Here half the arc is 18°.56½′
Log. sine 18°.56′ = 9·51117
Log. sine 18°.57′ = 9·51154
double sine 18°.56½′ = 19·02271
9·69897
Log.vers.sine 37°.53′ = 9·32374

CHAP. IV.

THE CONSTRUCTION AND USE OF THE PLAIN SCALE.

(W) *The Plain Scale* is a mathematical instrument of extensive use. The scale generally used at sea is a ruler of two feet in length, having drawn upon it equal parts, chords, sines, tangents, secants, &c. These are contained on one side of the scale, and the other side contains their logarithms.

(X) Describe a semicircle with any convenient radius CB (Fig. I. Plate II.); from the centre C draw CD perpendicular to AB, and produce it to F, &c.; draw BE parallel to CF, and join AD and BD.

(Y) *Rhumbs.* Divide the quadrantal arc AD into eight equal parts, with one foot of the compasses in A, transfer the distances A1, A2, A3, &c. to the straight line AD, and it will be a line of rhumbs containing eight points of the compass, or one-fourth of the whole circumference of the compass. By subdividing each of the divisions A1; 1, 2, &c. into four equal parts, and transferring them in the same manner to the line AD, it will contain the points, and half and quarter points.

(Z) *Chords.* Divide the arc BD into nine equal parts, with one foot of the compasses in B and the distances B10, B20, B30, &c.; transfer them to the straight line BD, which will be a line of chords constructed to every ten degrees. The single degrees are constructed by subdividing the arcs, B10; 10, 20; &c. into ten equal parts, and transferring the divisions in the same manner to the line BD.

(A) *Sines.* Through each of the divisions of the arc BD draw lines parallel to CD, such as 80, 10; 70, 20; &c. and the line CB will be divided into a line of sines reckoning from C to B (for CG is the cosine of the arc B80, or the sine of the arc D80, which is ten degrees); if this line be numbered from B towards C, it will become a line of *versed sines.*

(B) *Tangent.* From the centre C draw straight lines through the several divisions of the quadrantal arc BD, to touch the straight line BE, which will become a line of tangents.

(C) *Secants.* Transfer the distances between the centre C and the divisions of the line of tangents, to the line DF, and it will become a line of secants which must be numbered from D towards F, as in the figure.

(D) *Semi-tangents.* From A draw lines through the several divisions of the arc BD, and they will divide the line CD into semi-tangents, which must be marked with the corresponding figures of the arc 10, 20, &c.

The preceding are some of the principal lines on the plain scale, and to render them convenient for practice, they are transferred from a general figure, such as figure the 1st, to a plain scale or ruler, such as figure 2d, and arranged in the same order.

Besides the lines already mentioned, there are, on several scales, other lines derived from the circle, such as

(E) *Miles of Longitude.* Divide the line AC into 60 equal parts, through each of these draw lines *a* 10; *b* 20, &c. parallel to DC, and every tenth of them will cut the arc AD in the points *a*, *b*, *c*, *d*, &c.; from A as a centre, the several divisions of the arc AD may be transferred to the line AD, which will give the divisions of the line of longitude. If this line be laid upon the scale close to the line of chords (as in Fig. III.) both inverted, so that 60 on the scale of longitude be against 0 on the chords, &c. and any degree of latitude be counted on the chords, there will stand opposite to it, on the line of longitude, the miles contained in one degree of longitude in that latitude, a degree at the equator being 60 miles.

These lines are generally drawn to a larger radius than the lines above described, and consequently are applied to a larger scale of chords, as appears by Fig. III.

(F) *Equal Parts.* The divisions of the line AC to form the miles of longitude may be considered as a scale of equal parts, and on some plain scales are laid down as such, marked *Leag.* viz. Leagues or equal parts.

Some scales are divided into equal parts, as in Fig. IV.

The outer one is generally a scale of inches. The others are divided similar to those marked A, B, C, &c. The scale A contains 40 equal parts to an inch, the scale B 35, C 30, &c. But the most useful scale of equal parts is

(G) *The Diagonal Scale.* Draw eleven lines parallel to, and equidistant from, each other, as in Fig. V.; cut them at right angles by the lines BC; EF; 1, 9; 2, 7, &c. then will HG, &c. be divided into ten equal parts, divide the line OP into ten equal parts, also the line *m n*. From the points of division on the line *m n*, draw diagonals to the points of division on the line OP, viz. join *m* and the first division on OP, the first division on *m n*, and the second on OP, &c.

The chief use of such a scale as this, is to lay down any line from a given measure, or to measure any line. In doing which, the units are counted from *m* towards 0, the tens from *m* towards *n*, and the hundreds from *m* towards H.

Thus, for example, to take off the number 242, extend the compasses from *m* to 2, towards H; with one leg fixed in the point 2, extend the other till it reaches 4 in the line *m n*; move one leg of the compasses along the line 2, 7, and the other along the line 4, till you come to the line marked 2 in BC, and you will have the number required.

OF THE LOGARITHMICAL LINES, OR GUNTER'S SCALE.

Mr. Edmund Gunter was the first who applied the logarithms of numbers, and of sines and tangents to straight lines drawn on a scale or ruler; with which, proportions in common numbers, and trigonometry, may be solved by the application of a pair of compasses only. The method is founded on this property, *That the logarithms of the terms of equal ratios are equidifferent.* This was called Gunter's proportion, and Gunter's line; hence the scale is generally called the Gunter.

The logarithmical lines, on *Gunter's* scale, are the eight following:

(H) *S. Rhumb*, or sine rhumbs, is a line containing the logarithms of the natural sines of every point and quarter point of the compass, numbered from a brass pin on the right hand towards the left with 8, 7, 6, 5, 4, 3, 2, 1.

(I) *T. Rhumb*, or tangent rhumbs, also corresponds to the logarithms of the tangents of every point and quarter point of the compass. This line is numbered from near the middle of the scale with 1. 2. 3. 4 towards the right hand, and back again with the numbers 5, 6, 7 from the right hand towards the left. To take off any number of points below four, we must begin at 1 and count towards the right hand; but to take off any number of points above four, we must begin at four and count towards the left hand.

(K) * *Numbers*, or the line of numbers, is numbered from the left hand of the scale towards the right with 1, 2, 3, 4, 5, 6, 7, 8, 9, 1 which stands exactly in the middle of the scale; the numbers then go on 2, 3, 4, 5, 6, 7, 8, 9, 10 which stands at the right hand end of the scale. These two equal parts of the scale are divided equally, the distance between the first or left hand 1, and the first 2, 3, 4, &c. is exactly equal to the distance between the middle 1 and the numbers 2, 3, 4, &c. which follow it. The subdivisions of these scales are likewise similar, viz. they are each one-tenth of the primary divisions, and are distinguished by lines of about half the length of the primary divisions.

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* This line being the principal one on the scale, has been contrived various ways. Gunter first applied it to the two feet rulers, or rather to the cross-staff. Wingate drew the logarithms on two separate rulers sliding against each other, to save the use of compasses. Oughtred applied the logarithms to concentric circles. Mr. William Nicholson proposed another disposition of them on concentric circles; (vide Philosophical Transactions 1787, page 251;) his instrument consists of three concentric circles engraved and graduated on a plate of brass: from the centre, two legs proceed as radii, having straight edges; these

These subdivisions are again divided into ten parts, where room will permit; and where that is not the case, the units must be estimated, or guessed at, by the eye, which is easily done by a little practice.

The primary divisions on the second part of the scale, are estimated according to the value set upon the unit on the left hand of the scale: If you call it one, then the first 1, 2, 3, &c. stand for 1, 2, 3, &c. the middle 1 is 10, and the 2, 3, 4, &c. following stand for 20, 30, 40, &c. and the ten at the right hand is 100: If the first 1 stand for 10, the first 2, 3, 4, &c. must be counted, 20, 30, 40, &c. the middle 1 will be 100, the second 2, 3, 4, 5, &c. will stand for 200, 300, 400, 500, &c. and the ten at the right hand for 1000.

If you consider the first 1 as $\frac{1}{10}$ of an unit, the 2, 3, 4, &c. following will be $\frac{2}{10}$, $\frac{3}{10}$, $\frac{4}{10}$, &c. the middle 1 will stand for an unit, and the 2, 3, 4, &c. following will stand for 2, 3, 4, &c. also the division at the right hand end of the scale will stand for 10. The intermediate small divisions must be estimated according to the value set upon the primary ones.

(L) *Sine.* The line of sines is numbered from the left hand of the scale towards the right, 1, 2, 3, 4, 5, &c. to ten; then 20, 30, 40, &c. to 90, where it terminates just opposite 10 on the line of numbers.

(M) *Versed Sine.* This line is placed immediately under the line of sines, and numbered in a contrary direction, viz. from the right hand towards the left 10, 20, 30, 40, 50, to about 169; the small divisions are here to be estimated according to the number of them to a degree.

(N) *Tangent.* The line of tangents begins at the left hand, and is numbered 1, 2, 3, &c. to 10, then 20, 30, 45, where

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legs or rulers are moveable about the centre either singly or together, so that if one be placed at the first term of any proportion, and the other at the second, and they be then fixed at this angle and moved together till the leg which was fixed at the first term coincides with the third, the other leg will point out the fourth.

there is a little brass pin just under 90 in the line of sines; because the sine of 90° is equal to the tangent of 45°. It is numbered from 45° towards the left hand 50, 60, 70, 80, &c. The tangents of arcs above 45° are therefore counted backward on the line, and are found at the same points of the line as the tangents of their complements.

Thus, the division at 40 represents both 40 and 50, the division at 30 serves for 30 and 60, &c.

(O) *Meridional Parts.* This line stands immediately above a line of equal parts, marked *Equal Pt.* with which it must always be compared when used. The line of equal parts is marked from the right hand to the left with 0, 10, 20, 30, &c.; each of these large divisions represents 10 degrees of the equator, or 600 miles. The first of these divisions is sometimes divided into 40 equal parts, each representing 15 miles.

The extent from the brass pin on the scale of meridional parts to any *division* on that scale, applied to the line of equal parts, will give (in degrees) the meridional parts answering to the latitude of that *division.* Or the extent from any *division* to another on the line of meridional parts, applied to the line of equal parts, will give the meridional difference of latitude between the two places denoted by the *divisions.* These degrees are reduced to leagues by multiplying by 20, or to miles by multiplying by 60.

THE CONSTRUCTION OF THE LOGARITHMICAL LINES ON GUNTER'S SCALE.

(P) *The line of numbers* on which most of the others depend, sometimes called the *Line of Lines,* is constructed thus: Let a line equal in length to half the line of numbers be divided into 1000 equal parts, then since the logarithm of 1 is 0, the distance of 1 from the beginning of the line is 0, viz. 1 stands at the beginning of the line. And because the logarithm of 2 is ·301, when the logarithm of 10 is 1, or, which is the same thing, the logarithm of 2 is 301, when the logarithm of 10 is 1000; therefore the distance between 1 and 2 is 301 equal parts taken from a plain scale of equal parts; the distance 477, the logarithm of 3, must be set off from 1 to 3; and 602 equal parts, the logarithm of 4, must be set off from 1 to 4, &c. Thus are all the primary divisions on the line formed: And the intermediate divisions are formed in a similar manner, by taking the logarithms of the intermediate numbers, as for example, the logarithm of $1\frac{1}{10}$, is 41, the logarithm of $1\frac{2}{10}$ is 79, of $1\frac{3}{10}$ is 114, &c. These numbers set off in a successive order from 1, will divide the primary divisions into ten parts, &c.

(Q) *The line of sines* is constructed by taking the * arithmetical complements of the logarithmical sines from the same scale of equal parts which the line of numbers was constructed from, and setting them off from 90 backwards, or towards the left hand; thus the arithmetical complement of the logarithmical sines of 80°, 70°, 60°, 50°, 40°, 30°, 20°, 10°, (or, which is the same thing, their cosecants rejecting the indices,) are 7, 27, 116, 192, 301, 466, 760, and these taken from the same scale of equal parts with which the line of numbers was constructed, and set off from 90 towards the left hand, give the sines of the abovementioned degrees, respectively. The reason of setting off these numbers thus is obvious, for the sine of 90° is equal to the radius; and the several arithmetical complements are what the sines of the arcs want of radius, viz. their distances from radius or 90°.

In the same manner the intermediate degrees, and divisions of the degrees are found.

(R) *The line of sine rhumbs* is constructed in a similar manner to the line of sines, by taking the arithmetical complements of the logarithmical sines of the degrees and minutes which are contained in the several points, and quarter-points of the compass: and setting them off from the right hand end of the scale, towards the left.

(S) *The line of tangents* is constructed in the same manner as that of the sines, by setting off the arithmetical complements of the tangents under 45°, backward from 45° towards the left hand. For the tangent of 45° is equal to the radius, and the arithmetical complement of any logarithmical tangent under 45°, is what that tangent wants of radius. It has been observed that the division at 40 serves both for 40 and 50, that at 30 for 30 and 60, &c. The reason of this will appear if we consider that the tangent of any arc is to the radius, as the radius to the co-tangent; or, which is the same thing, $\frac{\text{tangent}}{\text{radius}} = \frac{\text{radius}}{\text{co-tangent}}$, or logarithmically speaking, the difference between the tangent and radius is equal to the difference between the radius and co-tangent. But the tangent of 45° is equal to the radius, therefore the difference between the tangent of an arc (below 45°) and 45°, is equal to the difference between the co-tangent of that arc and 45°; viz. they are both equidistant from 45°.

(T) *The line of tangent rhumbs* is constructed in a similar manner as the line of tangents, by taking the arithmetical complements of the logarithmical tangents of the degrees and minutes contained in the first four points of the compass, and setting them from the end of the line towards the left hand.

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* The arithmetical complement of any logarithm is what that logarithm wants of 10; or what a sine, &c. wants of the logarithmical radius. Thus the arithmetical complement of 2·56820, the logarithm of 370, is 7·43180; the arithmetical complement of 9·53405, the logarithmical sine of 20, is ·46595.

(U) *The line of versed sines* is constructed by the help of a table of logarithmical versed sines extending to 180°. Take the logarithmical versed sines of the supplements of the arcs, and subtract the logarithm of 2 from them, the arithmetical complements of the remainders, taken from the same scale of equal parts as the other lines were constructed from, and applied from the right hand towards the left, will give the divisions of the line of versed sines.

(W) The nature and use of this line are, I believe, very imperfectly understood; and in order to explain them clearly, we must have recourse to the inventor, viz. *Gunter;* he says, p. 231, Leyburn's edition 1673, that he contrived this line "for the more easy finding of an angle having three sides, or "a side having three angles of a spherical triangle given." He then gives the following proportions:

As radius	*As this fourth sine*
Is to sine of one of the sides containing any angle; so is the sine of the other containing side, to a fourth sine.	*Is to sine of half the sum of the three sides; so is the sine of this half sum diminished by the side opposite the given angle, to a* seventh sine.

The mean proportional between this seventh sine and the radius, gives the sine of the complement of half the angle required.—By the scale,

(X) Extend the compasses from the sine of 90 to the sine of one of the sides containing any angle; that extent applied the same way will reach from the sine of the other side to a *fourth sine.* From this fourth sine extend the compasses to the sine of half the sum of the three sides, and this extent applied the same way will reach from the sine of the difference between the half sum of the three sides and the side opposite the angle taken, to a *seventh sine;* immediately under which, stands the angle required in the line of versed sines.

It is for this reason that the line of sines and versed sines are placed so close together. This is * *Gunter*'s Rule, and it is more

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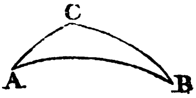
* As *Gunter's* works are merely practical, perhaps an investigation of these proportions will be acceptable to some readers.



Let ABC be any spherical triangle whatever, it is demonstrated in Spherical Trigonometry that sine AB × sine BC : square radius : : sine ½ (AB + BC + AC) × sine ½ (AB + BC + AC) — AC; square cosine ½ B. But square cosine ½ B $-\frac{\text{radius} \times \text{versed sine supp}^{t}\ \text{B}}{2}$, hence simple in its application than that which *Robertson* has given in his Navigation, without demonstration (Art. 29, Book IX.), for working an Azimuth.

(Y) *The line of meridional parts* is constructed by the help of a table of meridional parts. Take the meridional parts correspondent to the several degrees of latitude from the table, and divide them by 60; take these quotients from the scale of equal parts, already described under the article meridional parts, and set them off on the line of meridional parts from the right hand towards the left.

THE USE OF THE LOGARITHMICAL LINES ON GUNTER'S SCALE.

By these lines and a pair of compasses, all the problems of Trignometry and Navigation, &c. may be solved.

(Z) These problems are all solved by proportion: Now in natural numbers, the quotient of the first term by the second is equal to the quotient of the third by the fourth: therefore logarithmically speaking, the difference between the first and

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sine AB × sine BC : square rad. : : sine ½ (AB + BC + AC) × sine ½ (AB + BC + AC) −AC : $\frac{\text{rad.} \times \text{vers. sine supp}^{\text{t}}\ \text{B}}{2}$, or, sine AB × sine BC : rad. : : sine ½ (AB + BC + AC) × sine ½ (AB + BC + AC) −AC : $\frac{\text{vers. sine supp}^{\text{t}}\ \text{B}}{2}$, hence it follows that $\frac{\text{sine AB} \times \text{sine BC}}{\text{rad.}} \times \frac{\text{vers. sine supp}^{\text{t}}\ \text{B}}{2} = \text{sine}\ \tfrac{1}{2}(\text{AB}+\text{BC}+\text{AC}) \times \text{sine}\ \tfrac{1}{2}(\text{AB}+\text{BC}+\text{AC})-\text{AC}$.

Now Gunter's proportions are

rad. : sine AB : : sine BC : $\frac{\text{sine AB} \times \text{sine BC}}{\text{rad.}}$, *the fourth sine,*

and $\frac{\text{sine AB} \times \text{sine BC}}{\text{rad.}}$: sine ½ (AB + BC + AC) : : sine ½ (AB + BC + AC) −AC $\frac{\text{versed sine supp}^{\text{t}}\ \text{B}}{2}$, *the 7th sine*, and multiplying the extremes and means, we have the equation above. The mean proportion between *the 7th sine*, and the *radius*, gives the cosine of half the angle required; for, square cos. ½ B = $\frac{\text{rad.} \times \text{versed sine supp}^{\text{t}}\ \text{B}}{2}$, hence cosine ½ B = $\sqrt{\text{rad.} \times \frac{\text{versed sine supp}^{\text{t}}\ \text{B}}{2}} = \sqrt{\text{rad.} \times \textit{7th sine}}$, the mean proportional. "But because the finding the mean "proportional between the radius or sine of 90° and *the 7th sine*, is somewhat "troublesome," says Gunter, "I have added this line of versed sines, that hav- "ing found *the 7th sine*, you might look over against it and there find the angle." *Gunter* has neither shewn the construction of the line of versed sines nor given the investigation of the above proportions; but he has illustrated them by a practical example, wherein the three sides of a spherical triangle are given to find an angle.

second term is equal to the difference between the third and fourth; consequently on the lines on the scale, the distance between the first and second term will be equal to the distance between the third and fourth. And for a similar reason, because four proportional quantities are alternately proportional, the distance between the first and third terms, will be equal to the distance between the second and fourth. Hence the following

(A) General rule. The extent of the compasses from the first term to the second, will reach, *in the same direction*, from the third to the fourth term. Or, the extent of the compasses from the first term to the third, will reach *in the same direction*, from the second to the fourth.

By the same direction is meant that if the second term lie on the right hand of the first, the fourth will lie on the right hand of the third, and the contrary. This is true, except the first two or last two terms of the proportion are on the line of tangents, and neither of them under 45°; in this case the extent on the tangents is to be made in a contrary direction: For had the tangents above 45° been laid down in their proper direction, they would have extended beyond the length of the scale towards the right hand; they are therefore as it were folded back upon the tangents below 45°, and consequently lie in a direction contrary to their proper and natural order.

(B) If the last two terms of a proportion be on the line of tangents, and one of them greater and the other less than 45°; the extent from the first term to the second, will reach from the third beyond the scale. To remedy this inconvenience, apply the extent between the first two terms from 45° backward upon the line of tangents, and keep the left hand point of the compasses where it falls; bring the right hand point from 45° to the third term of the proportion; this extent now in the compasses applied from 45° backward will reach to the fourth term, or the tangent required. For, had the line of tangents been continued forward beyond 45°, the divisions would have fallen above 45° forward; in the same manner as they fall under 45° backward.

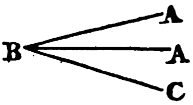
CHAP. V.

GEOMETRICAL DEFINITIONS, AND INTRODUCTORY PROBLEMS.

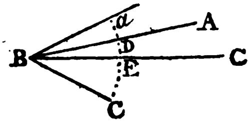
DEFINITIONS, &C. OF ANGLES.

(C) An angle is the inclination or opening of two straight lines meeting in a point as A.

(D) One angle is said to be less than another, when the lines which form it are nearer to each other. Take two lines AB and BC meeting each other in the point B; conceive these two lines to open like the legs of a pair of compasses, so as always to remain fixed to each other in B. While the extremity A moves from the extremity C, the greater is the opening or angle ABC; and, on the contrary, the nearer you bring them together, the less the opening or angle will be.



(E) The magnitude of an angle does not consist in the length of the lines which form it, but in the extent of their opening or inclination to each other. Thus the angle ABC is less than the angle *a*BC, though the lines AB and CB which form the former angle, are longer than the lines *a*B and CB which form the latter.

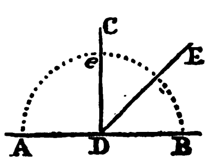


(F) When an angle is expressed by three letters, as ABC, the middle letter always stands at the angular point, and the other two letters at the extremities of the lines which form the angle; thus the angle ABC is formed by the lines AB and CB, and that of *a*BC by the lines *a*B and CB, &c.

(G) Every angle of a triangle is measured by an arc of a circle described about the angular point as a centre, thus the arc *a*C is the measure of the angle *a*BC and the arc DE is the measure of the angle ABC.

(H) The circumference of every circle is supposed to be divided into 360 equal parts called degrees, each degree into 60 equal parts called minutes, each minute into 60 equal parts called seconds. The angles are measured by the number of degrees cut from the circle by the lines which form the angles; thus, if the arc DE contain 20 degrees, or the 18th part of the circumference of the circle, the measure of the angle ABC is 20 degrees. Degrees, &c. are thus marked, $44^\circ\ 32'\ 21''\ 14'''$, &c. and read 44 degrees, 32 minutes, 21 seconds, 14 thirds, &c.

(I) When a straight line CD standing upon a straight line AB, makes the angles CDB and CDA on each side equal to one another, each of these equal angles is said to be a right angle, and the line CD is perpendicular to AB. The measure of a right angle is therefore 90°, or a quarter of a circle.

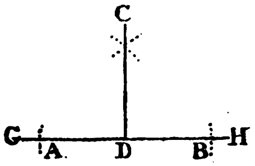


(K) An *acute* angle is less than a right angle, or 90°, as EDB.

(L) An *obtuse* angle is greater than a right angle, or 90°, as ADE.

(M) If ever so many angles are formed at the point D, on the same side of the line AB, they are altogether equal in measure to two right angles, or 180°.

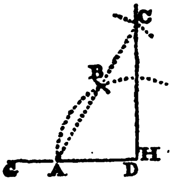
PROBLEM I.



(N) *To erect a perpendicular from a given point* D *in a given line* GH, *or to make a right angle.*

On each side of D take the equal distances AD and BD.

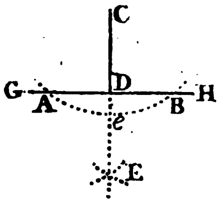
With any extent of the compasses greater than AD, and centres A and B, describe arcs crossing each other in C; a line CD, drawn through C and D, will be the perpendicular required.



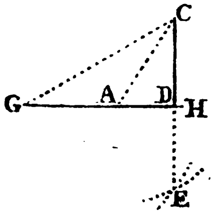
(O) *Otherwise.* When the point D is at the end of the line GH; with the centre D and any opening of the compasses describe an arc; set off the distance AD from A to B; with B as a centre, and the distance AB in your compasses describe another arc; through A and B draw the line ABC, cutting the second arc in C; lastly, through C and D draw the line CD, and it will be the perpendicular required.

PROBLEM II.

(P) *From a given point* C, *not in the straight line* GH, *to draw a staight line* CD *perpendicular to* GH.



Take any point *e* on the contrary side of GH to which the point C is, and with the distance C*e* and centre C describe an arc cutting GH in A and B; with A and B as centres, describe arcs crossing each other in E, a line CDE drawn through C and E will be the perpendicular required.

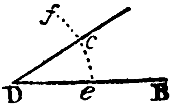


(Q) *Otherwise.* When the point C is near the end of the line GH. Take any points A and G in the line GH, with the centres A and G, and the distances AC and GC, describe arcs crossing each other in E, the line CDE drawn through C and E, will be the perpendicular required.

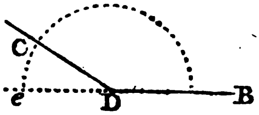
PROBLEM III.

(R) *To make an angle of any proposed number of degrees upon a given straight line, by the scale of chords.*

Upon the line DB to make an angle of 30°. Take the extent of 60° from the line of chords, with which and the centre D, describe the arc *ef*. Take 30° from the same scale of chords and set them off from *e* to C; through C draw the line DC, then CDB is the angle required.



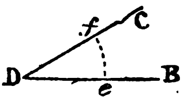
To make an angle of 150°. Produce the line BD to *e*, with the centre D and the chord of 60° describe a semicircle, take the given obtuse angle from 180° and set off the remainder, viz. 30° from *e* to C, through C draw CD, then CDB is the angle required.

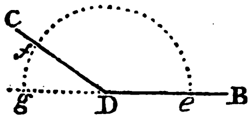


PROBLEM IV.

(S) *An angle being given, to find how many degrees it contains, by a scale of chords.*

With the chord of 60° in your compasses and centre D describe an arc *ef*, cutting DB and DC in *e* and *f*. Then take the distance *ef* in your compasses, and setting one foot on the brass pin at the beginning of the chords on your scale, observe how many degrees the other foot reaches to, and that will be the number of degrees contained in the arc *ef*, or angle CDB.





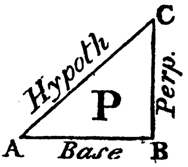
If the extent *ef*, reach beyond the scale, which will always be the case when the angle is obtuse, extend the line BD from D towards *g*, and measure the arc *gf* in the same manner, the degrees it contains deducted from 180°, will give the measure of the angle CDB.

DEFINITIONS, &c. OF TRIANGLES.

(T) *A plane triangle* is a space included by three straight lines, and contains three angles.

(U) Triangles are of various kinds, but in trigonometry only two kinds are considered, viz. right angled triangles, and oblique angled triangles; which indeed include the whole.

(W) *A right angled triangle** is that which has one right angle in it, as P. The longest side AC, or that opposite to the right angle B, is called the hypothenuse; the other two AB and BC, are called the legs, or sides, or the base and perpendicular.



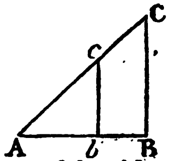
(X) The sum of the three angles of every plane triangle is equal to 180°, hence in a right angled plane triangle, if either acute angle be taken from 90°, the remainder will be the other acute angle.

(Y) *The complement* of an arc, or angle less than 90°, is what that angle wants of a quadrant, or 90°.

(Z) If one acute angle of a right angled triangle be 45°, or half a right angle, the other acute angle will also be 45° or half a right angle; and the base and perpendicular will be equal to each other.

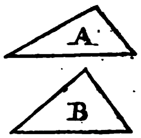
(A) If two right angled triangles ABC and A*bc*, have the angle A common, they are equiangular and similar; that is, the sides about the equal angles are proportional, viz.

AB : BC : : A*b* : *bc* and AB : BC : : AC : A*c*, &c.



Hence if a straight line be drawn parallel to one of the sides of a plane triangle, it will cut the other two sides proportionally.

(B) *An oblique angled triangle* is that which has not a right angle in it; hence two of its angles must necessarily be acute, or each less than a right angle, but the remaining angle may be either greater or less than a right angle; as A and B.



(C) Any one angle of an oblique plane triangle subtracted from 180°, leaves the sum of the other two angles. Or the sum of any two angles subtracted from 180°, leaves the third angle.

(D) *The supplement* of any angle is what that angle wants of 180°. Hence the supplement of any one angle is always equal to the sum of the other two.

(E) Any two sides of a triangle added together are greater than the third side. The greatest side of any triangle is opposite to the greatest angle; and the contrary, the greatest angle is opposite to the greatest side.

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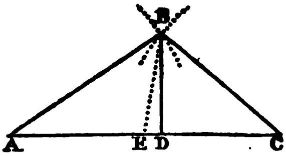
* An *equilateral* triangle has three equal sides; an *isosceles* triangle has two of its sides equal; an *acute angled* triangle has three acute angles; and an *obtuse angled* triangle has two acute angles, and one obtuse.

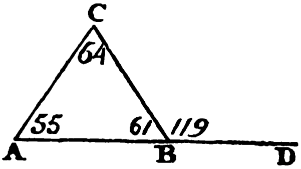
(F) Every triangle has two of its angles acute, and if the third angle be either a right angle, or an obtuse angle, it is opposite to the greatest side.



(G) If a perpendicular BD be drawn upon the longest side of any triangle, from the opposite angle, it will fall within the triangle; and the greater segment AD, will meet the greater (AB) of the other two sides and the less segment DC, will meet the less of these sides (BC).

(H) In an equilateral or isosceles triangle a perpendicular BD drawn from the vertical angle, will bisect both the base and the vertical angle.

(I) If any one side of a plane triangle be produced, or extended beyond the angular point, the outward angle will be equal to the two inward angles, opposite to the angular point where the side is extended. Thus in the triangle ABC, if the measure of the angles be as expressed in the triangle, and the side AB be produced to D; then will the angle CBD, be equal to the angles BAC and BCA together.



(K) In a right angled plane triangle, the hypothenuse is equal to the square-root of the sum of the squares of the base and perpendicular: and the square-root of the difference of the squares of the hypothenuse and either of the other sides, is equal to the remaining side.

(L) Each of the angles of an equilateral triangle is 60 degrees.

(M) Each of the angles at the base of an isosceles triangle is equal to half the supplement of the vertical angle, or to the complement of half the vertical angle.

(N) If two straight lines intersect each other, the vertical or opposite angles will be equal.

(O) If a stright line intersect two parallel straight lines it makes the alternate angles equal to each other, and the exterior angle equal to the interior and opposite on the same side of the line.

(P) All angles in the same segment of a circle are equal to each other.

(Q) An angle at the centre of a circle is double to an angle at the circumference when they both stand on the same arc.

(R) Every angle in a semicircle is a right angle.

BOOK II.

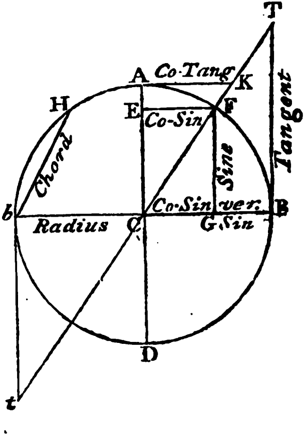
CHAPTER I.

DEFINITIONS OF PLANE TRIGONOMETRY, WITH THE INVESTIGATION OF GENERAL RULES FOR CALCULATING THE SIDES AND ANGLES OF PLANE TRIANGLES.

(A) PLANE TRIGONOMETRY is the art of measuring and calculating the sides and angles of triangles described on a plane surface, or of such triangles as are composed of straight lines. It likewise includes the relation between the radius of a circle and certain other straight lines described in and about a circle.

(B) The theory of triangles is the very foundation of all geometrical knowledge, for all sttraight lined figures may be reduced to trangles. The angles of a triangle determine only its relative species, and are measured in *degrees*, *minutes*, and *seconds* (H. 25); but the sides determine its absolute magnitude, and are expressed in *yards*, *feet*, *chains*, or any other lineal measure.

(C) *A circle* is a plain figure contained under one line called the circumference, to which all lines drawn from the centre are equal. Thus ABDBHA is the circumference; C the centre, and CA, CD, *cb*, CB, CF, are all equal to each other.



(D) The distance from the centre of a circle to the circumference is called the *radius*, thus CA, CB, CD, &c. are radii.

(E) A straight line drawn through the centre of a circle to touch the circumference in two points, is called a diameter, and is always double the radius. Thus AD and B*b* are diameters, and are each of them double of AC or of *b*C.

(F) The exact ratio between

the radius and the circumference of a circle being unknown, mathematicians were at a loss to form a comparison between the sides and angles of a triangle, since they could not compare a straight line with any part of the circumference of a circle. They were therefore under the necessity of determining the relation between the radius of a circle, and certain other straight lines described in and about a circle, called *chords*, *sines*, *tangents*, &c.

(G) *The chord* of an arc is a straight line drawn from one extremity of the arc to the other. Thus *b*H is the chord of the arc *b*H, or of the arc HAFBD*b*.

The chord of an arc of 60° is equal to the radius of the circle.[*]

(H) *The complement* of any arc is the difference between that arc and a quadrant. Or it is the difference between any angle and 90°. Thus the arc AF is the complement of the arc BF, or the angle ACF is the complement of the angle FCB.

(I) *The supplement* of any arc is the difference between that arc and a semi-circle. Or it is the number of degrees which any angle wants of 180°. Thus the arc BF is the supplement of the arc FAH*b*, or the angle FCB is the supplement of the angle FC*b*.

(K) *The sine* of an arc is a straight line drawn from one end of that arc, perpendicular to a diameter passing through the other end of the same arc. Thus FG is the sine of the arc BF, or it is the sine of the supplemental arc FAH*b*.

The sine of an arc of 90° is equal to the radius, for AC *is the sine of the arc* BA. *The sine of an arc of 30° is equal to half the radius.*[†]

(L) *The tangent*[‡] of an arc is a straight line drawn from one extremity of the arc, perpendicular to the diameter, and is terminated by a straight line drawn through the centre of the circle and the other extremity of the arc. Thus BT is the tangent of the arc BF, or of the angle BCF.

The tangent of any arc is equal to the tangent of the supplement

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[*] For (Euclid 15. of IV.) the side of a hexagon, which is the chord of 60°, is equal to the radius of the circumscribing circle.

[†] *The sine of any arc is equal to half the chord of double that arc;* thus let BF and BH (Plate I. Fig. 1.) be equal arcs, then FGH is the chord of the double arc FBH; and FGH is bisected in G (Euclid 3. of III). Therefore if FB be an arc of 30°, FG its sine will be half the chord of 60°, and the chord of 60° has been shewn to be equal to the radius, therefore the sine of 30° is equal to half the radius.

[‡] *The semi-tangent* of an arc is the tangent of half that arc. Let BF be any arc (Plate I. fig. 1.) then *mc* is the semi tangent of that arc. For the angle FCB is double the angle F*b*B (Euclid 20. of III.) consequently the arc OC, of which *cm* is the tangent, is the half of the arc BF.

of that arc.[*] *The tangent of 45° is equal to the radius,*[†] *and an arc of 90° has no tangent.*

(M) *The secant* of an arc is a straight line drawn from the centre of the circle, and produced till it meets the tangent. Thus CT is the secant of the arc BF, or of the angle BCF.

The secant of any arc is equal to the secant of the supplement of that arc.[‡] *The secant of 45° is equal to double the sine of 45°,*[§] *and an arc of 90° has no secant.*

(N) *The versed sine* of an arc is that part of the diameter which is contained between the sine and the arc. Thus BG is the versed sine of the arc BF, and *b*G is the versed sine of the supplemental arc FAH*b*.

The versed sine of the supplement of an arc, is the difference between the versed sine of that arc and the diameter.

For $bG = bB - GB$; or, $GB = bB - bG$.

(O) *The co-sine* of an arc is the sine of the complement of that arc; or it is that part of the diameter contained between the centre of the circle and the sine. Thus FE is the cosine of the arc BF, being the sine of AF, which is the complement of BF; or CG is the cosine of BF, because CG = FE.

The cosine of an arc is equal to the cosine of its supplement.

The cosine of an arc, less than a quadrant, is equal to the radius diminished by the versed sine. For CG = CB − GB.

(P) *The co-tangent* of an arc is the tangent of the complement of that arc. Thus AK is the co-tangent of the arc BF, being the tangent of the arc AF, which is the complement of BF.

The co-tangent of an arc is equal to the co-tangent of its supplement.

(Q) *The co-secant* of an arc is the secant of the complement of that arc. Thus CK is the co-secant of the arc BF, being the secant of the arc AF.

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[*] The supplemental tangent will fall on the contrary side of the diameter. Let BF be any arc, (Plate I. fig. 1.) its supplement is FA*fb*, *bt* is the tangent of the supplement, and it is equal to BT. For the right angled triangles CBT and *cbt* are equi-angular, and have the side CB equal to *cb*.

[†] For in this case the angle BCT, measured by the arc BF (Plate I. fig. 1.,) will be half a right angle; consequently the angle BTC will be half a right angle; the triangle is therefore isosceles, and BC = BT.

[‡] Though the secant of an arc, and the secant of its supplement, be the same in quantity, they have contrary directions like the tangents. For *bt* (Plate I. fig. 1.) has been shewn to be the tangent of the supplemental arc FA*fb*, and equal to TB; it is plain therefore that CT = C*t*.

[§] For if the arc BF = 45°, BC = BT; and (Euclid 47. *of* I.) $CB^2 + BT^2 = CT^2$ the square of the secant, $= CB^2 + AC^2 = AB^2$; therefore AB = CT, that is, the chord of 90° is equal to the secant of 45°. But the sine of 45° is equal to half the chord of 90° (K. 31. and Note), therefore the sine of 45° is equal to half its secant.

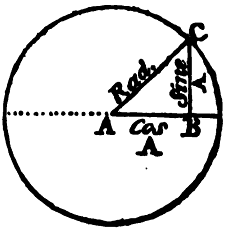
The co-secant of an arc, and the co-secant of its supplement, are of the same magnitude.

(R) *The co-versed sine* of an arc is the versed sine of the complement of that arc. Thus, AE is the co-versed sine of the arc BF, being the versed sine of the arc AF.

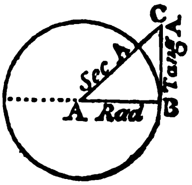
(S) *The co-chord* of an arc is the chord of the complement of that arc.

(T) From the preceding definitions it appears that the sine and co-sine of any arc can never exceed the radius; the secant and co-secant always exceed the radius, and the tangent and co-tangent admit of all possible degrees of magnitude. Also, that any side of a triangle may be considered as the radius of a circle, and that the other sides will necessarily become either sines, tangents, or secants.

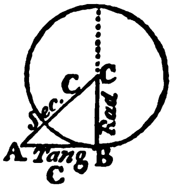
(U) Thus, 1st. *If the hypothenuse* AC be made the radius of a circle, it is evident that the perpendicular BC is the sine of the angle A, and that AB is the cosine thereof But the sine of either of the acute angles of a right angled triangle is the cosine of the other, and the contrary; therefore BC is the cosine of C, and AB is the sine thereof.



(W) 2dly. *If the base* AB be considered as the radius of a circle, BC is evidently the tangent of the angle A, and AC is the secant thereof. But the tangent of one angle is the co-tangent of the other; also the secant of one angle is the co-secant of the other: therefore BC is the co-tangent of C, and AC is the co-secant of C.



(X) 3dly. *If the perpendicular* BC be considered as the radius of a circle, it is evident that the base AB will be the tangent of the angle C, and the hypothenuse AC will be the secant thereof. But because the tangent, secant, &c. of one angle of a right angled triangle is the co-tangent, co-secant, &c. of the other; the base AB will be the co-tangent of the angle A, and the hypothenuse AC will be its co-secant.



INVESTIGATION OF RULES FOR CALCULATING THE SIDES AND ANGLES OF PLANE TRIANGLES.

PROPOSITION I. *(Plate I. Fig. 4.)*

(Y) In any right angled plane triangle, *if any of the three sides be made the radius of a circle; the other sides will be either sines, tangents, or secants, of the respective angle, correspondent to that radius; and will be similar to the sines, tangents, or secants in the tables, when compared with the tabular radius.*

The sines, tangents, &c. of all angles whatever, are calculated to the logarithmical radius of *ten thousand millions*, and arranged in tables for the convenience of calculation. Now as no triangle can be formed, but another may be formed similar to it, and that the sides about the equal angles of similar triangles are proportional; if A*d*, C*e*, A*b*, and C*b*, each represent the radius in the tables, then will AF and CG be the secants, *b*F and G*b* the tangents, and D*d* and E*e* the sines in the tables. Then,

I. *If the hypothenuse be the radius of a circle.*

Because the triangles *e* EC and ABC are similar,

C*e* : AC :: E*e* : AB

viz. radius : hypoth. :: sine of angle C = *cosine of* A : *base.*

Again, because the triangles AD*d* and ABC are similar,

A*d* : AC :: D*d* : BC

viz. radius : hypoth. :: sine of angle A = *cosine of* C : *perp.*

(Z) THEREFORE, the sine of any angle is to the side opposite to it, as the sine of any other angle is to its opposite side.

II. *If the base be the radius of a circle.*

Because the triangles A*b*F and ABC are similar,

AF : AC :: A*b* : AB

And AF : AC :: *b*F : BC

viz. secant of angle A = *cosec. of* C : *hypoth. :: radius : base,*

and, *secant of angle* A = *cosec. of* C : *hypoth. :: tang. of angle* A = *cotan. of* C : *perp.*

(A) HENCE. Radius : base :: tangent of the angle A, or co-tangent of the angle C : perpendicular.

III. *If the perpendicular be the radius of a circle.*

Because the triangles G*b*C and ABC are similar,

CG : AC :: *b*G : AB

And, CG : AC :: C*b* : BC

viz. secant of angle C = *cosec. of* A : *hypoth. :: tang. of angle* C = *cotan. of* A : *base.*

and, *secant of angle* C = *cosec. of* A : *hypoth. :: radius : perpend.*

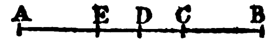
(B) HENCE. Tangent of angle C, or co-tangent of A : base :: radius : perpendicular.

Any of the foregoing quantities are proportional by inversion, &c.

PROPOSITION II.

(C) *The excess of the greater of two given magnitudes above half their sum, is equal to half the difference between those magnitudes: or half their sum increased by half their difference, gives the greater magnitude; and being diminished by half their difference, leaves the less magnitude.*

Let AC and CB be two unequal magnitudes, of which AC is the greater.

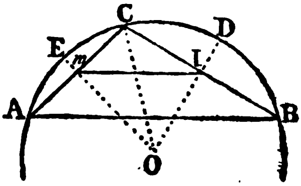


Bisect AB in D and make $AE = CB$, then will $AD =$ half the sum of AC and CB, and $DC =$ half the difference; for, since AB is bisected in D, and $AE = CB$, it follows that EC is bisected in D, and EC is the difference between AC and CB.

Hence $AC - AD = DC$; or $AD + DC = AC$, and $AD - (ED = DC) = AE\ (= BC)$. Q. E. D.

PROPOSITION III.

(D) In any plane triangle, *the sine of any angle, is to the side opposite to it, as the sine of any other angle, is to the side opposite to it, and the contrary.*



Since any triangle whatever may be inscribed in a circle, let ABC be the given triangle and O the centre of the circle. Draw OD and OE perpendicular to the sides of the triangle, then (by Euclid 3 *of* III. and 30 *of* III.) the sides of the triangle will be bisected, and also the arcs which these sides subtend. Now an angle at the centre of a circle is double to an angle at the circumference when they stand on the same arc, (Euclid 20 *of* III.) therefore an angle at the centre is equal to an angle at the circumference standing on the double arc; or in other words, *an angle at the circumference of a circle is measured by half the arc on which it stands, and an angle at the centre by the whole arc on which it stands:* hence the angle $COD =$ the angle CAB, and $COE = CBA$, &c. Now CI is the sine of the angle COD, and Cm is the sine of the angle COE. Draw mI, which will be parallel to AB, then per similar triangles $CI : CB :: Cm :$

AC, viz. the sine of the angle at A, is to the side BC, as the sine of the angle at B, is to the side AC, and the contrary by inversion.

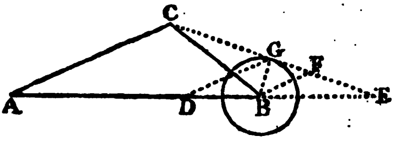
COROL. The sides of triangles are to each other as the chords of double their opposite angles.

PROPOSITION IV.

(E) 1. In any plane triangle, *the sum of any two sides is to their difference, as the tangent of half the sum of their opposite angles, is to the tangent of half their difference.*

Let ABC be any triangle; make BE = BC and join CE, then the angle CBE, being the exterior angle of the triangle ABC, is equal to CAB + BCA. (Euclid 32 *of* I.)

Bisect AE in D and CE in G, then join DG and it will be parallel to AC: draw BF parallel to DG, join BG, and with the centre B and radius BG describe a circle.



The angle CBG = the angle EBG equal to half the sum of the angles CAB and BCA; for the triangles CBG and EBG have the two sides BC and CG, equal to the two sides BE and EG, and the side BG common to both, therefore they are equal in all respects. (Euclid 8 *of* I.)

BF being parallel to AC, the angle FBE = the angle CAB, and the angle CBF = the angle ACB. (Euclid 29 *of* I.)

The angle GBF = half the difference between the angles CBF and FBE, for it is the excess of the greater above half their sum GBE (C. 35); hence GE = tangent of the half sum, and GF = tangent of the half difference between the two angles CBF and FBE. For the same reason, DB being the excess of the greater of the two given lines AB and BE or BC, above half their sum AD, it is equal to half their difference.

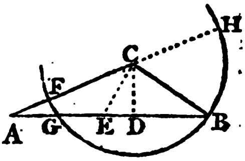
By similar triangles, $DE = \frac{1}{2}(AB + BC) : DB = \frac{1}{2}(AB - BC) :: GE = \text{tangent } \frac{1}{2}(ACB + CAB) : GF = \text{tangent } \frac{1}{2}(ACB - CAB)$ Q. E. D.

Half the sum of the angles ACB and CAB added to half their difference gives the angle opposite to the greater side (C. 35) for the greater *angle* is opposite to the greater side: and half the sum diminished by half the difference, gives the angle opposite to the less side; for the *less* angle is opposite to the less side. (Euclid 18 *of* I.)

(F) NOTE. Instead of the tangent of half the sum of the opposite angles, the tangent of half the supplement of the contained angle may be used; or the co-tangent of half the contained angle *, which are all equal to each other.

PROPOSITION V.

(G) In any plane triangle, *double the base or longest side, is to the sum of the other two sides, as their difference is to the distance of a perpendicular from the middle of the base; this distance, added to half the base, gives the greater segment, and being subtracted leaves the less.*



Let ABC be the triangle, with C as a centre, and the radius CB = the least side, describe a circle, produce AC to H; then because CF = CB = CH; AA = AC + CB the sum of the sides, and AF = AC − BC the difference between the sides.

Because CD is perpendicular to GB, GD = BD (Euclid 3 *of* III.) therefore AG is the difference between the segments of the base; but ED = ½ AG (C. 35.) for it is the excess of the greater segment AD above half their sum AE; AB being the base, or sum of the segments AD and DB, and AE its half.

Then (Euclid 36 *of* III. corol.) AH × AF = AB × AG, therefore AB : AH ∷ AF : AG, that is AB : AC + CB ∷ AC − CB : 2ED, or, 2AB : AC + CB ∷ AC − CB : ED the distance of the perpendicular CD from the middle E of the base; and that ED added to half the base AE gives AD the greater segment, &c. is obvious. Q. E. D.

PROPOSITION VI.

(H) *From half the sum of the three sides of any triangle subtract the side opposite to the angle sought, and note the half sum and remainder. Then, the rectangle of the sides containing that angle, is to the square of the radius, as the rectangle of the half sum and remainder is to the square of the cosine of half the angle taken.*

Let ABC be the triangle; bisect the angle CAB by the indefinite line AH, and draw EC perpendicular to AD. Draw DG parallel and equal to CB, join GB and produce it to H.

---

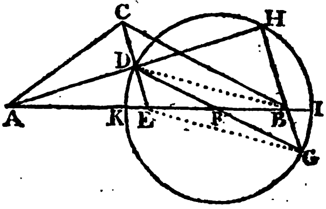
* Since the sum of the three angles of every triangle is equal to two right angles, it is plain that half the sum of any two angles is half the supplement of the third angle. And, that the tangent of half the supplement of any angle is equal to the co-tangent of half that angle may be thus shown:

Let HBF (Fig. I. Plate 1.) be any arc bisected in B, and FA*f*, its supplement, bisected in A; then AK, the co-tangent of the half arc BF, is evidently the tangent of the half supplement AF.

Now the triangles ADE and ADC have the two sides DC and DE equal to each other (by construction), and the side AD common, also the angles about D are right angles, therefore $AC = AE$, (Euclid 4 *of* I.) and consequently EB is equal to the difference between the sides AC and AB.

Because the side EC of the triangle CEB is bisected in D, and DF is parallel to CB (by construction), $EF = FB$, therefore $FB =$ half the difference between the sides AC and AB, and $AF =$ half their sum. (C. 35.)

By construction CB is equal and parallel to DG, therefore (Euclid 33 *of* I.) GB is equal and parallel to CD or DE, for the same reason DBGE is a parallelogram, and $FD = FG$, for the diagonals of a parallelogram bisect each other. And, since GB is parallel to ED, GH is also parallel to it, and the angle at D being a right angle, the angle at H is also a right angle. With F as a centre, and radius $FD = FG = \frac{1}{2} CB$ describe a circle, which will pass through the point H. (Euclid 31 *of* III.)



Since $FG = \frac{1}{2} CB$, $FI = \frac{1}{2} CB$, but $AF = \frac{1}{2}(AC + AB)$, to each of these equals add FI or the half of CB, then will $AI = \frac{1}{2}(AC + AB + BC)$, but $KI = DG = CB$, therefore $AK = AI - CB = \frac{1}{2}(AC + AB + BC) - BC$

$$\left\{\begin{array}{l}\text{radius} : AE :: \text{sine } AED = \text{cosine } DAE : AD \\ \text{radius} : AB :: \text{sine } ABH = \text{cosine } HAB = DAE : AH\end{array}\right\} \quad \text{(Z. 34.)}$$

Hence, square radius : $AE \times AB$ :: square cosine DAE : $AD \times AH$. But $AH \times AD = AI \times AK$ (Euclid 36 *of* III.), and $AE = AC$; therefore square radius : $AC \times AB$ :: square cosine DAE : $AI \times AK$; or $AC \times AB$: square radius :: $AI \times AK$: square cosine DAE; that is, $AC \times AB$: square radius :: $\frac{1}{2}(AC + AB + BC) + (\frac{1}{2} AC + AB + BC) - BC$: square cosine $DAE = \frac{1}{2} CAE$ or CAB. Q. E. D.

PROPOSITION VII.

(I) *From half the sum of the three sides of any plane triangle subtract the side opposite to the angle sought, and note the remainder: Also from the said half sum subtract each of the other sides of the triangle.* Then,

The rectangle contained by half the sum of the sides and the first remainder, is to the square of the radius; as the rectangle contained by the other two remainders is to the square of the tangent of half the angle sought.

The same construction being made as in the preceding proposition, $AI = \frac{1}{2}(AC + AB + BC)$, and $AK = \frac{1}{2}(AC + AB + BC) - BC$.

In the right angled triangles ADE and AHB.

$$\left.\begin{array}{l}\text{radius} : AD :: \text{tangent } DAE : DE \\ \text{radius} : AH :: \text{tangent } DAE : BH\end{array}\right\} \quad (A. 34.)$$

Hence, square radius : $AD \times AH$ :: square tangent DAE : $DE \times BH$. And $AD \times AH$: square radius :: $DE \times BH$: square tangent DAE.

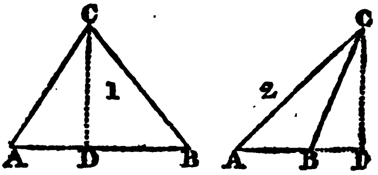
But $AD \times AH = AI \times AK$ (Euclid 36 *of* III.) and because $DE = GB$ being opposite sides of a parallelogram, we have $DE \times BH = GB \times BH = KB \times BI$. (Euclid 35 *of* III.) Again, because, KI is bisected in F, and that FE and FB are equal; KE is equal to BI, and therefore KB is equal to EI; but $EI = AI - AE = AI - AC$, consequently $KB = AI - AC$, and $BI = AI - AB$.

Hence $AI \times AK$: square radius :: $(AI - AC) \times (AI - AB)$: square tangent DAE, viz. $\frac{1}{2}(AC + AB + BC) \times (\frac{1}{2}\overline{AC + AB + BC} - BC)$: square radius :: $(\frac{1}{2}\overline{AC + AB + BC} - AC) \times (\frac{1}{2}\overline{AC + AB + BC} - AB)$: square tangent DAE. Q. E. D.

PROPOSITION VIII.

(K) *Twice the rectangle of any two sides of a plane triangle, is to the sum of the squares of the same two sides diminished by the square of the third side; as radius, is to the cosine of the angle contained between those sides.*

Let ABC be any plane triangle and CD a perpendicular from the vertical angle upon the base or upon the base produced.



1. If the perpendicular falls within the triangle $AC^2 = AB^2 + BC^2 - 2AB \times BD$. (EUCLID 13 *of* II.)

2. If the perpendicular falls without the triangle $AC^2 = AB^2 + BC^2 + 2AB \times BD$. (EUCLID 13 *of* II.)

From the first $BD = \dfrac{AB^2 + BC^2 - AC^2}{2AB}$.

From the second $BD = \dfrac{AC^2 - AB^2 - BC^2}{2AB}$.

But BC : rad. :: BD : cosine of the angle B (Z. 34.)

$\because BD = \dfrac{\cos. \angle B \times BC}{\text{rad.}}$, here $\angle B$ is acute in the first triangle, and obtuse in the second. By substitution,

$$\frac{\cos. \angle B \times BC}{\text{rad.}} = \frac{AB^2 + BC^2 - AC^2}{2AB},$$

$$\text{or, } \frac{-\cos. \angle B \times BC}{\text{rad.}} = \frac{AC^2 - AB^2 - BC^2}{2BA}:$$

from either of these equations, by reduction,

$$\cos.\angle\text{B}=\frac{\text{rad.}\times(\text{AB}^2+\text{BC}^2-\text{AC}^2)}{2\text{AB}\times\text{BC}},$$

or, $2\text{AB}\times\text{BC} : \text{AB}^2+\text{BC}^2-\text{AC}^2 :: \text{rad.} : \cos.\angle\text{B}$. Q. E. D.

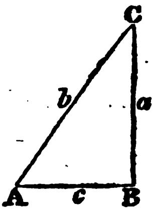
Note. If the sides of the triangle be large, the rule given above is inconvenient for arithmetical calculation.

SCHOLIUM.

(L) Various other propositions might be given, but the preceding are sufficient for solving every case of plane trigonometry. The analogies of plane trigonometry are only particular cases of spherical trigonometry, and may in general be deduced from thence by simple inferences only.

Vide Book III. Chap. V. Prop. xxviii. and the *Scholium.*

SOLUTIONS OF THE DIFFERENT CASES OF RIGHT ANGLED PLANE TRIANGLES.



Rules for solving the different cases both of right angled plane triangles, and oblique angled plane triangles, have already been investigated. The solutions here given are adapted to the table of natural sines*; and the method of notation is that used by Lagrange and Legendre, the three angles of the triangle being represented by A, B, C, and their opposite sides by a, b, c, as in the figure annexed; r=radius=sine of 90°

Case I. Given the angles and the hypothenuse, to find the base and perpendicular.

Solution. $$a=\frac{\text{sine A}\,.\,b}{r}=\frac{\cos\text{ C}\,.\,b}{r}=\frac{\text{tang A}\,.\,b}{\text{sec A}}=\frac{r\,.\,b}{\text{cosec A}}.$$

$$c=\frac{\cos\text{ A}\,.\,b}{r}=\frac{\text{sine C}\,.\,b}{r}=\frac{r\,.\,b}{\text{sec A}}=\frac{\cot\text{ A}\,.\,b}{\text{cosec A}}.$$

Case II. Given the angles and the base, to find the hypothenuse and the perpendicular.

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* Logarithmical formulæ are easily supplied, thus in the first case (log. sine A + log. b) − 10 = log. a. The introduction of such formulæ would only increase the size of the book, without any real advantage to the student.

SOLUTION. $b=\frac{r\,.\,c}{\cos A}=\frac{r\,.\,c}{\text{sine } C}=\frac{\sec A\,.\,c}{r}=\frac{\text{cosec } A\,.\,c}{\cot A}$.

$a=\frac{\text{sine } A\,.\,c}{\cos A}=\frac{\cos C\,.\,c}{\text{sine } C}=\frac{\text{tang } A\,.\,c}{r}=\frac{r\,.\,c}{\cot A}$

CASE III. Given the angles and the perpendicular, to find the base and the hypothenuse.

SOLUTION. $c=\frac{\cos A\,.\,a}{\text{sine } A}=\frac{\text{sine } C\,.\,a}{\cos A}=\frac{r\,.\,a}{\text{tang } A}=\frac{\cot A\,.\,a}{r}$.

$b=\frac{r\,.\,a}{\text{sine } A}=\frac{r\,.\,a}{\cos C}=\frac{\sec A\,.\,a}{\text{tang } A}=\frac{\text{cosec } A\,.\,a}{r}$

CASE IV. Given the hypothenuse and the base, to find the angles and the perpendicular.

SOLUTION. Cos $A=$ sine $C=\frac{r\,.\,c}{b}$; sec $A=$ cosec $C=\frac{r\,.\,b}{c}$.

tang $\frac{1}{2}A=r\sqrt{\frac{b-c}{b+c}}$

Having found the angles, the perpendicular may be found by Case I. or by Case II. OR, $a=\sqrt{b^2-c^2}=\sqrt{(b+c).(b-c)}$.

CASE V. Given the hypothenuse and the perpendicular, to find the angles and the base.

SOLUTION. Sine $A=$ cos $C=\frac{r\,.\,a}{b}$; sec $C=$ cosec $A=\frac{r\,.\,b}{a}$;

tang $\frac{1}{2}C=r\sqrt{\frac{b-a}{b+a}}$.

Having found the angles, the base may be found by Case I. Or, $c=\sqrt{b^2-a^2}=\sqrt{(b+a)\,.\,(b-a)}$.

CASE VI. Given the base and the perpendicular, to find the angles and the hypothenuse.

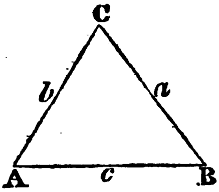
SOLUTION. Tang $A=$ cot $C=\frac{r\,.\,a}{c}$; cot $A=$ tang $C=\frac{r\,.\,c}{a}$;

$$=\sqrt{a^2+c^2}=a\sqrt{1+\frac{c^2}{a^2}}=c\sqrt{1+\frac{a^2}{c^2}}.$$

Or, having found the angles, the hypothenuse may be found by Case II. or by Case III.

SOLUTIONS OF THE DIFFERENT CASES OF OBLIQUE ANGLED PLANE TRIANGLES.

Case I. Given the angles and one side, to find the other two sides.



$$a=\frac{\text{sine A}\,.\,c}{\text{sine C}};\quad b=\frac{\text{sine B}\,.\,c}{\text{sine C}}$$

$$b=\frac{\text{sine B}\,.\,a}{\text{sine A}};\quad c=\frac{\text{sine C}\,.\,a}{\text{sine A}}\ \Big\|\ a=\frac{\text{sine A}\,.\,b}{\text{sine B}};\quad c=\frac{\text{sine C}\,.\,b}{\text{sine B}}.$$

Case II. Given two sides and an angle opposite to one of them, to find the other angles, and the third side.

Solution. $\text{Sine B}=\dfrac{\text{sine A}\,.\,b}{a}$; $\text{sine A}=\dfrac{\text{sine B}\,.\,a}{b}\ \Big\|\ \text{sine C}=\dfrac{\text{sine A}\,.\,c}{a}$; $\text{sine A}=\dfrac{\text{sine C}\,.\,a}{c}\ \Big\|\ \text{sine C}=\dfrac{\text{sine B}\,.\,c}{b}$; $\text{sine B}=\dfrac{\text{sine C}\,.\,b}{c}$.

Note. If the *given* ∠ be obtuse; or if the side opposite to the given ∠ be the *greater* of the two given sides, then the ∠ sought is always acute: in every other case it is ambiguous.

Two angles being now given, their sum deducted from 180°, leaves the third ∠; hence the remaining side may be found by Case I.

Case III. Given two sides and the angle contained between them, to find the other angles; and the third side.

Solution. $\text{Tang}\ \frac{1}{2}(\text{A}\sim\text{B})=\dfrac{a\sim b}{a+b}\cdot\ \cot\frac{1}{2}\text{C}$ (E. 36.)

Then $\frac{1}{2}(\text{A}\sim\text{B})+(90^\circ-\frac{1}{2}\text{C})=$ ∠ opposite to the greater side.
And $90^\circ-\frac{1}{2}\text{C})-\frac{1}{2}(\text{A}\sim\text{B})=$ ∠ opposite to the less side.

OR THUS,

$b : a :: r : \text{tang } \varphi$, and if a be the greater of the two given sides, φ will be greater than 45°.

$r : \text{tang } (\varphi - 45°) :: \cot \frac{1}{2} \text{C} : \text{tang } \frac{1}{2} (\text{A} \sim \text{B})$.

OR THUS,

$$\text{Tang B} = \frac{\text{sine C} . b}{a - \frac{\cos \text{C} . b}{r}} \text{ ; tang A} = \frac{\text{sine C} . a}{b - \frac{\cos \text{C} . a}{r}}$$

The remaining side may be found from the angles, by Case I.

OR THUS,

$$c = \sqrt{a^2 + b^2 - \left(\frac{2a . b}{r} . \cos \text{C}\right)} \text{; (K. 39.)}$$

CASE IV. Given the three sides, to find the angles.

$$\textbf{Solution. } \text{Cos} \tfrac{1}{2} \text{C} = r \sqrt{\frac{\frac{1}{2}(b+a+c) . \frac{1}{2}(b+a+c) - c}{ba}} \text{(H.37.)}$$

$$\text{Tang } \tfrac{1}{2} \text{C} = r \sqrt{\frac{\frac{1}{2}(b+a+c) - a . \frac{1}{2}(b+a+c) - b}{\frac{1}{2}(b+a+c) . \frac{1}{2}(b+a+c) - c}} \text{ (I. 38.)}$$

$$\text{Cos C} = \frac{a^2 + b^2 - c^2}{2ab} . r \text{ (K. 39.)}$$

$$\text{Vers C} = \frac{c^2 - \overline{b \sim a}^2}{2ab}.$$

CHAP. II.

I. PRACTICAL RULES FOR THE SOLUTION OF ALL THE DIFFERENT CASES IN RIGHT ANGLED PLANE TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

(M) In every right angled plane triangle there must be two *given* quantities, one of which must be a side, (together with the right angle,) to find the rest.

I. TO FIND A SIDE.

Call any one of the sides of the triangle radius, and write upon it the word radius; observe whether the other sides become sines, tangents, or secants, and write these words on them accordingly. Call the word written upon each side the name of that side. Then say,

As the *name* of the given side, is to the given side; so is the name of the required side, to the required side.

II. TO FIND AN ANGLE.

Call either of the *given* sides of the triangle radius, and write upon it the word radius; observe whether the other sides become sines, tangents, or secants, and write these words on them accordingly. Call the word written upon each side the name of that side. Then say,

As the side made radius, is to radius; so is the other *given* side to its name. The angles being found, the remaining side must be found by the first part of this rule.

(N) OR; IN ANY RIGHT ANGLED TRIANGLE.

I. The sine of any angle is to the side opposite to it, as the sine of any other angle is to its opposite side. *And,*

Any side is to the sine of its opposite angle, as any other side is to the sine of its opposite angle.

II. Radius is to the tangent of either of the acute angles, as the side adjacent to that angle is to the side opposite to it.

And, The side adjacent to either of the acute angles is to the side opposite to that angle, as radius is to the tangent of that angle.

III. Radius is to the co-tangent of either of the acute angles, as the side opposite to that angle is to the side adjacent to it.

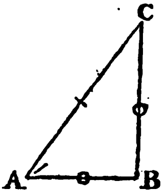
Also, the side opposite to either of the acute angles is to the side adjacent to that angle, as radius is to the co-tangent of that angle.

(O) Case I. *Given the angles* and the hypothenuse, to find the base and perpendicular.*

Given { The hypothenuse AC=480 } Required the base AB
{ The angle A=53°.8' } and perpendicular BC.

BY CONSTRUCTION.

Draw the line AB of any length, make the angle CAB=53°.8' (R.27) by a scale of chords, draw the hypothenuse AC=480 from a scale of equal parts, from C let fall the perpendicular CB (P. 26) then ABC is the triangle required. AB measured by the same scale of equal parts, which the hypothenuse was measured by, will be 288, and BC will be 384.



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* The *given* parts of a triangle are generally marked with a dash 1, and the *required* parts with an O, as in the figures annexed to the constructions.

BY CALCULATION.

I. *The hypothenuse radius*, BC will be the sine of the angle A, and AB the cosine.

To find BC.		To find AB.	
radius, sine of 90°,	10·00000	radius, sine of 90°,	10·00000
: hypoth. AC=480	2·68124	: hypoth. AC=480	2·68124
: : sine ∠ A=53°.8′	9·90311	: : cosine ∠ A 53°.8′	9·77812
: perpendicular BC=384	2·58435	: base AB=288	2·45936

II. *The base radius*, BC will be the tangent of the angle A, and AC will be the secant thereof.

To find BC.		To find AB.	
secant ∠ A=53°.8′	10·22188	secant ∠ A=53°.8′	10·22188
: hypoth. AC=480	2·68124	: hypoth. AC=480	2·68124
: : tang. ∠ A=53°.8′	10·12499	: : radius, sine of 90°,	10·00000
: perpendicular BC=384	2·58435	: base AB=288	2·45936

III. *The perpendicular radius*, the base AB will be the tangent of the angle C, or co-tangent of A, and the hypothenuse AC will be the secant of C, or the co-secant of A.

To find BC.		To find AB.	
co-secant ∠ A=53°.8′	10·09689	co-secant ∠ A=53°.8′	10·09689
: hypoth. AC=480	2·68124	: hypoth. AC=480	2·68124
: : radius, sine of 90°	10·00000	: : co-tangent ∠ A=53°·8′	9·87501
: perpendicular BC=384	2·58435	: base AB=288	2·45936

(P) BY GUNTER'S SCALE.

In working the several cases by Gunter's scale, we shall always suppose the hypothenuse radius, (where it can be done,) being the most simple of the three.

1. Extend the compasses from 90° to 53°.8′ on the line of sines, that extent will reach from 480 to 384 on the line of numbers.

2. Extend the compasses from 90° to 36°.52′ the complement of the angle A, that extent will reach from the hypothenuse AC=480, to the base AB=288, on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right angled plane triangle ABC,

Given { Hypoth. AC=645, ∠ A=39°.10′ } *Ans.* { AB=500, BC=407·37 }

Required the base and perpendicular.

2. In the right angled plane triangle ABC,

Given { Hypoth. AC=98, ∠ A=56°.48′ } *Ans.* { AB=53·66, BC=82·01 }

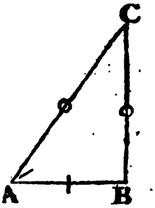
Required the base and perpendicular.

(Q) Case II. *Given the angles and the base, to find the hypothenuse and the perpendicular.*

$$\text{Given}\begin{cases}\text{The angle A} = 53^\circ.8' \\ \text{The base AB} = 288\end{cases}\text{Required the hypoth. AC, and the perpend. BC.}$$

BY CONSTRUCTION.

Draw the base AB, which make = 288, from a scale of equal parts, at B erect the perpendicular BC (N. 26), make the angle A = $53^\circ.8'$. (R. 27), and draw the hypothenuse AC to cut the perpendicular BC in the point C. Then will AC measure 480, and BC 384.



BY CALCULATION.

I. *The hypothenuse radius,* BC will be the sine of the angle A, and AB the cosine.

To find AC.		To find BC.	
cosine ∠ A = $53^\circ.8'$	9·77812	cosine ∠ A = $53^\circ.8'$	9·77812
: base AB = 288	2·45939	: base AB = 288	2·45939
: : radius, sine of 90°	10·00000	: : sine ∠ A = $53^\circ.8'$	9·90311
: hypoth. AC = 480	2·68127	: perpend. BC = 384	2·58438

II. *The base* AB *radius,* BC will be the tangent of A, and AC the secant thereof.

To find AC.		To find BC.	
radius	10·00000	radius	10·00000
: base AB = 288	2·45939	: base AB = 288	2·45939
: : secant ∠ A = $53^\circ.8'$	10·22188	: : tang. ∠ A = $53^\circ{\cdot}8'$	10·12499
: hypoth. AC = 480	2·68127	: perp. BC = 384	2·58438

III. *The perpendicular* BC *radius,* AB will be the tangent of C, or co-tangent of A, and AC will be the secant of C, or co-secant of A.

To find AC.		To find BC.	
co-tangent ∠ A = $53^\circ.8'$	9·87501	co-tangent ∠ A = $53^\circ.8'$	9·87501
: base AB = 288	2·45939	: base AB = 288	2·45939
: : co-secant ∠ A = $53^\circ.8'$	10·09689	: : radius	10·00000
: hypoth. AC = 480	2·68127	: perpend. BC = 384	2·58438

(R) BY GUNTER'S SCALE.

1st. Extend the compasses, from $36^\circ.52'$ the complement of A to 90, on the line of sines; that extent will reach from 288 to 480 = AC on the line of numbers.

2d. Extend from $36^\circ.52'$ to $53^\circ.8'$ on the line of sines; that extent will reach from 288 to BC = 384 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right angled plane triangle ABC,

Given $\begin{cases} \angle A = 39^\circ\ 10' \\ \text{Base } AB = 500 \end{cases}$ *Ans.* $\begin{cases} AC = 645 \\ BC = 407{\cdot}37 \end{cases}$

Required the hypothenuse and the perpendicular.

2. In the right angled plane triangle ABC,

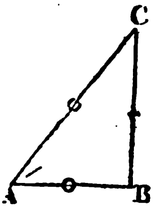
Given $\begin{cases} \angle A = 56^\circ.48' \\ \text{Base } AB = 53{\cdot}66 \end{cases}$ *Ans.* $\begin{cases} AC = 98 \\ BC = 82{\cdot}01 \end{cases}$

Required the hypothenuse and the perpendicular.

(S) CASE III. *Given the angles and perpendicular, to find the base and hypothenuse.*

Given $\begin{cases} \text{The angle } A = 53^\circ.8' \\ \text{The perpend. } BC = 384 \end{cases}$ $\begin{cases} \text{Required the hypothenuse} \\ AC, \text{ and the base } AB. \end{cases}$

BY CONSTRUCTION.



Draw the line AB of a sufficient length, at any point B erect the perpendicular BC (N. 26.) which make equal to 384 by a scale of equal parts, at C make the angle ACB = 36°.52′ (R. 27.), the complement of A, from C draw the hypothenuse, and it will cut the base AB in the point A. Then will AC measure 480, and AB 288.

BY CALCULATION.

I. *The hypothenuse radius,* BC will be the sine of A, and AB the cosine thereof.

To find AC.		To find AB.	
sine ∠ A = 53°.8′	9·90311	sine ∠ A = 53°.8′	3 90311
: perpend. BC = 384	2·58433	: perpend. BC = 384	2·58433
: : radius	10·00000	: : cosine ∠ A = 53°.8′	9·77812
: hypoth. AC = 480	2·68122	: base AB = 288	2·55934

II. *The base* AB *radius,* BC will be the tangent of A, and AC the secant thereof.

To find AC.		To find AB.	
tangent ∠ A = 53°.8′	10·12499	tangent ∠ = 53°.8′	10·12499
: perpend. BC = 384	2·58433	: perpend. BC = 384	2·58433
: : secant ∠ A = 58°.8′	10·22188	: : radius	10·00000
: hypoth. AC = 480	2·68122	: base AB = 288	2·45934

III. *The perpendicular* BC *radius,* the base AB will be the tangent of C, or the co-tangent of A; AC will be the secant of C, or co-secant of A.

To find AC.		To find AB.	
radius	10·00000	radius	10·00000
: perpend. BC = 384	2·58433	: perpend. BC = 384	2·58433
: : co-secant ∠ A = 53°.8′	10.09689	: : co-tangent ∠ A = 53°.8′	9·87501
: hypoth. AC = 480	2·68122	; base AB = 288	2·45934

(T) BY GUNTER'S SCALE.

1. Extend the compasses from 53°.8′ to 90° on the line of sines, that extent will reach from 384 to 480, the hypothenuse on the line of numbers.

2. Extend from 53° 8′.to 36°.52′, the complement of the angle A, on the line of sines, that extent will reach from 384 to 288 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right angled plane triangle ABC,

Given $\begin{cases} \angle A = 39°.10' \\ \text{Perp. } BC = 407{\cdot}37 \end{cases}$ *Ans.* $\begin{cases} AC = 645 \\ AB = 500 \end{cases}$

Required the hypothenuse and the base.

2. In the right angled plane triangle ABC,

Given $\begin{cases} \angle A = 56°.48' \\ \text{Perp. } BC = 82{\cdot}01 \end{cases}$ *Ans.* $\begin{cases} AC = 98 \\ AB = 53{\cdot}66 \end{cases}$

Required the hypothenuse and the base.

(U) Case IV. *Given the hypothenuse and base, to find the angles and the perpendicular.*

Given $\begin{cases} \text{The hypothenuse } AC = 480 \\ \text{The base } AB = 288 \end{cases}$ $\begin{cases} \text{Required the angles A} \\ \text{and C, and the perp. BC.} \end{cases}$

BY CONSTRUCTION.

Draw the base AB equal to 288 from a scale of equal parts, at B erect the perpendicular BC (N. 26); with the distance AC=480, taken from a scale of equal parts, cross BC in the point C. Then BC measured by a scale of equal parts will be 384, and the angles A and C measured by a scale of chords (S. 27.) will be 53°.8′ and 36°.52.

BY CALCULATION.

I. *The hypothenuse radius*, BC will be the sine of the angle A, and AB the cosine thereof.

To find the angle A.		To find the perpend. BC.	
hypoth. AC=480	2·68124	radius	10·00000
: radius	10·00000	: hypoth. AC=480	2·68124
: : base AB=288	2·45939	: : sine ∠ A=53°.8′	8·90311
: cosine ∠ A=53°.8′	9 77815	: perpend. BC=384	2·58435

II. *The base radius*, BC will be the tangent of A, and AC the secant thereof.

To find the angle A.		To find BC.	
base AB=288	2·45939	radius	10·00000
: radius	10·00000	: base AB=288	2·45939
: : hypoth. AC=480	2·68124	: : tangent ∠ A=53°.8′	10·12499
: secant ∠ A=53°.8′	10·22185	: perpend. BC=384	2·58435

NOTE. The side BC may be found (by Euclid 47 *of* I.) thus, $\sqrt{AC^2 - AB^2} = BC = 384$.

(W) From the examples hitherto given, it appears that when we want to find a side, the proportions begin with a *given* angle; and in the first three cases all the angles are given, therefore any side may be considered as the radius of a circle. But when we want to find an angle, the proportions begin with a *given* side; and as only two sides are given at once, it follows that these two *given* sides *only* can be considered as the radii of circles.

(X) BY GUNTER'S SCALE.

1. Extend the compasses from the hypothenuse 480 to the base 288 on the line of numbers, that extent will reach from 90° to 36°.52′, on the line of sines, the complement of the angle A.

2. Extend the compasses from 90° to 53°.8′ on the line of sines, that extent will reach from the hypothenuse 480 to BC the perpend. 384.

PRACTICAL EXAMPLES.

1. In the right angled plane triangle ABC,

Given { Hypoth, AC=645
Base AB=500
Ans. { ∠A=39°.10′
∠C=50°.50′
BC=407·37

Required the angles and the perpendicular.

2. In the right angled plane triangle ABC,

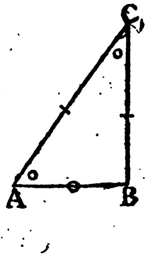
Given { Hypoth. AC=98
Base AB=53·66
Ans. { ∠A=56°.48′
∠C=33°.12′
BC=82·01

Required the angles and the perpendicular.

(Y) CASE V. *Given the hypothenuse and perpendicular, to find the angles and the base.*

Given { The hypothenuse AC=480
The perpendicular BC=384 } Required the angles A and C, and the base AB.

BY CONSTRUCTION.



Draw the base AB of an indefinite length, at B erect the perpendicular BC (N. 26.) which make equal to 384 by a scale of equal parts; take AC =480 from the same scale, with this extent in your compasses and centre C cross the base AB in A. Then AB measured by a scale of equal parts, will be 288, and the angles A and C measured by a scale of chords (S. 27.) will be 53°.8′. and 36°. 52′.

BY CALCULATION.

I. *The hypothenuse radius*, BC will be the sine of the angle A, and AB the cosine thereof.

To find the angle A.		To find the base AB.	
hypoth. AC=480 -	2·68124	radius - -	10·00000
: radius - -	10·00000	: hypoth. AC=480 -	2·68124
: : perpend. BC=384 -	2·58433	: : cosine ∠ A=53°.8′ -	9·77812
: sine ∠ A=53°.8′ -	9·90309	: base AB=288 -	2·45936

II. *The perpendicular* BC *radius*, AB will be the tangent of C, or the co-tangent of A, and AC will be the secant of C or co-secant of A.

To find the angle A.		To find the base AB.	
perpend. BC=384 -	2·58433	radius - -	10·00000
: radius - - -	10·00000	: perpend. BC=384 -	2·58433
: : hypoth. AC=480 -	2·68124	: : co-tangent ∠ A=53°.8′	9·87501
: co-secant ∠ A53°.8′ -	10·09691	: base AB=288 - -	2·45934

Or AB may be found thus, $\sqrt{AC^2-BC^2}=AB=288$.

(Z) BY GUNTER'S SCALE.

1. Extend the compasses from the hypothenuse 480 to the perpendicular 384 on the line of numbers, that extent will reach from 90° to 53°. 8′ on the line of sines.

2. Extend the compasses from 90° to 36°.52′, the complement of A, on the line of sines, that extent will reach from 480 to 288 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right angled plane triangle ABC,

$$\text{Given}\begin{cases}\text{Hypoth. AC}=645\\ \text{Perp. BC}=407{\cdot}37\end{cases}\qquad \mathit{Ans.}\begin{cases}\angle A=39^\circ.10'\\ \angle C=50^\circ.50'\\ AB=500\end{cases}$$

Required the angles and the base.

2. In the right angled plane triangle ABC,

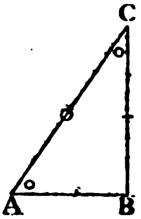
$$\text{Given}\begin{cases}\text{Hypoth. AC}=98\\ \text{Perpen. BC}=82{\cdot}01\end{cases}\qquad \mathit{Ans.}\begin{cases}\angle A=56^\circ.48'\\ \angle C=33^\circ.12'\\ AB=53{\cdot}66.\end{cases}$$

Required the angles and the base.

(A) CASE VI. *Given the base and perpendicular, to find the angles and the hypothenuse.*

$$\text{Given}\begin{cases}\text{The base AB}=288\\ \text{The perpend. BC}=384\end{cases}\text{Required the angles A and C, and hypothenuse AC.}$$

BY CONSTRUCTION.



Make the base AB equal to 288 by a scale of equal parts, at B erect the perpendicular BC, which make equal to 384 from the same scale; join AC. Then AC measured by a scale of equal parts will be 480, and the angles A and C measured by a scale of chords (S. 27.) will be 53°. 8′ and 36°. 52′.

BY CALCULATION.

I. *The base radius,* BC will be the tangent of the angle A, and AC will be the secant thereof.

To find the angle A.		To find the hypoth. AC.	
base AB=288 -	2·45939	radius -	10·00000
: radius - -	10·00000	: base AB=288 -	2·45939
: : perpend. BC=384 -	2·58433	: : secant ∠A=53°.8′	10·22188
: tangent ∠A=53°.8′ -	10·12494	: hypoth. AC=480 -	2·68127

II. *The perpendicular radius,* AB will be the tangent of C, or the co-tangent of A; and AC will be the secant of C, or the co-secant of A.

To find the angle A.		To find the hypoth. AC.	
perp. BC=384 -	2·58433	radius - -	10·00000
: radius - -	10·00000	: perpend. BC=384 -	2·58433
: : base AB=288 -	2·45939	: : co-secant ∠ A 53°.8′	10·09689
: co-tangent ∠A=53°.8′	9·87506	: hypothenuse AC=480	2·68122

Or AC may be found thus, $\sqrt{AB^2+BC^2}=AC=480$.

(B) BY GUNTER'S SCALE.

1. Extend the compasses from 384 to 288 on the line of numbers, that extent will reach from 45° to 53°.8′ on the line of tangents.

2. Extend the compasses from 53°.8′ to 90° on the line of sines, that extent will reach from 384 to 480 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the right angled plane triangle ABC,

Given $\begin{cases}\text{Base AB}=500\\ \text{Perp. BC}=407{\cdot}37\end{cases}$ *Ans.* $\begin{cases}\angle A=39°.10'\\ \angle C=50°.50'\\ AC=645\end{cases}$

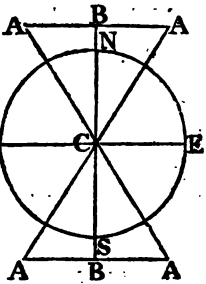
Required the angles and the hypothenuse.

2. In the right angled plane triangle ABC,

Given $\begin{cases}\text{Base AB}=53{\cdot}66\\ \text{Perp. BC}=82.01\end{cases}$ *Ans.* $\begin{cases}\angle A=56°.48'\\ \angle C=33°.12'\\ AC=98\end{cases}$

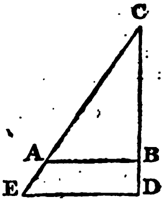
Required the angles and the hypothenuse.

SCHOLIUM.



(C) Plane sailing in navigation is nothing more than the practice of right-angled trigonometry; calling the hypothenuse the distance sailed, the perpendicular the difference of latitude, the base the departure, and the angle opposite to the base the course.

In the annexed figure, WESN represents the horizon of the place C, from whence the ship sails; CA the point of the compass or rhumb she sails on, and A the place sailed to. Then WE represents the parallel of latitude sailed from, and AA the parallel of latitude arrived at. Hence BC becomes the difference of latitude, AB the departure, CA the distance sailed, the angle ACB the course, and the angle BAC the complement of the course. And this will always be the case whether the ship sails between the north and west, viz. between N and W; the north and east, viz. between N and E; the south and west, viz. between S and W; or south and east, viz. between S and E.



(D) The practice of Mercator's sailing is only the application of right-angled trigonometry, and similar triangles (Y. 34). What is called the difference of latitude in plane sailing, is here called the *proper* difference of latitude, to distinguish it from the line CD, which is called the meridional difference of latitude. The part BD, by which CB is enlarged, is found by help of a table of meridional parts; being the sum, or difference of the meridional parts answering to the latitude sailed from and bound to; according as they are on different sides, or on the same side of the equator. ED is the difference of longitude.

II. PRACTICAL RULES FOR SOLVING ALL THE CASES OF OBLIQUE TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

(E) I. *When two of the three * given parts are a side and its opposite angle.*

Any one side of a triangle,
Is to the sine of its opposite angle;

---

* In every plane triangle there must be three parts given, to find the rest; and of these three given parts, one, at least, must be a side, because the same angles are common to an infinite number of triangles.

As any other side,
Is to the sine of its opposite angle.
And, The sine of any angle,
Is to its opposite side;
As the sine of any other angle,
Is to its opposite side. (D. 35.)

(F) An *angle* found by this rule is *sometimes* ambiguous, for trigonometry gives us only the sine of an angle, and not the angle itself, and the sine of every angle is also the sine of its supplement. The tables give only the *acute* value of an angle; the *obtuse* value is the supplement thereof.

When the *given* side, opposite to the *given* angle, is greater than the other *given* side; then the angle opposite to that other given side is always acute. But when the *given* side opposite to the *given* angle, is less than the other given side, then the angle opposite that other given side may be either acute or obtuse, and consequently it is ambiguous.

(G) II. *When two sides and the angle contained between them are given.*[*]

The sum of the two given sides,
Is to their difference;
As the co-tangent of half the contained angle,
Is to the tangent of half the difference between the other two angles.

This half difference added to the complement of half the contained angle, gives the angle opposite to the greater side; and being subtracted, leaves the angle opposite to the less side,

OR,

The sum of the two given sides,
Is to their difference;
As the tangent of half the supplement of the contained angle
Is to the tangent of half the difference between the other two angles.

This half difference, added to the half supplement, give the angle opposite to the greater side; and being subtracted, leaves the angle opposite to the less side. (E. 36.)

The remaining side of the triangle may be found by Rule I.

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[*] If the two given sides be equal to each other, the triangle is isosceles, and each of the remaining angles will be equal to half the supplement of the given angle. If the given angle be 90°, the required parts may be found by Case VI. of right angled triangles.

(H) III. *When the three sides of a triangle are given, to find the angles.* *

Double the base, or longest side,
Is to the sum of the other two sides;
As the difference between those two sides,
Is to the distance of a perpendicular from the middle of the base.

This distance added to half the base gives the greater segment, and being subtracted therefrom, leaves the less segment. (C. 35.)

The triangle being thus divided into two right angled triangles, each of which contains two given sides, the remaining angles may be found by Rule I.

OR,

From half the sum of the three sides of any plane triangle, subtract the side opposite to the angle sought, and note the half sum and the remainder. Then, to the logarithm of the half sum, add the logarithm of the remainder, and increase the index of the sum by 20; from the sum, thus increased, subtract the sum of the logarithms of the two sides containing the required angle, the remainder, divided by two, will give the logarithm cosine of half the angle required. (H. 37.)

The remaining angles may either of them be found by Rule I.

OR,

From half the sum of the three sides of any plane triangle subtract each of the sides containing the angle sought, and note the two remainders.

Then, add the logarithms of these two remainders together, and increase the index of their sum by 20; also, from half the sum of the three sides, subtract the side opposite to the angle required, and add the logarithm of the half sum to the logarithm of the remainder; the difference between these logarithmical sums, divided by 2, will give the logarithmical tangent† of half the angle sought. (I. 38.)

Either of the remaining angles may be found by Rule I.

---

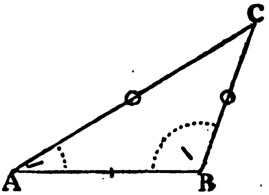
* If the triangle be equilateral, each of its angles will be 60°; if it be isosceles, the perpendicular will bisect the base, and the angles may be found by Case IV. of right angled triangles.

† If an angle be near 90°, or if it be a very small angle, it ought to be determined by a rule which produces a tangent or a co-tangent. For the sines of arcs near 90° vary by almost imperceptible increments; and the co-sines of very small arcs, or those near 180°, vary by decrements which are not easily assigned. To illustrate these observations by an example; suppose that in working any problem, by a table calculated to seconds, and carried to 7 places of decimals, the result of the operation should produce 9·9999998 a sine or co-sine. By the tables this is the sine of any arc be-

(I) CASE I. *Given the angles and one side of a triangle, to find the rest.*

$$\text{Given}\begin{cases}\text{The angle } A = 32^\circ.15' \\ \text{The angle } B = 114^\circ.24' \\ \text{The side } AB = 98\end{cases}\text{ Required the sides AC and BC}$$

BY CONSTRUCTION.



Make AB equal to 98 by a scale of equal parts, draw BC making the angle B equal to 114°.24′. (R. 27); also make the angle A equal to 32°.15′, draw AC meeting BC in C. Then AC measured by a scale of equal parts is 162, and BC 95.

BY CALCULATION. *See Rule I.*

To the angle A = 32°.15′ add the angle B = 114°.24′, the sum is 146°.39′, which subtract from 180° leaves the angle C = 33°.21′.

To find AC.			To find BC.		
sine ∠ C=33°.21′	-	9·74017	sine ∠ C=33°.21′	-	9·74017
: AB=98	-	1·99123	: AB=98	-	1·99123
: : sine ∠ B114°.24′	-	9·95937	: : sine ∠ A=32°.15′	·	9·72723
: side AC=162·34	-	2·21043	: side BC=95·12	-	1·97829

(K) BY GUNTER'S SCALE.

1. Extend the compasses from 33°.21′ to 65°.36′ the supplement of B, on the sines, that extent will reach from 98 to 162 on the line of numbers.

2. Extend the compasses from 33°.21′ to 32°.15′ on the line of sines, that extent will reach from 98 to 95 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

$$\text{Given}\begin{cases}\angle A = 27^\circ.59' \\ \angle B = 44^\circ.12' \\ AC = 104\end{cases} \qquad \textit{Ans.}\begin{cases}AB = 142\cdot02 \\ BC = 70 \\ \angle C = 107^\circ.49'\end{cases}$$

Required the other parts.

---

tween 89°.56′.19″, and 89°.57′.8″; or the co-sine of any arc between 2′.52″ and 3′.41″; here is therefore a choice of 49″ or arcs, and no reason can be given why one should be preferred to another: but the logarithmical differences between the tangents and co-tangents are never so small as to leave any doubt respecting the accuracy of the result, the smallest difference for 1″ being 42, at which time the arc is 45°.

2. In the plane triangle ABC,

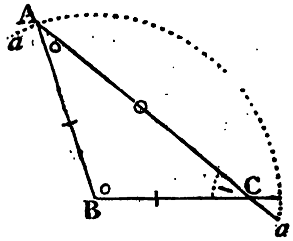
Given $\begin{cases} \angle A = 79^\circ.23' \\ \angle B = 54^\circ.22' \\ BC = 125 \end{cases}$ *Ans.* $\begin{cases} AC = 103{\cdot}4 \\ AB = 91{\cdot}87 \\ \angle C = 46^\circ.15' \end{cases}$

Required the other parts.

(L) CASE II. *Given two sides and an angle opposite to one of them, to find the rest.*

Given $\begin{cases} \text{The angle } C = 33^\circ.21' \\ \text{The side } AB = 98 \\ \text{The side } BC = 95{\cdot}12 \end{cases}$ Required the angles A and B and the side AC.

BY CONSTRUCTION.



Make BC=95·12 by a scale of equal parts, draw CA making the angle C=33°.21′ (R. 27.) With the side AB in your compasses, taken from the same scale of equal parts, and B as a centre, describe the arc *aa* cutting AC in the point A. Then AC measured by the scale of equal parts will be 162, and the angles A and B measured by a scale of chords (S. 27.) will be 32°.15′ and 114·24.

Here agreeably to the observation (F. 53.) the angle A is acute and not ambiguous; but had AB been less than BC, the arc *aa* would evidently have cut AC in two points on the same side of BC.

BY CALCULATION. *See Rule I.*

AB=98	1·99123	sine ∠C=33°,21′		9·74017
: sine ∠C=33°.21′	9·74017	: AB=98		1·99123
: : BC=95·12	1·97827	: : sine ∠B=114°.24′ or	}	9·95937
: sine ∠A=32°.15′	9·72721	sine 65°.36′	}	
hence ∠B=114°.24′		: side AC=162·34		2·21043

(M) BY GUNTER'S SCALE.

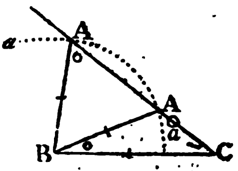
1. Extend the compasses from 98 to 95 on the line of numbers, that extent will reach from 33°.21′ to 32°.15′ on the line of sines.

2. Add the angles A = 32°.15′ and C = 33°.21′ together, and the sum will be 65°.36′; then extend the compasses from 33°.21′ to 65°.36′ on the line of sines, that extent will reach from 98 to 162·3 on the line of numbers.

Given $\begin{cases} \text{The angle } C = 33^\circ.21' \\ \text{The side } BC = 95^\circ.12' \\ \text{The side } AB = 60 \end{cases}$ Required the angles A and B and the side AC.

BY CONSTRUCTION.

This construction is exactly the same as in the preceding example: only AB, being shorter than BC, cuts AC in two points on the same side of BC, hence the angle A may be either acute or obtuse.



BY CALCULATION. *See Rule I.*

AB = 60 - - 1·77815
: sine ∠ C = 33° 21′ - 9·74017
: : BC = 95·12 - 1·97827
$: \text{sine} \angle A \left\{ \begin{array}{l} 60^\circ.38' \textit{ acute} \\ 119^\circ\ 22' \textit{ obtuse} \end{array} \right\} 9{\cdot}94029$

The sum of the angles C and A subtracted from 180° leaves the angle B = 86°.1′ if A be acute; or 27°.17′ if A be obtuse.

sine ∠ C = 33°.21′ - 9·74017
: AB = 60 - - 1·77815
: : sine ∠ B = 86°.1′ - 9·99896
: AC = 108·87 - 2·03693

OR,

sine ∠ C = 33°.21′ - 9·74017
: AB = 60 - - - 1·77815
: : sine ∠ B = 27°.17′ - 9·66124
: AC = 50·03 - - 1·69922

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

$$\text{Given}\begin{cases} AC = 104 \\ BC = 70 \\ \angle B = 44^\circ.12' \end{cases} \qquad \textit{Ans.}\begin{cases} \angle A = 27^\circ.59' \\ \angle C = 107^\circ.49' \\ AB = 142{\cdot}02 \end{cases}$$

Required the other parts.

2. In the plane triangle ABC,

$$\text{Given}\begin{cases} AC = 104 \\ AB = 142{\cdot}02 \\ \angle B = 44^\circ.12' \end{cases} \qquad \textit{Ans.}\begin{cases} \angle C = 72^\circ.11' \textit{ or } 107^\circ.49' \\ BC = 134{\cdot}77 \textit{ or } 70 \\ \text{ambiguous.} \end{cases}$$

Required the other parts.

3. A man travels from A to B, 3·7 miles, then turning a little to the left hand goes from B to C, which is 4·7 miles: at C he observes that A and B make an angle of 29°.16′. What is the distance from A by the shortest cut?

Answer, 7 miles.

4. A man travels from A to B, 3·7 miles; returning in a mist, he loses his way, and going a little too much towards the left hand, comes to C, which is 4·7 miles from B. It now clearing up he could see both A and B, and observes that they make an angle of 29°.16′. Pray how far is he from A?

Answer, 1·201 mile.

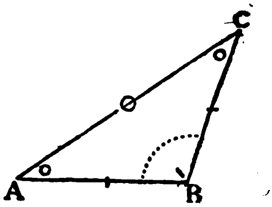
(N) Case III. *Given two sides and the angle contained between them, to find the rest.*

Given { The side AB = 98
The side BC = 95·12
The angle B = 114°.24′ }

{ Required the angles A and C, and the side AC.

BY CONSTRUCTION.

Make AB = 98 by a scale of equal parts, and the angle B = 114°.24′ by a scale of chords (R.27.); draw BC, which make equal to 95·12, and ABC is the triangle required. The angles A and C will measure 32°.15′ and 33°.21′, and AC will be 162.



BY CALCULATION. *See Rule II.*

To find the angles.

AB + BC = 193·12 -	2·28583
: AB − BC = 2·88 -	0·45939
: : co-tangent ½ B = 57°.12′	9·80919
: tangent ½ (A ∼ C) = 0°. 33′	7·98275
½ B = 57° 12′ its comp. 32°.48′	

sum 33·21 = ∠ C

diff. 32·15 = ∠ A.

Or thus,

AB + BC = 193·12 -	2.28583
: AB − BC = 2·88 -	0·45939
: : tang. ½ supp. B = 32°.48′	9·80919
: tangent ½ (A ∼ C) = 0°.33′	7·98275

sum 33·21 = ∠ C

diff. 32·15 = ∠ A.

To find the side AC.

sine ∠ C = 33°.21′ - - -	9·74017
: AB = 98 - - -	1·99123
: : sine ∠ B = 114°.24′, or sine 65°·36′ -	9·95337
; sine AC = 162·34 - - -	2·21043

(O) BY GUNTER'S SCALE.

1. Extend the compasses from 193·12 to 2·88 on the line of numbers, that extent will reach from 32°.48′ to 0°.33′ on the line of tangents. This is the method of working such examples as this, but so small an angle as 33′ is not contained on the scale.

2. Extend from 33°.21′ to 65°.36′ on the line of sines, that extent will reach from 98 to 162 on the line of numbers.

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

Given { AB = 345
AC = 174·07
∠ A = 37°.20′ }

Ans. { ∠ B = 27°.4′
∠ C = 115°.36′
BC = 232 }

Required the other parts.

2. In the plane triangle ABC,

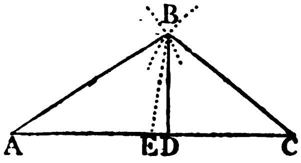
$$\text{Given}\left\{\begin{array}{l} AB=103\\ AC=126\\ \angle A=56°.30' \end{array}\right. \qquad \textit{Ans.}\left\{\begin{array}{l} \angle B=72°.20\\ \angle C=51°.10\\ BC=110{\cdot}3 \end{array}\right.$$

Required the other parts.

(P) CASE IV. *Given the three sides, to find the angles.*

$$\text{Given}\left\{\begin{array}{l} \text{The side } AB=98\\ \text{The side } BC=95{\cdot}12\\ \text{The side } AC=162{\cdot}34 \end{array}\right\}\text{Required the angles } A, B, \text{and } C.$$

BY CONSTRUCTION.



Draw the longest side AC = 162·34 from a scale of equal parts; with AB = 98 in your compasses (taken from the same scale) and one foot in A describe an arc; with BC = 95·12 in your compasses cross it in B; then ABC is the triangle required. The angles measured by a scale of chords (S. 27.) will be A = 32°.15′, B = 114°.24′, and C = 33°.21′.

BY CALCULATION. *See Rule III.*

Let E be the middle of the base AC, and DB perpendicular to AC.

To find the segments AD *and* DC *of the base.*

double AC = 324·68 - - - 2·51145
: AB + BC = 193·12 . - - 2·28583
: : AB − BC = 2·88 - - - 0·45939
to ED = 1·713 - - - - 0·23377

Then ½ AC + ED = 81·17 + 1·713 = 82·883 = AD, the greater segment, and ½ AC − ED = 81·17 − 1·713 = 79·457 = DC, the less segment.

To find the angles in the right angled triangles ADC *and* CDB.

AB = 98 - -	1·92123	BC = 95·12 - -	1·97827
: radius, sine 90° -	10·00000	: radius, sine of 90° -	10·00000
: : AD = 82·883 - -	1.91846	: : CD = 79·457 -	1·90018
: sine DBA = 57°·45′ or co-sine DAB = 32°.15′	9·92723	: sine DBC = 56°.39′ or co-sine DCB = 33°.21′	9·92186

Then DBA + DBC = 57°.45′ + 56°.39′ = 114°.24′ = ABC.

OR THUS,

$$\left.\begin{array}{rl} \tfrac{1}{2}(AC+AB+BC) = & 177{\cdot}73 \text{ log.} = 2{\cdot}24976\\ \tfrac{1}{2}(AC+AB+BC)-BC = & 82{\cdot}61 \text{ log.} = 1{\cdot}91703 \end{array}\right\} \text{sum} + 20 = 24{\cdot}16679$$

$$\left.\begin{array}{rl} AB = & 98{\cdot} \quad \text{log.} = 1{\cdot}99123\\ AC = & 162{\cdot}34 \text{ log.} = 2{\cdot}21042 \end{array}\right\} \text{sum} = 4{\cdot}20165$$

2)19·96514

½ ∠A = 16° 7′ 30″……co-sine 9·98257
2

∠A = 32° 15′ 0″

OR THUS,

$$\left.\begin{array}{l}\frac{1}{2}(\mathrm{AC}+\mathrm{AB}+\mathrm{BC})-\mathrm{AC}=15{\cdot}39 \text{ log.}=1{\cdot}18724\\ \frac{1}{2}(\mathrm{AC}+\mathrm{AB}+\mathrm{BC})-\mathrm{AB}=79{\cdot}73 \text{ log.}=1{\cdot}90162\end{array}\right\}\text{sum}+20=23{\cdot}08886$$

$$\left.\begin{array}{l}\frac{1}{2}(\mathrm{AC}+\mathrm{AB}+\mathrm{BC})\quad=\ 177{\cdot}73 \text{ log.}=2{\cdot}24976\\ (\mathrm{A}+\mathrm{AB}+\mathrm{BC})-\mathrm{BC}=82{\cdot}61 \text{ log.}=1{\cdot}91703\end{array}\right\}\text{sum}\quad=\ 4{\cdot}16672$$

$$2)\,18{\cdot}92207$$

$$\tfrac{1}{2}\angle\mathrm{A}=16^\circ\ 7'\ 27''\ldots\ldots\text{tangent}\quad 9{\cdot}46103$$

$$\begin{array}{r}2\\ \hline 32^\circ\ 14'\ 54''\end{array}$$

The remaining angles may be found by Rule I.

(Q) BY GUNTER'S SCALE, *according to the first method.*

1. Extend the compasses from 324·68 to 193·12 on the line of numbers, that extent will reach from 2·88 to 1·713 the distance of a perpendicular from the middle of the base.

2. Extend from 98 to 82·883 on the line of numbers, that extent will reach from 90° to 57°.45′ on the line of sines.

3. Extend from 95·12 to 79·457 on the line of numbers, that extent will reach from 90° to 56°.39′ on the line of sines.

BY GUNTER'S SCALE, *according to the second method.*

1. Extend the compasses from half the sum of the three sides 177·73 to one of the containing sides AB=98, that extent will reach from AC=162·34 the other containing side, to a fourth number 89·5, on the line of numbers.

2. Extend the compasses from this fourth number 89·5 to (the difference between the half sum of the three sides and the side opposite to the angles sought), 82·61 on the line of numbers, that extent will reach from 90° on the line of sines to the required angle 32°, on the line of versed sines, immediately under the line of sines.

This is derived from the proportions in the investigation of Gunter's Rule (X. 22, and note).

PRACTICAL EXAMPLES.

1. In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l}\mathrm{AB}=142{\cdot}02\\ \mathrm{BC}=70\\ \mathrm{AC}=104\end{array}\right.\qquad \textit{Ans.}\left\{\begin{array}{l}\angle\mathrm{A}=27^\circ.59'\\ \angle\mathrm{B}=44^\circ.12'\\ \angle\mathrm{C}=107^\circ.49'\end{array}\right.$$

Required the angles.

2. In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{l}\mathrm{AB}=112{\cdot}65\\ \mathrm{BC}=112\\ \mathrm{AC}=120\end{array}\right.\qquad \textit{Ans.}\left\{\begin{array}{l}\angle\mathrm{A}=57^\circ.27'\\ \angle\mathrm{B}=46^\circ.34'\\ \angle\mathrm{C}=57^\circ.59'\end{array}\right.$$

SCHOLIUM.

(R) There are some authors and teachers of trigonometry, who make no distinction of cases between right and oblique-angled triangles, but divide the whole into three cases; because the three rules necessary for solving the problems that occur in oblique trigonometry, are sufficient for solving those which occur in right-angled trigonometry. For instance, Rule I. (E. 52.) will solve all the cases in right-angled triangles (except the 6th), and the first and second case in oblique-angled triangles: Rule II. (G. 53.) will solve the 6th case in right-angled triangles and the 3d case in oblique; and Rule III. (H. 54.) will solve the last case in oblique triangles.

CHAP. III.

THE APPLICATION OF PLANE TRIGONOMETRY TO THE MENSURATION OF DISTANCES, HEIGHTS, &C.

(S) The mensuration of distances and heights depends upon the rules of plane trigonometry already explained, together with the use of certain instruments for taking angles.

(T) Horizontal and vertical angles are usually measured with a theodolite furnished with one or two telescopes, and a vertical arc; and if the horizontal and vertical arcs of the instrument be described with a radius of not less than 3½ inches, the observed angles may be measured to half a minute, or the 120th part of a degree.

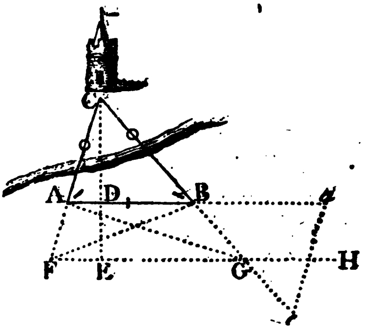
(U) Angles which are oblique to the horizon are generally taken with a sextant; which must be held in such a position, that its plane may coincide with the two objects and the eye. When vertical angles are taken with this instrument, an artificial horizon must be used, and the reflected image of the object from the glasses of the sextant must be brought to coincide with the reflected image of the same object in the artificial horizon.

(V) Base lines are generally measured with rods, or the *four pole* Gunter's chain; but common tape of 50 or 100 feet in length is often preferred both for accuracy and expedition: especially if it be kept dry, and the ground be tolerably level.

(W) The use of instruments must be acquired under the direction of a person well skilled in their several adjustments, as but little information can be obtained from written description*, and even the most expert observer will find it necessary, in several cases, to apply corrections to the different angles according to the situations of the objects. See Chapter IV.

EXAMPLE I.

Being on one side of a river, and wanting to know the distance of a fort, or other object, on the other side, suppose I measured 500 yards along the side of the river in a straight line AB and found the two angles between this line and the object to be CAB = 74°.14′, and CBA = 49°.23′. Required the distance between each station and the object?



180° − ∠ (CAB + CBA) = ∠ACB = 56°.23′		sine ∠ ACB = 56°.23′	9·92052
sine ∠ ACB = 56°.23′	9·92052	: AB = 500	2·69897
: AB = 500	2·69897	: : sine ∠ CAB = 74°.14′	9·98334
: : sine ∠ CBA = 49°.23′	9·88029	: BC = 577·8	2·76179
: AC = 455·8	2·65874		

(X) Or, produce AB till B*a* be equal to it, and CB till the angle *a* be equal to the angle A; then will the distances B*c* and *ac* be equal to the distances BC and AC.

(Y) Or, without any instrument to measure an angle, the distances AC and BC may be found. Make AF of any length in a line with AC, and BG in a line with BC; measure AF, AB, BG, FB, and AG; then the three sides of the triangles BAF and ABG will be given, with which the angles BAF and ABG may be found; their supplements will give the angles CAB and CBA, and hence the distances AC and BC may be found as in the example.

(Z) The perpendicular distance DC may be readily determined, for having found BC, the angle B and BC will be given; or find AC, then AC and the angle A will be given to find DC.

The perpendicular distance DC may be found without any instrument for taking angles. With a cross-staff fixed at D observe the object at C, and set up a staff in a line with it at E, and another at B at right angles to CE, move the cross-staff from

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* Copper-plates, well executed, of theodolites, sextants, quadrants, and other instruments for taking angles, are given in Adams' Geometrical and Graphical Essay, edited and published by Mr. Wm. Jones, mathematical instrument maker in Holborn, London.

D to E, and set up a pole or staff at H perpendicular to EC. Measure the distances DB, DE, and also EG in the direction EH till G, B, and C make one straight line. Then

EG—DB : DE :: EG : EC from which take DE.

Or, EG—DB : DE :: DB : DC.

The four following examples are referred to the foregoing figure, and serve to exercise the above observations.

EXAMPLE II.

An engineer wanted to know the breadth of a river (over which the general intended to pass the whole army), in order to determine how many pontoons of five feet broad and six feet asunder he should have occasion to make use of. Perceiving an object (C) on the opposite bank of the river close to the edge of the water, he measured 144 paces or 360 feet along the edge of the river (as AB). The angle at A between the object and the base line was 83°.57′, and at B it was 80°.32′; required the perpendicular breadth (DC) of the river, and the number of pontoons sufficient to form a passage for the troops?

Answer. The perpendicular breadth of the river is 528 paces or 1320 feet, a military pace being 2½ feet; and 120 pontoons will answer the purpose.

EXAMPLE III.

Two ships of war intending to cannonade a fort at C, separated from each other 500 yards, and coming as near to the shore as possible without being in danger of running aground, the officers in each ship observe the angles formed between the other ship and the fort, and find them, viz. A=38°.16′, and B=37°.9′. Are the ships at a proper distance for commencing a cannonade, the most convenient distance being about 300 yards?

Answer. A is 312 yards from the fort, and B 320.

EXAMPLE IV.

Wanting to know my distance from an object at C, I measured from E to D in a line with the object 29 poles, from D, I measured DB at right angles to ED=43 poles, and coming to E I measured EG in the direction EH=54 poles, what is my distance DC from the object?

Answer. 113$\frac{4}{11}$ poles.

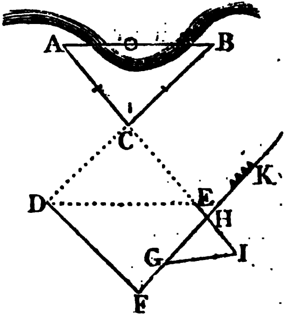
EXAMPLE V.

Wanting to know the distance of two objects A and B (which are 500 yards from each other) from a tower at C on the opposite side of a river, and, having no other instrument than a chain of 22 yards long, I measured AF in a straight line with AC, and found it to be 300 yards long. I likewise measured FB and found it to be 649·2 yards, I then measured BG = 600 yards and AG = 1000 yards, required the distances AC and BC?

Answer. AC = 455·8 yards, and BC = 577·8.

EXAMPLE VI.

Suppose I wanted to know the distance between two places A and B, accessible at both ends of the line AB, and that I measured AC = 735 yards, and BC = 840, also the angle ACB = 55°.40′. What is the distance between A and B?



BC + AC = 1575	3·19728
: BC − AC = 105	2·02119
: : tangent ½ supp. ∠ ACB = 62°.10′	10·27738
: tangent 7°.11′.47″	9·10129
angle A = 69°.21′.47″	
angle B = 54°.58′.13″	
sine ∠ B = 54°.58′.13″	9·91332
: AC = 735	2·86629
: : sine ∠ C = 55°.40′	9·91686
: AB = 741	2·86983

(A) If the lines AC and BC be produced till DC be equal to CB and EC to AC, then will the distance DE be equal to the distance AB without calculation.

EXAMPLE VII.

Wanting to know the angle ACB formed by two walls AC and BC, I measured DC = 840 yards, EC = 735 yards, and DE = 741 yards: hence the angle ACB = DCE is required?

Answer. ACB = 55°.40′.

EXAMPLE VIII.

Suppose ACB to represent the flanked angle of a bastion, (which of course will be inaccessible on account of the covert-way, ditch, &c.) and it be required to determine the measure of that angle.

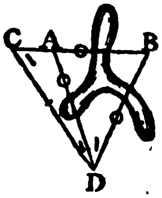
Place a pole at D, in the direction of the face BC of the bastion (out of the reach of musket shot). With a cross-staff at D direct a person to set up another pole at F perpendicular to DC. Measure DF = 375 paces or 937½ feet, make FK perpendicular to DF, and set up another pole at H in the direction of the face AC of the bastion. Prolong CH, and measure the three sides of the triangle GHI; suppose GH = 10 yards, GI = $12\frac{3}{25}$ yards, and HI=10 yards, then will the angle GHI be equal to the flanked angle ACB. Required this angle?

Answer. The angle ACB=74°.36′, the proper flanked angle of a regular pentagon.

(B) In the same manner (except measuring the triangle GHI,) we may draw a line KF parallel to the inaccessible face BC of the bastion, in order to place a battery at K, to produce the greatest effect; viz. at about 375 paces distance, and that the direction of the fire may be nearly perpendicular to $\frac{1}{3}$ of the face BC.

EXAMPLE IX.

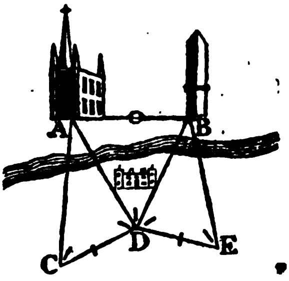
Wanting to know the distance of an object at D, from two others A and B, and also the distance between A and B, I set up a pole at C in a right line with AB, at C the angle ACD was 57°.0′; I then measured CD=784·8 yards, and at D the angles CDA and ADB were 14°.0′ and 41°.30′. Required the several distances above mentioned.



Answer. AD=696·1, DB=712·4, and AB=499·3 yards.

EXAMPLE X.

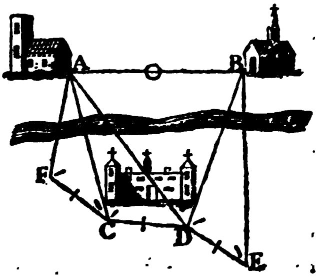
Wanting to know the distance between a church at A and an obelisk at B; both objects could only be seen from a particular place as D; at D I took the angles ADC=89°.0′, ADB =72°.30′, and BDE=54°.30′; I then measured DE=200 yards, and at E I took the angle BED=88°.30′; and, lastly, I measured DC=200 yards, and at C took the angle DCA=50°.30′. Required the distance AB?



Answer. AD=237·627, DB=332·215, and AB=345·5 yards.

EXAMPLE XI.

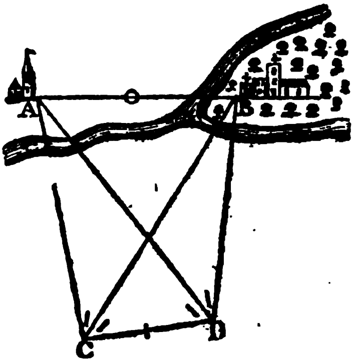
I wanted to know the distance between two places A and B, but could not meet with any station from whence I could see both objects. I measured a line CD= 200 yards; from C the object A was visible, and from D the object B was visible, at each of which places I set up a pole. I also measured FC=200 yards, and



DE = 200 yards, and at F and E set up poles.—I then took the angle AFC = 83°, ACF = 54°.31′, ACD = 53°.30′, BDC = 156°.25′, BDE = 54°.30′, and BED = 88°.30′. Required the distance AB?

Answer. DB = 332·215, AD = 237·627, AC = 293·93; the angle ADC = 83°.55′, consequently ADB = 72°.30′, hence AB = 345·5.

EXAMPLE XII.



Wanting to know the distance between two inaccessible objects A and B, I measured a base line CD = 300 yards: at C the angle BCD was 58°.20′ and ACB 37°; at D the angle CDA was 53°.30′ and ADB 45°.15′. Required the distance AB?

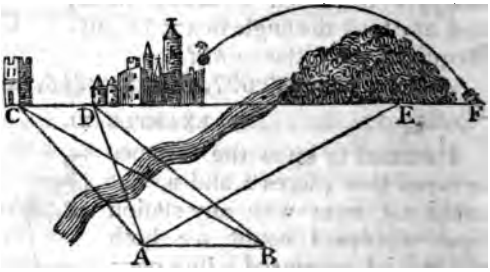
Answer. First find CA = 465·97 by Example I, and also CB || 761·46; and then find AB = 479·79 yards by Example VI.

EXAMPLE XIII.

Suppose the base line in the foregoing example had been 908·36 feet, the angle ACB = 14°.34′, and ACD = 60°.50′; also the angle ADC = 96°.44′, and BDC = 115°.23′; what would have been the distance AB in this case?

Answer. 674·62 feet.

EXAMPLE XIV.



Wanting to prolong the inaccessible line CD, in order to place a mortar battery behind the obstacle E in the direction of CD, I measured a base line AB = 414·2 yards, at A, I took the angles CAD = 49°.13′, CAB = 123°.45′, and the angle CAE = 100°, having first set up a pole at E behind the obstacle. At B, I took the angles DBC = 33°.45′ and DBA = 67°.30′. Hence it is required to find the angle AEF, which will determine the position of EF with respect to CD?

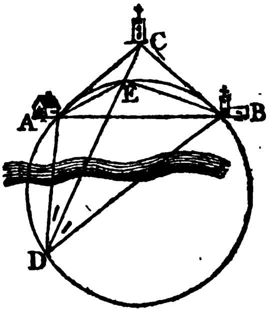
Answer. Exactly in the same manner as in the two preceding examples find AC $= 601 \cdot 6$, AD $= 622$, and the angle ACD $= 67^\circ.30'$, then ACD + CAE $=$ AEF $= 167^\circ.30'$; if therefore at E, an angle of $167^\circ.30'$ be made with A, the point F will be determined, in the same line with CD. The distances CD or CE are easily calculated if necessary.

EXAMPLE XV.

From a station at D I perceived three objects, A, B, C, whose distances from each other I knew to be as follow, AB $= 12$ miles, BC $= 7\frac{1}{5}$ miles, and AC $= 8$ miles: at D, I took the angles CDB $= 25^\circ$, and ADC $= 19^\circ$. Hence it is required to find my distance from the objects.

CONSTRUCTION.

Make a triangle ABC whose three sides are 12, $7\frac{1}{5}$, and 8 (P. 59), viz. make AB $= 12$, BC $= 7\frac{1}{5}$, and AC $= 8$. At B make the angle EBA $=$ ADE $= 19^\circ$, and at A make the angle EAB $=$ BDE $= 25^\circ$; draw AE and BE, and through the point of intersection E and the points A, B, describe a circle. From C through E draw CED, join BD and AD, then will AD, CD, and BD be the distances required. From this construction it appears, that if the angles at the station D had been greater than the angles A and B of the triangle ABC, the point E would have fallen beyond C, but this would not alter the manner of construction, neither would it materially affect the method of calculation.



CALCULATION.

I. *In the triangle* ABC are given AB $= 12$, BC $= 7\frac{1}{5}$, and AC $= 8$ to find the angle BAC $= 35^\circ.34'.38''$.

II. *In the triangle* AEB are given all the angles, viz. EAB $= 25^\circ$, EBA $= 19^\circ$, and AEB $= 136^\circ$, and the side AB $= 12$, to find AE $= 5 \cdot 624$, and BE $= 7 \cdot 3$.

III. *In the triangle* CAE we have given the side AC $= 8$, AE $= 5 \cdot 624$, and the angle CAE $=$ BAC $\sim$ BAE $= 10^\circ.34'.38''$, hence the angle AEC will be found $= 146^\circ.45'.7''$, but because exterior angle of the triangle ADE, if we subtract the from it ($= 19^\circ$) we shall have the angle EAD $= 127^\circ$ if to this angle we add the angle CA [illegible] °.34′. have the angle CAD $= 138^\circ.19'.45''$.

IV. *In the triangle* DAC all the angles are given and the side AC, to find AD and CD, viz. CAD = 138°.19′.45″, and ADC = ADE = 19°, the sum of these taken from 180° leaves ACD = 22°.40′.15″ and AC = 8. Hence AD = 9·4711 miles and DC = 16·3369 miles.

V. *In the triangle* DEB the side BE and all the angles are given. Because the opposite angles of every quadrilateral inscribed in a circle are equal to two right angles, the supplement of EAD = 127°.45′.7″ gives the angle EBD = 52°.14′.53″, and the sum of EBD and EDB subtracted from 180° leaves DEB = 102°.45′.7″, and BE has been found = 7·3. Hence DB = 16·8485 miles.

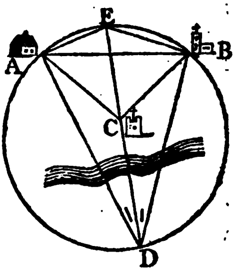
EXAMPLE XVI.

Suppose three objects A, B, C, were seen from D, (and stood as *per* figure Example XV), and that their distances were AB = 12 furlongs, BC = 9, and AC = 6 furlongs. The angle BDC = 33°.45′ and ADC = 22°.30′. It is required to determine the distances AD, DC, and DB?

Answer. This example is of the same kind as the 15th, and the distances may be found in a similar manner, AD = 14·01, DC = 15·64 and DB = 10·6.

EXAMPLE XVII.

From a station D, I perceived three objects A, B, and C (situated as in the annexed figure) whose distances were as follow, AB = 12 miles, AC = 8 miles, and BC = 7½ miles. At D I took the angles CDB = 25°, and ADC = 19°. Hence it is required to find my distance from the objects.



CONSTRUCTION.

This is constructed exactly as Example XV, except turning the vortex of the triangle ACB towards the place of observation D.

CALCULATION.

I. *In the triangle* ABC are given AB = 12, BC = 7½, and AC = 8, to find the angle BAC = 35°.34′.38″.

II. *In the triangle* AEB are given the angle EAB = 25°, EBA = 19°, and AEB = 136°, and the side AB = 12, to find AE = 5·624 and BE = 7·3.

III. *In the triangle* CAE there is given AE = 5·624, AC = 8, and the angle EAC = EAB + BAC = 60°.34′.38″, to find the angle ACE = 43°.5′.14″, if from this angle (being the exterior angle of the triangle DCA) we subtract the angle $\mathrm{ADC}=19^{\circ}$ the remainder $24^{\circ}.5'.14''$ will be the angle DAC.

IV. *In the triangle* ACD there is given the side AC and all the angles, viz. $\mathrm{AC}=8$, $\mathrm{ADC}=19^{\circ}$, $\mathrm{DAC}=24^{\circ}.5'.14''$, and $\mathrm{ACD}=136^{\circ}.54'.46''$, to find $\mathrm{AD}=16{\cdot}7857$, and $\mathrm{CD}=10{\cdot}0286$.

V. *In the triangle* EBD all the angles and the side EB are given, to find BD, viz. $\mathrm{EDB}=\mathrm{CDB}=25^{\circ}$, $\mathrm{EBD}=180^{\circ}-\mathrm{EAD}=180^{\circ}-\overline{\mathrm{DAC}+\mathrm{CAE}}=95^{\circ}.20'.8''$, and $\mathrm{BED}=59^{\circ}.39'.52''$, and $\mathrm{EB}=7{\cdot}3$. Hence $\mathrm{BD}=14{\cdot}9095$.

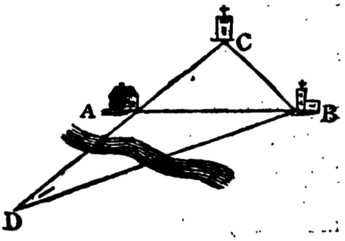
EXAMPLE XVIII.

Suppose the three objects A, B, C were seen from D (and stood as *per* figure to Example XVII. and that their distances were $\mathrm{AB}=106$, $\mathrm{AC}=65\frac{1}{8}$, and $\mathrm{BC}=53\frac{1}{4}$. The angle $\mathrm{BDC}=13^{\circ}.30'$, and $\mathrm{ADC}=29^{\circ}.50'$. It is required to find the distances DA, DC, and DB?

Answer. This example is of the same kind as the 17th, $\mathrm{DA}=131$, $\mathrm{DC}=107{\cdot}4$, and $\mathrm{DB}=151{\cdot}4$.

EXAMPLE XIX.

From a station at D, I perceived three objects, A, B, C, (situated as in the annexed figure,) whose distances were as follow: $\mathrm{AB}=12$ miles, $\mathrm{AC}=8$, and $\mathrm{BC}=7\frac{1}{4}$. At D the angle ADB was $17^{\circ}.47'.19''$. Required the distances DA, DC, and DB?

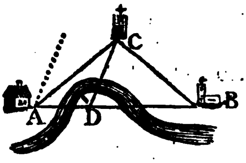


Answer. With the three given sides of the triangle ABC find the angle $\mathrm{CAB}=35^{\circ}.34'.38''$, the supplement of this $\mathrm{angle}=\mathrm{BAD}=144^{\circ}.25'.22''$, also $\mathrm{CAB}-\mathrm{ADB}=17^{\circ}.47'.19''=\mathrm{ABD}$. Hence all the angles of the triangle ADB are given, and one side AB, to find the remaining sides.

Viz. $\mathrm{AD}=12$, and $\mathrm{BD}=22{\cdot}85$, and $\mathrm{DC}=20$.

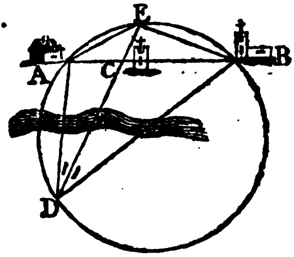
EXAMPLE XX.

Three objects A, B, and C, whose distances were $\mathrm{AC}=8$, $\mathrm{BC}=7\frac{1}{5}$, and $\mathrm{AB}=12$ miles, were visible from one station at D, from whence the angle ADC was observed to be $107^{\circ}.56'.13''$. Required the distances DA, DC, and DB?



Answer. With the three sides of the triangle ABC find the angle $\mathrm{BAC}=35^{\circ}.34'.38''$. Then in the triangle ADC all the angles will be given, and the side AC, to find AD=5, and DC=4·892; hence DB=7 miles.

EXAMPLE XXI.



Three objects A, B, and C, in a straight line, whose distances were AC = 3·626, AB = 12, and BC = 8·374 miles were visible from one station at D, from whence the angle ADC was 19°, and BDC=25°. Required the distances DA, DC, and DB?

CONSTRUCTION.

The construction to this example is almost exactly the same as the construction to Example XV.

CALCULATION.

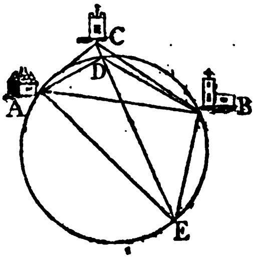
I. *In the triangle* EAB are given the angle EAB=EAC=25°, EBA=19°, and consequently AEB=136°, and the side AB=12, to find AE=5·624, and BE=7·3.

II. *In the triangle* AEC there is given AC=3·626, AE=5·624, and the contained angle EAC=25°, to find the angles about C, viz. ACE=DCB=121°.45'.7" the supplement of which is ACD=ECB=58°.14'.53".

III. *In the triangle* ACD, all the angles and the side AC are given, to find AD=9·4711 miles, and DC=10·861.

IV. *In the triangle* ADB or DCB all the angles and two sides are given, to find DB=16·8485 miles.

EXAMPLE XXII.



Three objects A, B, and C, forming a triangle, were visible from one station at D within the triangle; the angle ADB was observed to be 123°.45', CDB = 132°.22', and consequently ADC = 103°.53'; the distance AB was known to be 12 miles, from B to C = 9, and from A to C = 6 miles; required the distance of each object from the station at D?

CONSTRUCTION.

Make a triangle whose three sides are AB=12, BC=9, and AC=6. At B make the angle ABE=the supplement of ADC=76°.7'=ADE, and at A make the angle BAE=the supplement of BDC=47°.38′=BDE, through the intersection E of AE and BE, and the points A, and B, describe a circle. Then DA, DC, DB, are the distances required.

CALCULATION.

I. *In the triangle* AEB all the angles and one side are given, to find AE. Thus the angle ABE=76°.7′, and BAE=47°.38′ by construction, hence AEB=56°.15′, and AB=12; hence AE =14·011.

II. *In the triangle* ABC all the signs are given, hence the angles may be found, thus CAB=46°.34′, this added to the angle BAE gives CAE=94°.12′.

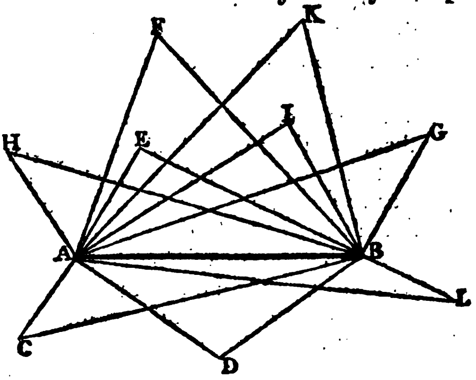
III. *In the triangle* CAE, are given AC = 6, AE = 14·011, and the angle CAE = 94°.12′, to find the angle ACE = 63°. 18′.21″.

IV. *In the triangle* DAC all the angles are given, viz. ACD= 63°.18′.21″, ADC = 103°.53′, and DAC = 12°.48′.39″, and the side AC=6, to find AD=5·523 and DC=1·372.

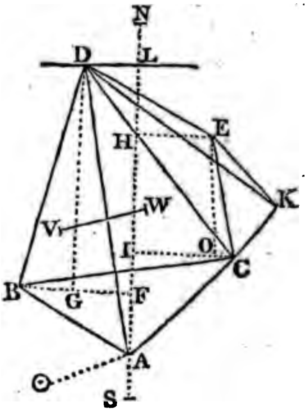
V. *In the triangle* BDC, two sides BC and CD, and an angle BDC opposite to one of them are given, to find the side BD = 8·018.

SCHOLIUM.

(C) Various other problems concerning distances might be formed, from the different situations which objects may be supposed to have, but they are only applications of the preceding examples.—For instance, the distances of the most remarkable places in a town, the distances of several villages from each other, the plan of a camp, &c. &c. may be taken from what has already been explained.



Let A, B, C, D, E, F, G, H, I, K, L, be several remarkable objects whose situations are to be laid down in a map. Choose a convenient situation AB for a base line, from whence you can see all the objects; let this base line be as long as possible, in proportion to the most distant object. From the extremity A measure the angles GAB, IAB, KAB, EAB, FAB, HAB, &c. from B measure the angles GBA, KBA, IBA, FBA, EBA, HBA, &c. Now it is evident from what has been already shewn, that having the base AB given, and the several angles, it will be easy to find the sides, or distances, AG, AI, AK, &c. &c. By Example I.



(D) A country may be surveyed, and the distances of remote objects may be obtained by means of a series of triangles. Thus, measure a base AB, and let C and D be two objects which can be seen from A and B. Measure the horizontal angles BAC, BAD, ABC, ABD, and calculate the distance DC, as in Example XII., and likewise the distances DB, DA, BC, AC. In like manner if E and K be two objects visible from C and D, the distance EK may be found, and its position with respect to CD; and thus the measurement may be continued to any distance. But, in order that the conclusion may be more accurate, the mensuration from one base to another may be carried on by different sets of triangles; for instance, two objects V and W might have been chosen instead of A and B, by means of which the distance EK might have been ascertained; and, it is plain, that by taking several series of triangles which lead to the same two objects, the mean of the results will be more accurate than the measurement obtained from one triangle. When a series of triangles has been carried on for a considerable distance, the interval of two objects, whose distance has been determined by calculation, should be actually measured, in order to detect any error which may have been made in the calculated distance. This is called a *Base of verification*. The trigonometrical survey of England and Wales had its first commencement in 1784; and from a base of 27404·2 feet measured on *Hounslow Heath*, and continued by a series of triangles to *Salisbury Plain*, the medium distance between *Beacon Hill* and *Old Sarum* was 36574·3 feet, and from actual admeasurement the distance was 36574·4 feet, a difference of little more than *one* inch.*

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* Trigonometrical Survey of England and Wales, Vol. I. pages 279 and 280, published by Mr. Faden, Charing Cross, London.

(E) By the assistance of a survey, executed with such accuracy, the length of an arc of a meridian may be measured. Let NS be the meridian of any place A, and let ⊙ represent the point of the horizon where the sun sets, measure the angle BA ⊙; now, by having the latitude of the place A given, the amplitude NA ⊙ is easily ascertained[*], and consequently their difference BAN is given; also if from BAC there be taken BAN, the remainder CAN is given. Let BF, CI be drawn perpendicular to the meridian, then because AB and the angle BAN are given, AF and FB are easily found: in the same manner, because AC and the angle CAN are given, the lines AI and IC may be found.

If from the angle ABC there be taken the angle ABF, the remainder FBC, added to DBC, gives DBF; by means of which, and the given side DB, DG or FL will be found, and likewise BG. By adding AF to FL we get AL, and by deducting BG from BF we shall have GF = DL.

In a similar manner the several distances from the point A to the perpendiculars, let fall from each of the angular points of the figure upon the meridian NS, may be determined, as also the perpendiculars themselves. By this method the extensive survey of the kingdom of France was carried on from the measure of nineteen bases.[†]

The measures being all reduced to the meridian NS, the difference of latitude between A and D may be determined. Find, from observation with a zenith sector[‡], the true zenith distance of a star at each place, the difference of these zenith distances will be the difference of latitude between A and D. Perhaps it may be proper to remark that the perpendicular DL does not accurately coincide with a circle[§], or parallel, of latitude, and where the utmost exactness is required the difference must be calculated and applied to AL; the true difference of latitude between A and L will then be obtained. This difference of latitude will be to 1° as the length of the meridian AL is to the length of a degree. The difference of latitude between A and each of the other places B, C, K, &c. may be found by a similar process.

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[*] Or if the sun be above the horizon, its altitude may be taken in the direction A ⊙, and thence its azimuth ⊙ AS may be computed.

[†] Traité de Trigonométrie, par M. Cagnoli, Chap. XI. Art. 341.

[‡] Trigonometrical Survey of England and Wales, Vol. II. Part II. published by Mr. Faden.

[§] The difference is not material, except the distance DL be very great. See Chapter IV. (G. 83.)

(F) In small surveys, military sketches, &c. where great accuracy is not required, angles may be taken with a good pocket compass, having two sights, and being fitted to the top of a staff, sharp at one end so that it may be stuck in the ground, and allow the needle to move freely about.

Every circle is divided into 360 degrees; the compass is divided into 32 points, therefore 1 point is 11°.15'; ¼ point=2°.48'.45''; ½ point=5°.37'.30'' and ¾ point=8°.26'.15''. These points are reckoned from the north and south, towards the east and west, as in the following table:

North.		South.		Points.	Degrees.
N.b.E.	N.b.W.	S.b.E.	S.b.W.	1	11°.15'
N.N.E.	N.N.W.	S.S.E.	S.S.W.	2	22 30
N.E.b.N.	N.W.b.N.	S.E.b.S.	S.W.b.S.	3	33 45
N.E.	N.W.	S.E.	S.W.	4	45 0
N.E.b.E.	N.W.b.W.	S.E.b.E.	S.W.b.W.	5	56 15
E.N.E.	W.N.W.	E.S.E.	W.S.W.	6	67 30
E.b.N.	W.b.N.	E.b.S.	W.b.S.	7	78 45
E.	W.	E.	W.	8	90 0

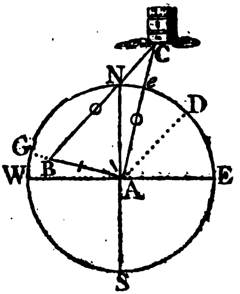
The variation of the magnetic needle ought to be allowed for, in measuring angles with a compass.

At London the north point of the needle rests in a direction of about 24°.30' westward of the true north.

EXAMPLE XXIII.

Having occasion to travel through the counties of Kent and Surrey, I perceived the fort built by Lady James on Shooter's Hill, which bore from me N.N.E., and travelling 20 miles in a W.N.W. direction, I perceived the fort again, which bore N.E.b.E. Required my distance from it at each station?

CONSTRUCTION.



With the chord of 60° describe a circle, draw the meridian NS, and EW at right-angles to it.

Set off the N.N.E. point of the compass, or 22°.30'. from N to *e*, draw AC; set off the W.N.W. point of the compass, viz. 67°.30' from N to G, and draw AG, on which set off AB = 20 miles from a scale of equal parts. Lastly, set off the N.E.b.E. point = 56°.15' from N to D, draw AD, and parallel thereto, through B, draw BC; then AC and BC are the distances required, and C the fort.

CALCULATION.

Because AD is parallel to BC, the angle CAD = BCA = the difference between ND and Ne = 33°.45′. The angle BAC = the sum of NG and Ne = 90°, hence the angle CBA = 56°.15′. By case second, right angled triangles, AC = 29·93 and BC = 36 miles.

EXAMPLE XXIV.

I observed a church which bore from me N.E.b.N. and going 2½ miles in a N.N.W. direction, the church then bore E.S.E. Required my distance from it at each observation?

Answer. At the first observation the distance was 1·8 mile, and at the second 2·1 miles.

EXAMPLE XXV.

From a ship at sea I observed a point of land to bear E.b.S. and after sailing N.E. 12 miles, it bore S.E.b.E. Required the distance of the last place of observation from the point of land?

Answer. 26 miles.

EXAMPLE XXVI.

From a station at H, an object G, the distance of which I knew to be 4 miles, bore N.E.b.N½E.; and another C, distant 2¼ miles, bore S.E.b.S.¼E. Required the distance between C and G?

Answer. CG = 4·923 miles.

EXAMPLE XXVII.

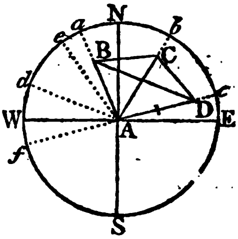
Saint Paul's church at London bore from me N.b.E. and travelling W.N.W. 15 miles, it bore N.E.¼E. Required my distance from St. Paul's at the last place of observation?

Answer. 23·19 miles.

EXAMPLE XXVIII.

Coasting along the sea-shore I observed two headlands, the first bore N.N.W. the second N.N.E.¾E. Then steering E.N.E.½E. 16 miles, the first headland bore W.N.W. the second N.W.b.N.¼W. Required the bearing and the distance of the two headlands from each other?

CONSTRUCTION.



With the chord of 60° describe a circle; draw the meridian NS and EW at right-angles to it.

1. Set off the N.N.W. point of the compass, or 22°.30′ from N to *a*, and draw A*a*.

2. Set off the N.N.E.¾E. point, or 30°.56′.15″ from N to *b*, and draw A*b*.

3. Set off the E.N.E.½E. point, or 73°.7′.30″, from N to *c*, draw A*c*, on which set off AD = 16 miles from a scale of equal parts.

4. Set off the W.N.W. point from N to *d* = 67°.30′ draw A*d*, and parallel thereto, through D, draw DB, then B is the situation of the first headland.

5. Set off the N.W.b.N.¼W. point of the compass, or 36°.33′.45″ from N to *e*, draw A*e* and DC parallel to it, then C is the situation of the second headland. Join BC, which will be the distance required.

CALCULATION.

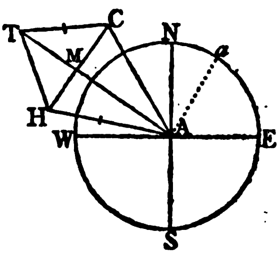
I. *In the triangle* BAD, the angle BAD = arc *ac* = 95°.37′.30″, the angle ABD = angle *d*A*a* = arc *da* = 67°.30′. − 22°.30′ = 45°, for ABD and *d*A*a* are alternate angles. Hence the angle BDA = 39°.22′.30″, and the side AD = 16. The distance AB may be found = 14·35, and BD = 22·52 miles.

2. *In the triangle* ADC the angle CAD = arc *bc* = 73°.7′.30″. − 30°.56′.15″ = 42°.11′.15″. Because A*e* is parallel to DC, and AD is produced to *f*, the angle *e*A*f*, measured by the arc *ef*, or the supplement of *ec* = ADC = 70°.18′.45″; hence the angle ACD = 67°.30′; hence AC may be found = 16·31 miles, and DC = 11·63.

3. *In the triangle* BAC are given AB = 14·35, AC = 16·31, and the angle BAC, measured by the arc *ab* = 53°.26′.15″; hence the angle ABC will be found = 70°.29′, and the distance BC = 13·9 miles.

If from the supplement 109°.31′ of ABC, the angle BAN be taken, the remainder 87°.1′ will be the bearing of the second headland C from the first B, viz. N.87°.1′ E.; or of the first headland B from the second C, viz. S. 87°.1′W.

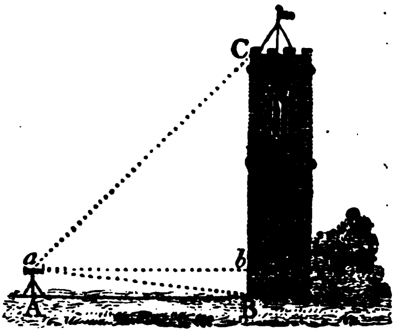
EXAMPLE XXIX.



The distance between a tower T, and a church C, was known to be 20 miles; from a ship at anchor A, I saw a windmill at M in a right line with the tower, the bearing of both was N.W.b.W. the church at the same time bore N.N.W.½W. After which we weighed anchor, and sailed upon a W.b.N. course HA = 25 miles, and the mill and the church were in a straight line HC, and bore N.E.b.N. Required the distance between the church, tower, and windmill, at each station, or place of observation A and H?

Answer.	AM = 23·09	HM = 9·6
	AC = 26·19	HC = 21·94
	AT = 38·82	HT = 18·41

EXAMPLE XXX.



From the bottom B of a tower, I measured 200 feet in a direct line AB on an horizontal plane. I then took the angle C *a* *b* = 46°.30′*, the height A*a* of my instrument was 5 feet. Required the height BC of the tower?

Answer. Here are given the base *ab* and the angle *cab* of a right-angled triangle, to find the perpendicular *bc*=210·7, to which add the height of the instrument, then BC=215·7 feet. For it is evident that $Aa = Bb$, since ab is parallel to AB and equal thereto, and the angle aBA equal to the angle baB.

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* In *Crakelt*'s translation of *Mauduit*'s Trigonometry, page 182, it is shewn, *that the error of an altitude* BC, *is to the error committed in taking the angle cab, as double the height* BC *is to the sine of double the observed angle cab.* The error, therefore, which can prevail in the determination of the said height, will be the least possible, when the *sine* of double the observed angle is the greatest possible; that is, when this angle is 45°. Hence, says he, when we want to find the height of any object, we must so contrive it that the observed angle be the nearest possible to 45°. He then shews that in observing an object at an angle of 45°, if an error of one minute be made in the determination of the angle, the error in the altitude will be $\frac{1}{1719}$ part of the altitude: also if the angle observed be greater or less than 45°, the error in the height will be increased in the ratio of radius to the sine of double the said angle.

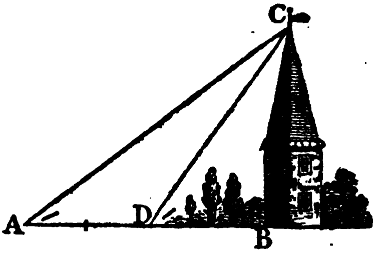
EXAMPLE XXXI.

Wanting to know the height of a church BC, I measured 45 yards from the bottom B upon a horizontal plane AB, at A, I took the angle $cab = 48^\circ.12'$, the height of my instrument was 5 feet. Required the height of the church?

Answer. The height $bc = 50{\cdot}33$ yards, to which add Bb the height of the instrument, then $BC = 51{\cdot}99$ yards the height of the church.

EXAMPLE XXXII.

Wanting to know the height of a steeple BC, inaccessible on account of several small trees on one side, and uneven ground on the other sides; at D, I took an angle $CDB = 51^\circ.30'$, and measuring the distance $AD = 75$ feet, at A, I took the angle $CAB = 26^\circ.30'$. Required the height of the steeple, and the distance of the first station from its base?



SOLUTION.

Subtract the angle CAB from the angle CDB, the remainder $25^\circ =$ the angle ACD: for ACD is the difference between ACB and DCB, the complements of the two observed angles, and the difference between the complements of two angles must be equal to the difference between the angles themselves. Then in the oblique triangle ADC all the angles are given to find $DC = 79{\cdot}18$, and in the right-angled triangle DBC, the hypothenuse DC, and angle CDB are given, to find $BC = 61{\cdot}97$, and $DB = 49{\cdot}29$.

EXAMPLE XXXIII.

Wanting to know the height of an inaccessible object BC; at A, I took the angle $CAB = 28^\circ.34'$, and measuring in a straight line towards the object from A to D 30 yards, the angle CDB was $50^\circ.9'$. Required the height of the object, and my distance from it at the second station?

Answer. The height $BC = 30$ yards, and distance $DB = 24{\cdot}99$.

EXAMPLE XXXIV.

Two observers at the distance of half a mile from each other, on an horizontal plane, took the altitude or angle of elevation of a cloud at the same moment of time, the same point of the cloud was observed by both, and in the same direction: the angles were 35°. and 64°. Required the perpendicular height of the cloud, and its distance from each observer?

Answer. The perpendicular height was 935·75, the distance from the observers 1041·1, and 1631·4 yards.

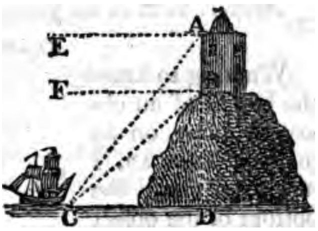
EXAMPLE XXXV.

In the year 1784 two observers on Blackheath, at the exact distance of a mile from each other, observed the angle of elevation of Lunardi's balloon; the angle at the nearest station to it was 36°.52′ and the other angle 30°.58′. Required the perpendicular height of the balloon, and its distance from each station?

Answer. The balloon's distance from the nearest station was 5·006 miles, from the other station 5·837 miles, and the perpendicular altitude 3·003 miles.

EXAMPLE XXXVI.

From the top (A) of Flamborough-head light-house, the angle of depression (EAC) of a ship at anchor was 3°.38′; and at the bottom (B) of the light-house, the angle of depression (FBC) was 2°.43′; required the horizontal distance (DC) of the vessel, and the height (DB) of the promontory above the level of the sea; the light-house (AB) being 85 feet high.

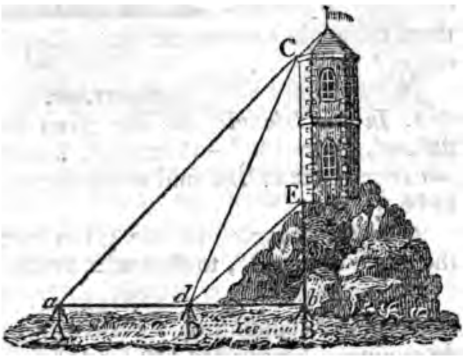


The $\angle \mathrm{EAC} = \angle \mathrm{ACD} = 3°.38'$; the $\angle \mathrm{FBC} = \angle \mathrm{BCD} = 2°.43'$; and $\mathrm{AB} = 85$ feet.

Hence CD will be found $= 5296·4$ feet, and $\mathrm{DB} = 251·31$ feet. *Answer.*

EXAMPLE XXXVII.

Wanting to know the height of a tower EC, which stood upon a hill: at A, I took the angle of elevation CAB = 44°; I then measured AD = 134 yards in a straight line towards the tower, at D the angle CDB was 67°.50′, and CDE = 16°.50′. Required the height of the



tower, and the height of the hill? The height of the instrument being five feet.

SOLUTION.

1. Subtract the angle CAB from CDB, the remainder 23°.50′ =ACD, hence all the angles in the triangle CAD are given, and one side AD, to find DC=230·4 yards.

2. *In the triangle* CDE all the angles are given, viz. CDE= 16°.50′, DCE=90°−67°.50′=22°.10′, hence the angle CED= 141°. and CE=106 yards; also DE=138·13 yards.

3. *In the right-angled triangle* EDB all the angles are given, viz. EDB=CDB−CDE=51°, and DEB=39°, also DE is given; hence BE=107·3 yards, to which add the height of the instrument, and you have 108·96 yards for the height of the hill.

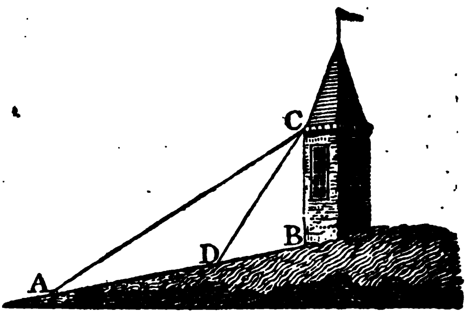
EXAMPLE XXXVIII.

Wanting to know the height of a castle standing upon an eminence, I took an angle of elevation CAD=40°, and measuring 40 yards in a straight line towards the castle, the angle CDB was 63°.20′, and CDE 14°.30′: what was the height of the castle EC, and the distance DB?

Answer. EC=24·69 yards, and DB=29·13 yards.

EXAMPLE XXXIX.

Wanting to know the height of an object standing on an inclined plane AB, I measured from the bottom of the object a distance BD = 40 feet, at D, I took the angle CDB = 41°; going on in the same direction 60 feet farther, the angle CAB was 23°.45′: Required the height of the object?



SOLUTION.

1. *In the triangle* ADC are given the angles, viz. CAB = 23°.45′, ADC=180°−41°=139°, hence ACD=17°.15′, or CDB −CAB=ACD=17°.15′ and AD being=60 feet, DC will be found to be 81·488.

2. *In the triangle* CDB are given CD=81·488, DB=40 and the angle CDB=41°, to find BC=57·64.

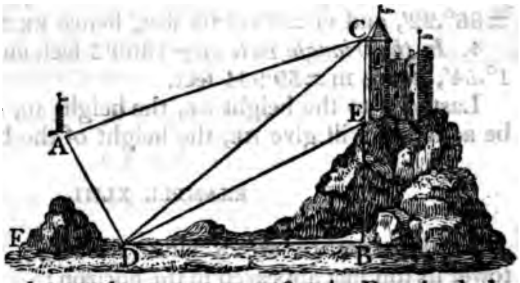
EXAMPLE XL.

An object standing on a declivity whose height I wished to determine,—I measured 150 yards from the base of it, and then took an angle CDB=47°.50′, and in the same direction I measured 80 yards farther, and then took the angle CAB=38°.30′: Required the height of the object?

Answer. DC=307·1, and BC the height=234·4 yards.

EXAMPLE XLI.

Wanting to know the height of a castle CE standing upon a hill, at D the angle CDB was 58°, CDE =25°, and ADC =72°.10′. The ground not permitting me to retreat in a straight line towards F, I therefore measured from D to A 52 yards; at A the angle CAD was 64°.30′: Required the height of the castle EC, and the height of the hill BE, above the level of the first station D?



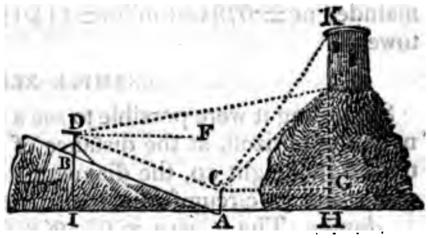
SOLUTION.

1. *In the triangle* ADC are given the angles ADC=72°.10′, CAD=64°.30′, and consequently ACD=43°.20′, and the side AD=52 yards, to find DC=68·392 yards.

2. The remaining part of the solution is the same as in Example XXXVII; CE will be found = 34·463 and BE = 23·536 yards.

EXAMPLE XLII.

Wanting to know the height of a tower EK standing upon a hill, the height HE of the hill, and the horizontal distance AH from the station A.—At A the angle GCK was 3°.38′, and the angle GCE was 2°.43′; then having set up a staff AC, whose height was equal to that of the theodolite, I measured 1809·5 feet up the sloping ground AB, as a base, in a direct line with the tower, keeping the points K, E, C, B in the same vertical plane. At B I took the following angles, viz. FDC=BAI=1°.54′; and EDF=1°.33′. Required the horizontal distance AH, the height HE of the hill, the height EK of the tower, and IB the elevation of the station B above that at A.



SOLUTION.

1. *In the triangle* EDC, the ∠ EDC = 3°.27′, ECD = 175°.23′. DEC = 1°.10′, and the side DC = 1809·5 feet, hence CE = 5348·08 feet.

2. *In the triangle* GCE, the ∠ GCE = 2°.43′, and CE = 5348·08 feet, hence CG = AH = 5342·07 feet, and GE = 253·48 feet.

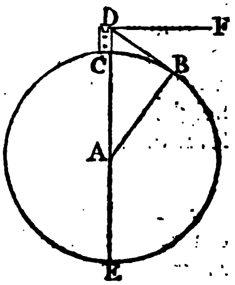
3. *In the triangle* KCE, the ∠ KCE = 0°.55′, CEK = 92°.43′, CKE = 86°.22′, and CE = 5348·08 feet, hence EK = 85·731 feet.

4. *In the triangle* BAI, AB = 1809·5 feet, and the angle BAI = 1°.54′, hence BI = 59·994 feet.

Lastly, if to the height GE, the height AC, of the instrument be added, it will give HE, the height of the hill.

EXAMPLE XLIII.

At the distance of 25 miles from a tower its top just appeared in the horizon; required its height. The diameter of the earth being 7964 miles, and its circumference 25019·7024 miles.



SOLUTION.

25019·7024 miles : 360° :: 25 miles : 21′.34″ = ∠ DAB. In the right-angled triangle ABD, AB and the ∠ DAB are given, to find AD = 3982·078446 miles, from which take AC = AB, the remainder DC = ·078446 miles = 414·19 feet, the height of the tower.

EXAMPLE XLIV.

Supposing it were possible to see a light-house or other object D, in the horizon, at the distance of 200 miles, it is required to find its height CD, the diameter EC of the earth being 7964 miles, and its circumference 25019·7024 miles.

Answer. The ∠ DAB = 2°.52′.39″, AD = 3987·0272 miles, and DC = 5·0272 miles = 26543·616 feet, being 5910·616 feet higher than *Chimboraço*, the highest of the *Andes*.

EXAMPLE XLV.

The Peak of Teneriffe is said to be 2½ miles above the level of the sea; at what distance can it be seen, supposing the radius of the earth to be 3982 miles?

Answer. $\sqrt{\text{ED} \times \text{DC}}$ = DB (Euclid 36 *of* III.) = 141·12 miles.

EXAMPLE XLVI.

If the Peak of Teneriffe be 2½ miles high, and the angle CDB made by a plumb-line, and a line DB meeting the surface of the sea in the farthest visible point B, be equal to 87°.58′.13″, it is required to find the diameter CE of the earth, supposing it to be a perfect sphere, and the utmost distance DB that can be seen from the top D of the mountain.

Answer. $\frac{(\text{rad.}+\text{sine CDB})\times \text{DC}}{\text{cosine CDB}}=\text{DB}=141{\cdot}12$ miles, and CE $=7964$ miles.

CHAP. IV.

OBSERVATIONS ON THE ADMEASUREMENT OF A BASE LINE.

(G) Where the ground is perfectly level, the manner of measuring a straight line from one object to another appears to be simple and easy; yet, on account of the curvature of the earth, no two points on its surface can be exactly situated in the same horizontal line; the chord of the arc, and not the arc itself, being the horizontal distance. Now the radius of one circle is to the radius of any other circle, as any arc of the former is to a similar arc of the latter. If we take, for instance, the base line measured on *Hounslow-heath*, (D. 72.) 27404·2 feet, the radius of the earth 3982 miles, or 21024960 feet, we shall have 21024960 feet : 27404·2 feet :: 1 : $\frac{27404{\cdot}2}{21024960}$ feet, the length of the measured arc in terms of the radius 1. But the difference between any arc and its chord, the radius being 1, is $\frac{1}{24}$ of the cube of the length of the arc *; hence $\left.\frac{27404{\cdot}2}{21024960}\right|^{3} \times \frac{1}{24} = {\cdot}00000000000092258$ will express the difference in terms of the radius 1, which multiplied by 21024960 feet, the radius of the earth, produces ·001939, the extent in feet by which a terrestrial arc of 27404·2 feet exceeds the chord of the same arc, a difference scarcely worth notice, even where the greatest accuracy is required.

(H) When the ground on which a base line is to be measured is sloping, it will be necessary, in some cases, to reduce it to a horizontal level. Thus, after having determined the direction of the base AF†, by poles LA, I*m*, H*n*, G*o*, pointed at one end and fixed perpendicularly in the ground by means of a plumb-

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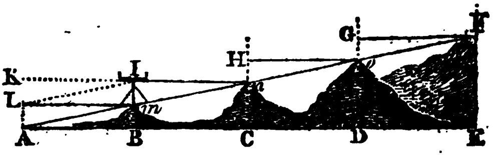
* See Chapter V. following.

† The point A, and the summits of the hills *m*, *n*, *o*, F, should be connected as to form a regular slope, AF.



line; the sum of the horizontal distances Lm, In, Ho, GF will evidently be equal to the whole horizontal distance AE; and if the heights AL, mI, nH, Go be successively measured, their sum will give the whole height EF.

(I) If the ground be irregular, or if it rise and descend alternately, it is evident that the difference between the heights of the poles must be added when ascending, and subtracted when descending, in order to determine the different elevations and depressions of the ground.

(K) Surveyors generally ascertain the altitudes of irregular hills by the assistance of a spirit-level, and perpendicular poles placed at convenient distances from each other. This practice is called levelling.

(L) A base line on a sloping ground may likewise be measured by taking angles at its extremities with a theodolite. Thus, let Im represent a theodolite, AL a pole fixed perpendicular to the horizon and equal in length to the height of the instrument; also, let KI be a horizontal line (which may be ascertained by the bubble of air in the spirit-level of the telescope resting in the middle) and KIL the angle of depression between the top of the pole AL and the horizontal line KI.

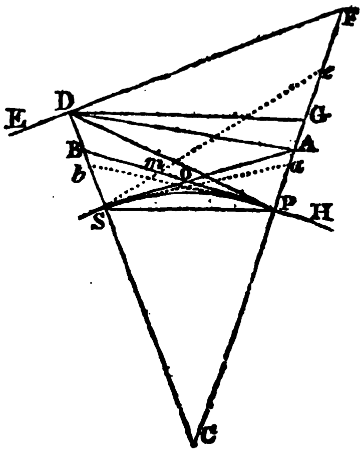
Then, because KI is parallel to AB, the angle KIL is equal to the angle mAB; then (O.44.), rad. : Am :: cosine ∠ KIL : AB, or AB $= \frac{\text{A}m \times \text{cosine} \angle \text{KIL}}{\text{rad.}}$.

(M) If Am = 400 yards, and ∠ KIL = 4°, AB will be 399·025 yards, hence the difference between Am and AB is less than 1 yard. It appears from this example, that when the measured base is inclined to the horizon in a small angle, a reduction of this kind will be unnecessary, except in cases where great accuracy is required.

OF THE ERRORS WHICH OCCUR IN TAKING ANGLES OF ELEVATION AND DEPRESSION WITH A THEODOLITE.

(N) When the observer is at a considerable distance from the object, the altitude taken with a theodolite will require correction. In the first place the horizon of the observer and that of the object observed are not the same. Let C be the centre of the earth, D the summit of a mountain, and HPOB the horizon of the observer. Through D draw EDF perpendicular to DC, and it will be the horizon of the point D. Now SD will be the true height of the mountain above the horizontal line PS; DPS the true angle of elevation and EPB the observed angle, from which the height of the mountain is determined to be BD, instead of SD.

Again, the $\angle$ FBD is supposed to be a right angle, in ordinary calculations, but in reality it is equal to the sum of the two interior angles BPC and PCB (Euclid 32 *of* I.) $=90^\circ + \angle$ C.



(O) The French academicians measured an inclined base at Peru, the length PD of which was found to be 6274·057 toises*, the angle of elevation DPB was 1°.5′.43″ (the effect of refraction being deducted). Now, rad. : PD :: sine DPB : DB, hence $\text{DB} = \frac{\text{PD} \times \text{sine DPB}}{\text{rad.}} = 119{\cdot}93$ toises. In order to calculate BS; PD may be used instead of PB, and CS and CB may be considered as equal to each other without sensible error. And as $(2\text{CS}+\text{BS}) \times \text{BS} = \text{PB}^2$ (Euclid 36 *of* III.), it will follow that $2\text{CS} \times \text{BS} = \text{PB}^2$, and hence $\text{BS} = \frac{\text{PB}^2}{2\text{CS}}$.

(P) If the diameter of the earth be taken $=6543373$ toises, as deduced from the admeasurements in Lapland, Paris, and Peru; BS will be found $= \frac{\overline{6274{\cdot}057}|^2}{6543373} = 6{\cdot}0158$ toises. By adding BS to BD the true height SD of the mountain $=125{\cdot}9458$ toises. Had the height DS been determined from the triangle DPS, by the most exact calculation, it would have been $=125{\cdot}97$ toises. Hence it appears that in most cases the angle at C may be rejected, that PB may be taken $=$ PD, the $\angle$ PBD $\doteq$ a right angle, and CS $=$ CB without material error.

(Q) Angles of elevation or depression taken with a theodolite may be corrected thus: Let D be the place of the telescope when the theodolite stands on the vertical line CD: P the situation of the telescope on the vertical line CP. Then if the telescope at P be directed to an object at D, the elevation of that object above the horizontal line HPB is the ∠ DPB; and when the telescope is at D and directed to an object at P, the angle of depression, below the horizontal line EDF, is the ∠ FDP.

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* Bouguer, Figure de la Terre. A toise=6 French feet, and 107 French feet=114 English feet.

Now, bcause PB touches the circle, and PS cuts it, the ∠ BPS (measured by half the arc PS *) = ½ ∠ C; therefore ∠ DPS = ½ ∠ C + the ∠ of elevation DPB.

Through D draw DG parallel to SP, then the ∠ GDP = ∠ DPS (Euclid 29 *of* I.), hence ∠ GDP = ½ ∠ C + the ∠ of elevation DPB; and because the triangles CPS and CDG are isosceles, and the ∠ C is common to both of them, the ∠ CPS = ∠ CDG, but the angles CPB and CDF are right angles, therefore the ∠ BPS) = ½ ∠ C) = ∠ FDG.

Again, ∠ FDP = ∠ GDP + FDG = ½ ∠ C + the ∠ of elevation DPB + ½ ∠ C; therefore ∠ FDP = ∠ C + the ∠ of elevation DPB; to each of these equals add the ∠ DPB, then the ∠ of depression FDP + the ∠ of elevation DPB = ∠ C + twice that of elevation DPB; hence $\frac{(\angle \text{ of depr. FDB} + \angle \text{ of elev. DPB}) - \angle \text{C}}{2}$ = the true ∠ of elevation.

When the angles are both elevations or both depressions, their difference must be diminished by the ∠ C, and half the remainder will be the true ∠ of elevation of the higher of the two objects.

The ∠ C is generally very small, and where the measured base does not exceed six or seven hundred yards, it may be rejected.

EXAMPLE XLVII.

(R) Suppose D and P to be two objects fixed exactly at the same height above the ground as the height of the telescope of the theodolite; now if the ∠ FDP of depression be 26′, and the ∠ DPB of elevation 14′, what will be the error in observation? The arc PS, or distance of the stations, being 8000 feet.

The length of a degree in latitude 51°.9′ is 364950 feet †; 364950 feet : 60′ :: 8000 feet : 1′.19″ nearly = ∠ C. Then $\frac{(26' + 14') - 1'.19''}{2} = 19'.20''\frac{1}{2}$ the true ∠ of elevation DPB, hence

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* *The angle formed between the tangent of any arc and its chord, is measured by half that arc.*

The ∠ TBF (Plate I. fig. 1.) is measured by half the arc BF. For the ∠ TBF is equal to the ∠ F*b*B in the alternate segment (Euclid 33 *of* III.), and the ∠ F*b*B = ½ ∠ FCB (Euclid 20 *of* III.) therefore the ∠ TBF = ½ ∠ FCB; but the ∠ FCB is measured by the arc BF, therefore the ∠ TBF is measured by half the arc BF. Q. E. D.

† Trigonometrical Survey of England and Wales, Vol. II. Part II. page 113.

$19'.20''\frac{1}{2} - 14' = 5'.20''\frac{1}{2}$ is the error of the instrument, or the quantity by which the ∠ of elevation was too small, or the ∠ of depression too large.

THE NATURE OF TERRESTRIAL REFRACTION AND ITS EFFECTS ON ANGLES OF ELEVATION.[*]

(S) As terrestrial refraction arises from the gross vapours, and exhalations of various kinds, which are suspended in the air near the surface of the earth, and which are perpetually changing, it is very difficult to ascertain the exact quantity of it at any particular time.

(T) The course of a ray of light in its passage through the atmosphere is, in general, that of a curve which is concave towards the earth, and the observer views the object in the direction of a tangent to this curve; hence the apparent, or observed angle of elevation is always greater than the true angle.

(U) The altitudes of the heavenly bodies when within 5° or 6° of the horizon, should never be used where a very accurate result is required. The figures of the sun and moon, when near the horizon, are sometimes elliptical, having the minor axis perpendicular to the horizon, and the major axis parallel to the horizon. This change of figure arises from the refraction of the under limb being greater than that of the upper. But a perpendicular object, situated on the surface of the earth, will not have its length altered by refraction, the refraction of the bottom being the same as that of the top.

(W) The allowances usually made for refraction are too uncertain for any reliance to be placed on them, as scarcely two writers agree on this subject. Dr. Maskelyne makes it $\frac{1}{10}$ of the intermediate arc PS between the observer and the object; Bouguer $\frac{1}{9}$; Legendre $\frac{1}{14}$; General Roy from $\frac{1}{2}$ to $\frac{1}{24}$; and in the second volume of the Trigonometrical Survey, the variation is found to be from $\frac{1}{5}$ to $\frac{1}{32}$ of the intermediate arc.[†] This difference does not arise from inaccuracy of observation, but from circumstances which cannot be avoided, as the evaporation of rains, dews, &c. which produce variable and partial refractions.

(X) The following method is used in the Trigonometrical Survey[‡] for ascertaining the quantity of refraction.

Let C be the centre of the earth, P and S two stations on its surface; PB, SA the horizontal lines at right angles to CA and CB; also suppose A and B to be the true places of the objects observed, and *a* and *b* their apparent places. Then the ∠*b*PS will be the refraction at P, and the ∠*a*SP that at S.*

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[*] See a paper by Mr. Huddart, in the Philosophical Transactions for 1797, page 29; and another by the Rev. S. Vince, 1799, page 13. Also the Trigonometrical Survey of England and Wales, Vol. I. page 175.

[†] Page 177 and 178, Part I.

[‡] Volume I. page 175.

In the quadrilateral figure CSOP, the angles at P and S are right angles, therefore the ∠SOP + ∠C = two right angles, but the three angles of the triangle SOP = two right angles, hence ∠OSP + ∠SOP + ∠OPS = ∠SOP + ∠C, consequently ∠OSP + ∠OPS = ∠C, which is measured by the intermediate arc PS.

Now (∠ASP + ∠BPS) − (∠AS*a* + ∠BP*b*) = (∠OSP + ∠OPS) − (∠AS*a* + ∠BP*b*) = ∠C − (∠AS*a* + ∠BP*b*) = ∠*a*SP + ∠*b*PS the sum of both the refractions. Hence the following

(Y) RULE. Subtract the sum of the two depressions from the contained arc, and half the remainder is the mean refraction.

(Z) If one of the objects (A) instead of being depressed, be elevated, suppose to the point *e*, the ∠ of elevation being *e* SA; then the sum of the angles *m*SP, *m*PS will be greater than ∠OPS + ∠OSP (the ∠C, or contained arc PS) by the ∠ of elevation *e* SA. Hence ∠C + ∠*e* SA = ∠*m*SP + ∠*m*PS, from each of these equals take the ∠BP*b*, then ∠C + ∠*e*SA − ∠BP*b* = ∠*m*SP + ∠*b*PS the sum of the two refractions; that is, subtract the depression from the sum of the contained arc and elevation, and half the remainder is the mean refraction. Perhaps it may be necessary to remark, that previous to the observations the error of the instrument must be accounted for. (Q. 85.)

EXAMPLE XLVIII.

(A) The refraction between Dover castle and Calais church was thus determined.†

Let C be the centre of the earth, PS the surface; D the station on Dover castle; A the top of the great balustrade of Calais steeple; EDF the horizontal line; also let PG = SD; then the ∠FDG = ½∠C, or half the arc PS (Q. 86.). The distance from Dover to Calais is 137455 feet, hence 364950 feet : 60′ :: 137455 feet : 22′.35″ the ∠C, hence ∠FDG = 11′.17″½.

The height of D above low-water spring tides = 469 feet.

The height of A (communicated from France) = 140½ feet.

AG = 328½

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* In observing these angles two instruments are used, one at P and another at S; and the reciprocal observations are made at the same instant of time by means of signals, or by watches previously regulated for that purpose. The observer at P takes the depression of S, at the same moment which the observer at S takes the depression of P.

† Trigonometrical Survey, Vol. I. page 178.

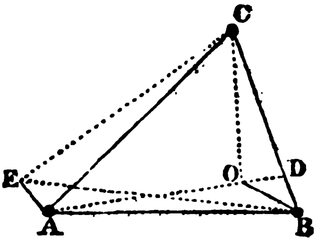
The triangle DGA may be considered as isosceles, and DG or DA = 137455 feet, the distance between Dover and Calais. Hence ½ AG : rad. :: DA : secant ∠ DAG = 89°.55′.52″.2, the double of which, deducted from 180°, leaves 8′.15″ for the ∠ GDA, to which add the ∠ FDG = 11′.17½″, and the whole angle FDA = 19′.32½″ supposing there was no refraction; but the ∠ FDA was determined from observation to be 17′.59″, hence the refraction was (19′.32½″ − 17′.59″ =) 1′33½″, being about $\frac{1}{15}$ of the contained arc.

(B) Mr. Huddart is of opinion, that a true correction for the effect of terrestrial refraction cannot be obtained by taking any part of the contained arc *; for different points, though nearly at the same distance from the observer, will have various refractions.

OF THE REDUCTION OF ANGLES TO THE CENTRE OF THE STATION.

(C) In surveys of kingdoms and counties, where signals on the steeples of churches, vanes of spires, &c. are used for points of observation, the instrument cannot be placed exactly at the centre of the signal, and consequently the angle observed will be different from that which would have been found at the centre. The correction is generally very small, and is only necessary where great accuracy is required.

The observer may be considered in three different positions with respect to the centre, viz, he is either in a line with the centre and one of the objects, or a line drawn from the centre through his situation would, if produced, pass between the objects; or a line drawn from the centre to the place of the observer, when produced, would pass without the objects.



(D) FIRST, let the observer be at D, in a line between the objects B and C, viz. on one side of the triangle ABC; B being the proper centre of the station and the ∠ ABC that required. It is plain that the ∠ CDA, being the exterior ∠ of the triangle ADB, is too large by the interior ∠ DAB.

Therefore ∠ ABC = ∠ CDA − ∠ DAB.

(E) SECONDLY. Let the observer be at O, within the triangle ABC, and let B be the centre of the station, and the ∠ ABC that required. Now ∠ AOC + ∠ OAC + ∠ OCA = ∠ ABC + −

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* Philosophical Transactions for 1797, page 29th et seq.

$\mathrm{OAB} + \angle \mathrm{OAC} + \angle \mathrm{OCA} + \angle \mathrm{OCB}$, each of the sums being equal to two right angles; therefore the $\angle \mathrm{AOC} = \angle \mathrm{ABC} + \angle \mathrm{OAB} + \angle \mathrm{OCB}$, that is, the $\angle \mathrm{AOC}$ is greater than the $\angle \mathrm{ABC}$ by the sum of the angles OAB and OCB. Therefore $\angle \mathrm{ABC}$* $= \angle \mathrm{AOC} - (\angle \mathrm{OAB} + \angle \mathrm{OCB})$.

(F) Thirdly. Let the observer be at E, without the triangle ABC, and let A be the centre of the station, and the $\angle \mathrm{CAB}$ that required.

Now $\angle \mathrm{CEB} + \angle \mathrm{ECA} + \angle \mathrm{ACO} + \angle \mathrm{OCB} + \angle \mathrm{EBO} + \angle \mathrm{OBC} = \angle \mathrm{CAB} + \angle \mathrm{ACO} + \angle \mathrm{OCB} + \angle \mathrm{ABE} + \angle \mathrm{EBO} + \angle \mathrm{OBC}$, each of the sums being equal to two right angles; therefore

$\therefore \angle \mathrm{CAB} + \angle \mathrm{ABE} = \angle \mathrm{CEB} + \angle \mathrm{ECA}$, and consequently

$\therefore \angle \mathrm{CAB} = \angle \mathrm{CEB} + \angle \mathrm{ECA} - \angle \mathrm{ABE}$.

EXAMPLE XLIX.

Let A and B represent the vanes on two steeples; E the situation of the theodolite upon the steeple A, and O its situation upon the steeple B. Then, suppose

$\mathrm{AE} = 12$ feet	$\mathrm{BO} = 10\ 5$ feet
$\angle \mathrm{CEB} = 74°.32'$	$\angle \mathrm{AOC} = 49°.27'$
$\angle \mathrm{CEA} = 139°.39'$	$\angle \mathrm{COB} = 137°.55'$.

It is required to find the angles CAB and ABC, the distance AB being 5000 feet.

SOLUTION.

The $\angle \mathrm{CEB} = \angle \mathrm{CAB}$ nearly, and $\angle \mathrm{AOC} = \angle \mathrm{ABC}$ nearly, with these angles and AB, find AC and BC (as in Example I. Chap. III.) $= 4581{\cdot}8$ and $5811{\cdot}6$. Then,

$\mathrm{AC} : \sin. \angle \mathrm{CEA} :: \mathrm{AE} : \sin. \angle \mathrm{ECA} = 5'.50''$

$\mathrm{AB} : \sin. \angle \mathrm{BEA} :: \mathrm{AE} : \sin. \angle \mathrm{ABE} = 7'.29''$.

Hence $\angle \mathrm{CAB} = 74°.30'.21''$ (F. 90.)

$\mathrm{BC} : \sin. \angle \mathrm{COB} :: \mathrm{BO} : \sin. \angle \mathrm{OCB} = 4'.10''$.

$\mathrm{AB} : \sin. \angle \mathrm{AOB} :: \mathrm{BO} : \sin. \angle \mathrm{OAB} = 0'.56''$.

Hence $\angle \mathrm{ABC} = 49°.21'.54''$ (E. 89.).

With the corrected angles CAB, ABC, and the distance AB, the sides AC and BC may be determined.

OF THE REDUCTION OF ANGLES FROM ONE PLANE TO ANOTHER.

(G) Angles which are inclined to the horizon, may be reduced to the corresponding horizontal angles, in cases where very great accuracy is required. Let the lines PS, PB, BS be three chords of terrestrial arcs, that is, let the points P, B, and S, be all equally distant from the centre of the earth, and let the point D be elevated so as to be farther from the centre of the earth than any of the points P, B, S, by the quantity SD, it is required to reduce the triangle BDP to the triangle BSP.

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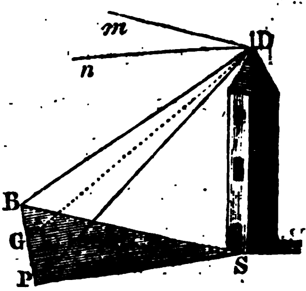
* If the proper centre of the station were at O, and the observer at B, it is plain that the angles OAB and OCB must be added to the $\angle \mathrm{ABC}$ to obtain the $\angle \mathrm{AOC}$.

(H) The line SD may be supposed to be perpendicular to each of the chords SB and SP without sensible error, though strictly speaking the angles DSB and DSP are each equal to $90^\circ + \frac{1}{2}$ the arc which the chords SB and SP subtend (Q. 86. and Note.)



Likewise the chords SB and SP may be used instead of their corresponding arcs. (G. 83.) By inspection of the figure it is plain that BD is greater than BS, and PD greater than PS; but the base PB is common to the two triangles BDP and BSP, therefore the ∠ BSP is greater than the ∠ BDP. (EUCLID 21 of I.)

(I) Now if a perpendicular be drawn from D upon the base PB, the segment $\text{GB} = \frac{\text{BD} \times \text{cos. DBP}}{\text{rad.}}$ (K. 39.); and if a perpendicular be drawn from the point S upon the base PB, the segment intercepted between PS and the perpendicular will be $= \frac{\text{BS} \times \text{cos. SBP}}{\text{rad.}}$ (K. 39.)

(K) Hence cos. $\text{DBP} \times \text{BD} = \text{cos. SBP} \times \text{BS}$, and $\text{cos. SBP} = \text{cos. DBP} \times \frac{\text{BD}}{\text{BS}}$, also BD : rad. :: BS : cos. DBS, for the triangles BSB and DSP are right angled at S.

Conseq. $\text{cos. SBP} = \frac{\text{cos. DBP} \times \text{rad.}}{\text{cos. DBS.}}$, or cos. DBS : cos. DBP :: rad. : cos. SBP.

(L) That is, *The cosine of the ∠ of elevation : the cosine of the ∠ inclined to the horizon :: radius : cosine of the horizontal ∠.*

Exactly in the same manner the ∠ SPB may be found.

Then $180^\circ - (\angle \text{SBP} + \angle \text{SPB}) = \angle \text{BSP}$.

(M) But the ∠ BDP taken on the elevated situation D, may be reduced to its corresponding ∠ BSP, by using only the observed ∠ BDP, and the angles of depression *m*DB, and *n*DP. The

$\angle\,\text{NDB} = \angle\,\text{DBS}$ and the $\angle\,\text{NDP} = \angle\,\text{DPS}$; hence the three angles necessary for this reduction are BDP, DBS, and DPS.

It is shewn (K. 39.) that $\cos.\angle\,\text{DBP} = \dfrac{\text{PB}^2 + \text{BD}^2 - \text{DP}^2}{2\text{BD} \times \text{PB}} \times \text{rad.}$

Hence $\text{PB}^2 + \text{BD}^2 - \left(2\text{BD} \times \text{PB} \times \dfrac{\cos.\ \text{DBP}}{\text{rad.}}\right) = \text{DI}^2.$

(N) That is, *The square of the side subtending any acute ∠ of a triangle, is less than the squares of the sides containing that acute ∠; by double the rectangle of those sides, multiplied by the cosine of the acute ∠ dividend by radius.*

Therefore, in the triangle BDP

$$\text{PB}^2 = \text{BD}^2 + \text{PD}^2 - \left(2\text{BD} \times \text{PD} \times \frac{\cos.\ \text{BDP}}{\text{rad.}}\right).$$

And, in the triangle SPB

$$\text{PB}^2 = \text{BS}^2 + \text{PS}^2 - \left(2\text{BS} \times \text{PS} \times \frac{\cos.\ \text{BSP}}{\text{rad.}}\right).$$

By making the values of PB^2 equal to each other, and reducing the equation, we get $(\text{BD}^2 - \text{BS}^2) + (\text{PD}^2 - \text{PS}^2) = \left(2\text{BD} \times \text{PD} \times \dfrac{\cos.\ \text{BDP}}{\text{rad.}}\right) - \left(2\text{BS} + \text{PS} + \dfrac{\cos.\ \text{BSP}}{\text{rad.}}\right)$. But $\text{BD}^2 - \text{BS}^2 = \text{PD}^2 - \text{PS}^2 = \text{SD}^2$ (Euclid 47 *of* I.), therefore $\text{SD}^2 \times \text{rad} =$

$$(\text{BD} \times \text{PD} \times \cos.\ \text{BDP}) - (\text{BS} \times \text{PS} \times \cos.\ \text{BSP}),$$

And hence $\cos.\ \text{BSP} = \dfrac{(\text{BD} \times \text{PD} \times \cos.\ \text{BDP}) - (\text{SD}^2 \times \text{rad.})}{\text{BS} \times \text{PS}}.$

$$\text{rad.} : \text{BD} :: \text{sine DBS} : \text{SD}$$

$$\text{rad.} : \text{PD} :: \text{sine DPS} : \text{SD}.$$

Hence, $\text{rad.}^2 : \text{BD} \times \text{PD} :: \text{sine DBS} \times \text{sine DPS} : \text{SD}^2$. (KEITH's *Geom.* 14 *of* VII.)

And, $\text{SD}^2 \times \text{rad.} = \text{BD} \times \text{PD} \times \text{sine DBS} \times \text{sine DPS} \times \dfrac{1}{\text{rad.}}.$

Again, $\text{rad.} : \text{BD} :: \cos.\ \text{DBS} : \text{BS}$

$\text{rad.} : \text{PD} :: \cos.\ \text{DPS} : \text{PS}$

Hence $\text{rad.}^2 : \text{BD} \times \text{PD} :: \cos.\ \text{DBS} \times \cos.\ \text{DPS} : \text{BS} \times \text{PS}$,

And $\text{BS} \times \text{PS} = \text{BD} \times \text{PD} \times \cos.\ \text{DBS} \times \cos.\ \text{DPS} \times \dfrac{1}{\text{rad.}^2}$

Therefore by substitution the cosine of the $\angle\,\text{BSP} =$

$$\frac{\cos.\ \text{BDP} - \left(\text{sine DBS} \times \text{sine DPS} \times \frac{1}{\text{rad.}}\right)}{\cos.\ \text{DBS} \times \cos.\ \text{DPS} \times \frac{1}{\text{rad.}^2}} =$$

$$\frac{(\cos.\ \text{BDP} \times \text{rad.}^2) - (\text{sine DBS} \times \text{sine DPS} \times \text{rad.})}{\cos.\ \text{DPS} \times \cos.\ \text{DPS}}.$$

(O) M. de Lambre gives the following formula * for finding the angle BSP, which he demonstrates by spherical trigonometry, viz.

$$\text{Sine } \tfrac{1}{2}\ \text{BSP} = \text{rad.}\sqrt{\frac{\text{sine } \frac{1}{2}(\text{BDP} + \text{DBS} - \text{DPS}) \times \text{sine } \frac{1}{2}(\text{BDP} + \text{DPS} - \text{DBS})}{\text{cos. DBS} \times \text{cos. DPS}}}$$

The sides BS and PS of the triangle BSP are easily found, from the height DS and the angles DBS and DPS being given.

EXAMPLE L.

Wanting to know the height of an object SD (standing on the horizontal plane PSB), and also the horizontal distances SP and SB, I measured a base PB=300 yards, at P the ∠ of elevation DPS was 8°.15′, and the ∠ DPB, inclined to the horizon, was 57°.15′: at B the ∠ of elevation DBS was 8°.45′.50″, and the ∠ DBP, inclined to the horizon, was 63°.15′. Required the horizontal ∠ BSP, the horizontal distances SP, and SB, and the height of the object above the plane.

Answer. ∠ BPS=56°.51′.50″; ∠ PBS=62°.54′.30″; (L. 91.) BS=289·41; PS=307·69; and DS=44·614 yards.

The height DS, and the distances BS and PS, may be found without reducing the angles. For, in the triangle DBP all the angles are given, and the side BP, whence BD and PD may be found; then in the triangle DPS, the ∠ DPS and the side PD are given to find PS and SD; also, in the triangle DBS, DB and the ∠ DBS are given, from which BS may be found.

EXAMPLE LI.

From the top of a tower 44·614 yards high, the angle BDP, subtended by two distant objects B and P, was 59°.30′; and the angles of depression *m*DB = 8°.45′.50″, and *n*DP = 8°.15′. Hence it is required to find the horizontal distances PB, BS, and PS.

Answer. The ∠ *m*DB = ∠ DBS = 8°.45′.50″; and ∠ *n*DP = ∠ DPS = 8°.15′, and ∠ BDP = 59°.30′. By the formula (N. 92 or O. 93.) cos. ∠ BSP = ·49655, hence ∠ BSP = 60°.13′.40″·4. and BS=289·41, PS=307·69, and PB = 300. The same answer may be obtained without reducing the angles, and without the formula.

(P) The formulæ at the conclusion of N. 92, and O. 93, may be applied to any two triangles, whether their bases be in the same horizontal plane and their vertices be elevated, as in Examples L. and LI. or their vertices be in the same horizontal plane, and the extremities of the base of the one triangle be elevated above the extremities of the base of the other.

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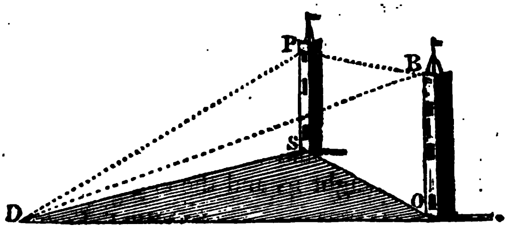
* This formula is demonstrated Prop. 7th, Chap. XI. Book III.

In the annexed figure, if the angle PDB be measured with a sextant, and the vertical angles BDO, PDS be measured with a theodolite,



$$\text{cosine} \angle \text{SDO} = \frac{(\text{cos. PDB} \times \text{rad.}^2) - (\text{sine PDS} \times \text{sine BDO} \times \text{rad.})}{\text{cos. PDS} \times \text{cos. BDO}}$$

or by M. de Lambre's formula*, viz.

$$\text{sine } \tfrac{1}{2} \text{ SDO} = \text{rad.} \sqrt{\frac{\text{sine } \frac{1}{2} (\text{PDB} + \text{PDS} - \text{BDO}) \times \text{sine } \frac{1}{2} (\text{PDB} + \text{BDO} - \text{PDS})}{\text{cos. PDS} \times \text{cos. BDO}}}$$

EXAMPLE LII.

From a station at D in the horizontal plane DSO, I took the angle PDB, subtended by the tops of two towers, $=37°.53'.20''$; and also the angles of elevation BDO $=4°.23'.55''$, and PDS $=4°.17'.21''$. The height of the tower BO is known to be 40 yards, and that of PS 30 yards; from which it is required to find the horizontal distance of my station from each of the towers, and their horizontal distance from each other.

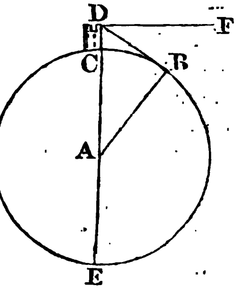
Answer. By the formula (P. 94.) $\angle$ SDO $=37°.59'$; DS $=400$ yards; DO $=520$; and SO $=320$ yards.

(Q) The same answer may be found without the formula. For, with the $\angle$ BDO and height BO, find DB $=521{\cdot}0536$, and DO $=520$; with the $\angle$ PDS and height PS, find DP $=401{\cdot}1234$, and DS $=400$. Then with DP, DB and the $\angle$ PDB, find PB $=320{\cdot}1562$; lastly, $\sqrt{\text{PB}^2 - \overline{\text{BO} - \text{PS}}^2} = \text{SO}$, the horizontal distance $=320$ yards.

OF THE DIP, OR DEPRESSION OF THE HORIZON AT SEA.

(R) The dip or depression of the horizon at sea, is the angle contained between the horizon of the observer, and the farthest visible point on the surface of the sea.

For, if an observer whose eye is situated at D, takes the altitude of a celestial object by a sextant, or Hadley's quadrant, and brings that object to the surface of the water at B, instead of the horizon DF, he evidently makes the altitude too great by the $\angle$ FDB.



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* Traité de Trigonométrie par M. Cagnoli, Appendice, page 467; or, Prop. 7. Book III. following.

Now the $\angle$ ADF is a right angle, and the $\angle$ ABD is likewise a right angle, therefore the $\angle$ FDB $=$ $\angle$ DAB. Hence, in the right-angled triangle ABD, there is given AB the radius of the sphere, AD$=$AC$+$DC, the radius of the sphere increased by the height of the eye above the surface of the sea, to find the $\angle$ DAB, or quantity to be subtracted from the observed altitude of any celestial object. Hence cos. $\angle$ DAB $=$ cos. $\angle$ FDB$=\dfrac{\text{rad.} \times \text{AB}}{\text{AC} + \text{CD}}$. Where AC$=$AB is the radius of the sphere, and DC the height of the observer's eye above the surface of the sea.

OR,

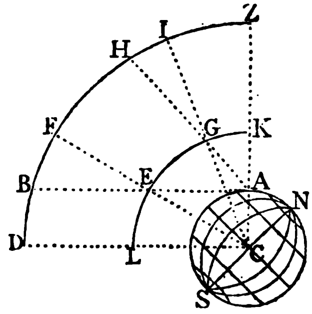
$(\text{EC}+\text{CD}) \times \text{DC} = \text{DB}^2$ (Euclid 36 *of* III.), that is $(\text{BC} \times \text{DC}) + \text{DC}^2 = \text{DB}^2$; or $(2\text{AC} \times \text{DC}) + \text{DC}^2 = \text{DB}^2$, and rejecting DC^2 as being small $\sqrt{2\text{AC} \times \text{DC}} = \text{DB}$, but AB $=$ AC : rad. :: DB $= \sqrt{2\text{AC} \times \text{DC}}$: tangt. DAB$=$tangt. $\angle$ FDB, therefore

$$\text{tangent} \angle \text{ FDB} = \frac{\text{rad.} \times \sqrt{2\text{AC}}}{\text{AC}} \times \sqrt{\text{DC}} = \sqrt{\frac{2 \times \text{rad.}^2}{\text{AC}}} \times \sqrt{\text{DC}}.$$

Now the first of these terms is a constant quantity, and if the diameter of the earth be 7964 miles, the logarithm of this quantity in feet will be 6·48915, hence log. tangt. $\angle$ FDB $=$ 6·48915$+\frac{1}{2}$ log. DC in feet.

OF THE PARALLAX OF THE CELESTIAL BODIES.

(S) That part of the heavens in which a planet would appear, if viewed from the surface of the earth is called its *apparent place:* and the point in which it would be seen at the same instant from the centre of the earth, is called its *true place*, the difference between the *true* and *apparent* place is called the parallax in altitude.



Let C be the centre of the earth, A the place of an observer on its surface, whose visible horizon is AB, true horizon CD, and zenith Z. Let ZIFD be a portion of a great circle in the heavens, and E the absolute place of any object in the visible horizon; join CE and produce it to F; then F is the true place of the object, and B its apparent place in the heavens; and the angle BEF$=$AEC is the parallax.

The parallax is the greatest when the object is in the horizon; for the angle AEC is greater than AGC, hence th [illegible] evated an object is (its distance from the earth's cen [illegible] the same) the less is the parallax. When the object is in the zenith the parallax vanishes, for then AC and AZ are in the same straight line CZ.

Since the apparent place (H) of a planet is more distant from the zenith (Z) then the true place (I) it therefore follows, that the parallax in altitude must be added to the observed altitude, in order to obtain the place of a planet as seen from the earth's centre. The stars on account of their immense distance from the earth have no sensible parallax, and the sun's mean parallax is only 8″.6. The moon's greatest horizontal parallax is 61′.32″. least 54′.4″.

(T) *The horizontal parallax being given, to find the parallax at any given altitude.*

In the right-angled triangle EAC

Radius : EC : : sine AEC : AC

And in the triangle GAC

GC=EC : sine GAC : : AC : sine AGC

And by comparing these two proportions

radius : sine GAC : : sine AEC : sine AGC.

But the sine of an arc is equal to the sine of its supplement, therefore sine GAC=sine GAK, and GAK is the complement of GAE, therefore sine of GAK=cosine GAE, hence

radius : cosine GAE : : sine AEC : sine AGC.

The last two terms being small, the arcs may be substituted for their sines without sensible error.

Hence the following rule:

Radius
: *Cosine of the apparent altitude,*
: : *The horizontal parallax in seconds,*
: *The parallax in altitude in seconds.*

EXAMPLE LIII.

The apparent altitude of the moon's centre is 24°. 29′.44″, the horizontal parallax 55′.2″. Required the parallax in altitude.

Radius, sine of 90° . . . . . 10·
: cosine moon's altitude 24°.29′.44″. . 9·95904
: : horizontal parallax 55′.2″=3302″log.= 3·51878
: parallax in altitude 3005″ log.= . . 3·47782

Hence the parallax in altitude is 3005″=50′.5″.

OF THE ADMEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

(U) One of the most simple and easy practical rules for measuring the elevations and depressions of objects by means of the barometer and thermometer, is that given in the Encyclopædia Brit. Article Pneumatics, or in Dr. Rees' New Cyclopædia, under the word Barometer. The directions are as follow — Let the observers be provided with two portable barometers, each of the same construction, with a nonius properly adapted to the scale, and a thermometer attached to each, having their bulbs each of the same diameter, nearly, as the diameters of the barometric tubes.

Place one of these barometers in the shade at the top of the eminence, with a detached thermometer near it; and let the other barometer be placed below in like manner, with a detached thermometer near it. When the thermometers have acquired the temperature of the air, that is, when the fluid in each becomes stationary, the observers must note down the temperatures shown by the thermometers, and the heights of the mercurial columns in the barometers.

Then the elevation of the higher barometer above the lower may be determined by the formula, which is deduced from the following observations.

(W) 1. The height through which we must rise in order to produce any fall of the mercury in the barometer, is inversely proportional to the density of the air; that is to the height of the mercury in the barometer.

2. When the barometer stands at 30 inches, and the air and quicksilver are of the temperature 32°, we must rise through 87 feet, in order to produce a depression of $\frac{1}{10}$th of an inch.

3. But if the air be of a different temperature, the 87 must be increased or diminished by ·21 of a foot for every degree of difference of the temperature from 32°.

4. Every degree of difference of the temperatures of the mercury at the two stations makes a change of 2·833 feet, or 2 feet 10 inches in the elevation. Hence,

If d be the difference between 32° and the mean temperature of the air, D the difference between the barometric heights in tenths of an inch, δ the difference between the mercurial temperatures, m the mean barometric height, and E the correct elevation. Then $\mathrm{E} = \frac{30\mathrm{D}\,(87 \pm \cdot 21d)}{m} \pm \delta \times 2{\cdot}833$

This formula may be given in words, thus:

(X) 1. Multiply the difference between 32° and the mean temperature of the air by ·21, and add the product to 87, if the mean temperature be above 32°, but subtract it if below.

2. Multiply the sum or difference, found above, by thirty times the difference, between the barometric heights, in tenths of inches, and divide the product by the mean of the barometric heights, the quotient will give the approximated elevation.

3. Multiply the difference between the mercurial temperatures by 2·833 feet, and add this product to the approximated elevation, if the upper barometer has been the warmer of the two, otherwise subtract it, the result will be the corrected elevation in feet.

EXAMPLE LIV.

Suppose that the mercury in the barometer at the lower station was at 29·4 inches, that its temperature was 50° of Fahrenheit's thermometer, and the temperature of the air 45°; and let the height of the mercury at the upper station be 25·19 inches, its temperature 46°, and the temperature of the air 39°, what would be the elevation of the higher barometer above the lower?

1. $\frac{45^\circ+39^\circ}{2}=42^\circ$ mean temperature of the air.

$42^\circ-32^\circ=10^\circ\ (=d)$.

$(\cdot21\times10)+87=89{\cdot}1\ (=87+\cdot21d)$.

2. $(29{\cdot}4-25{\cdot}19)\times10=42{\cdot}1$ diff. of barometric heights in *tenths* of inches $(=\text{D})$

$\frac{29{\cdot}4+25{\cdot}19}{2}=27{\cdot}295$ mean barometric height $(=m)$

$\frac{42{\cdot}1\times30\times89{\cdot}1}{27{\cdot}295}=4122{\cdot}854$ feet, the approximated elevation $=\frac{30\text{D}\,(87+\cdot21d)}{m}$.

3. $50^\circ-46^\circ=4^\circ$ diff. of mercurial temperatures $(=\delta)$

$2{\cdot}833\times4=11{\cdot}332\ (=\delta\times2{\cdot}833)$.

Then $4122{\cdot}854-11{\cdot}332=4111{\cdot}522$ feet, the correct elevation.

(Y) If several sets of observations be made at each station, after short intervals of time, and the mean of the separate results be taken, the conclusion will probably be more accurate than that derived from a single observation.

CHAP. V.

OF THE SIGNS OF TRIGONOMETRICAL QUANTITIES, &c.

(Z) In the investigation of some theorems in spherical trigonometry, and in the application of trigonometry to several astronomical and analytical problems, the changes of the signs of the tangents, sines, &c. from affirmative or +, to negative or −, ought to be particularly attended to. They likewise serve to illustrate the use of the negative sign in the application of algebra to geometry.

(A) The tangents, sines, and versed sines are counted from, or have their origin at B, the beginning of the arc (Plate I. fig. 1.) The cosines, secants, and co-secants, originate at the centre C.

(B) The co-tangents and co-versed sines originate at the end of the first quadrant, as at A.

(C) The *sine* increases from nothing at B till it becomes equal to the radius, at the end of the first quadrant AB; from hence it decreases along the second quadrant from A to *b*, and vanishes at the point *b*, shewing the sine of a semicircle to be nothing. After this, the sine will lie on the contrary side of the diameter: therefore being reckoned affirmative before, must now be counted negative. During the 3d quadrant *b*D the sine I*h* increases negatively, till it becomes equal to the radius; after which it decreases negatively, till it arrives at the point B, where it is nothing as before.

(D) The *cosine* is equal to the radius when the arc is nothing; but decreases through the first quadrant BA, at the end of which it is nothing: during the second quadrant it is negative; for the cosine CI will lie in an opposite direction to the cosine CG. Through the third quadrant *b*D the cosine decreases negatively, at D the negation is destroyed; and in the fourth quadrant DB it again becomes affirmative.

(E) The *tangent* at the beginning of the arc is nothing, and increases to infinity during the first quadrant BA, at the point A there is no tangent, for CA produced can never meet BT (L. 31.) In the second quadrant A*b* the tangent B*t* is negative; for the tangents being reckoned from the point B, the tangent B*t* will lie in an opposite direction from BT. During this second quadrant the tangent decreases negatively, from infinity to nothing. In the third quadrant the tangent is again affirmative, and increases from nothing to infinity. In the fourth quadrant it decreases from an infinite negative to nothing, just the same as in the second quadrant.

(F) The *co-tangent* at the beginning B, of the arc, that is when the arc is very small, is infinite affirmative, but in the second quadrant it will fall on the contrary side of AD, and consequently will be negative. In the third quadrant it will again be affirmative, and in the fourth negative, exactly the same as the tangent.

(G) The *secant* at the commencement of the first quadrant, when the arc is nothing, is equal to the radius, and increases affirmatively to the end of the first quadrant BA, where it ceases to exist (M. 32.) In the second quadrant it is negative, for the *revolving radius* falls on the contrary side of AD, and continues so through the third quadrant; but in the fourth quadrant it is again affirmative, or falls on the same side of AD as in the first quadrant.

(H) The *co-secant* will agree with the sine, for the same reason that the secant agrees with the cosine.

(I) The *versed sine* increases from nothing during the first two quadrants, till it becomes equal to the diameter its utmost limit. It decreases for the last two quadrants till it becomes nothing, but being always counted in the same direction from B to *b* is always affirmative.

(K) Hence,	From 0° to 90°	From 90° to 180°	From 180° to 270°	From 270° to 360°
Sine . .	+ . .	+ . .	− . .	−
Cosine .	+ . .	− . .	− . .	+
Tangent	+ . .	− . .	+ . .	−
Cotangent	+ . .	− . .	+ . .	−
Secant .	+ . .	− . .	− . .	+
Cosecant	+ . .	+ . .	− . .	−
Versed sine	+ . .	+ . .	+ . .	+

(L) In analytical enquiries, arcs of all magnitudes*, whether greater or less than 180°, are frequently used; but in trigonometry, every arc or angle made use of must be less than 180°. It has already been shewn that equal arcs have equal sines, tangents, &c.; but if any arc be considered as affirmative, its equal arc in a contrary direction will be negative.

Let BF and BH be two equal arcs (*Plate I. Fig.* 1.); now if the arc BF be considered as positive, the arc BH in a contrary direction will be negative. The arcs BF and BH are equal, and have the same cosine GC, and the same versed sine BG: but the sine HG, lying in a contrary direction to the sine FG, becomes negative, or contrary to the sine FG of the positive arc BF.

It is shown in the XIth proposition following, that if A = any arc, tang. $\text{A} = \frac{\text{sine A}}{\text{cos. A}} \times \text{rad}$; cot. $\text{A} = \frac{\text{cos. A}}{\text{sine A}} \times \text{rad}$; and cosec. $\text{A} = \frac{\text{rad.}^2}{\text{sine A}}$. These quantities will consequently have a different sign for a negative arc, to that which they will have for a positive arc; but as sec. $A = \frac{\text{rad.}^2}{\text{cos.}\,A}$, the secant will have the same sign for a negative arc as it will have for an affirmative arc.

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* For if to any arc BF, there be added one or more circumferences of the circle, they will terminate exactly in the point F, and the augmented arc will have the same positive or negative sine, cosine, &c. with the arc BF. Thus if C denote an entire circumference, or 360°, sine x = sine $(\text{C}+x)$ = sine $(2\text{C}+x)$ = sine $(3\text{C}+x)$, &c. and the same will take place with respect to the cosine, tangent, &c. *Legendre's* Geometry, 6th edit. page 336.

(M) If the side, tangent, cotangent, or cosecant of $A-B$ be found at the conclusion of a problem, and the arc A be less than the arc B, the signs of the sine, tangent, cotangent, or cosecant of $A-B$, will be contrary to those which are given in the foregoing table. It may also be remarked, that in trigonometrical calculations, when a required quantity comes out a sine, the case is frequently ambiguous; because the sine of an arc and the sine of its supplement are equal, and have the same sign. But when the required quantity is expressed by a cosine, tangent, or a cotangent, there is no ambiguity, for a positive cosine, tangent, or cotangent, shows the arc to be less than 90°; and a negative cosine, tangent, or cotangent, denotes an arc between 90° and 180°.

(N) Since the sine of an arc = sine of its supplement, tangent of an arc = tangent of its supplement, &c.; if A represent any arc less than 90°, we shall have

$\text{Sine } (90^\circ + A) = \text{sine } (90^\circ - A) = \text{cos. } A$
$\text{Tang. } (90^\circ + A) = \text{tang. } (90^\circ - A) = \text{cot. } A$
$\text{Sec. } (90^\circ + A) = \text{sec. } (90^\circ - A) = \text{cosec. } A$
$\text{Cos. } (90^\circ + A) = \text{cos. } (90^\circ - A) = \text{sine } A$
$\text{Cot. } (90^\circ + A) = \text{cot. } (90^\circ - A) = \text{tang. } A$
$\text{Cosec. } (90^\circ + A) = \text{cosec. } (90^\circ - A) = \text{sec. } A$

Also if A represent any arc greater than 90°.

$\text{Cos. } A = -\text{cos. } (180^\circ - A)$
$\text{Tang. } A = -\text{tang. } (180^\circ - A)$
$\text{Cot. } A = -\text{cot. } (180^\circ - A)$

(O) The cosine of half an arc is equal to the sine of half the supplement of that arc, or the sine of half an arc is equal to the cosine of half the supplement of that arc, &c. for the tangent and secant.

For if $\frac{1}{2}A$ be substituted for A (N. 101.) we shall have,

$$\text{Cos. } \tfrac{1}{2}A = \text{sine } (90^\circ - \tfrac{1}{2}A) = \text{sine } \frac{180^\circ - A}{2}$$

$$\text{Sine } \tfrac{1}{2}A = \text{cos. } (90^\circ - \tfrac{1}{2}A) = \text{cos. } \frac{180^\circ - A}{2}$$

$$\text{Cot. } \tfrac{1}{2}A = \text{tang. } (90^\circ - \tfrac{1}{2}A) = \text{tang. } \frac{184^\circ - A}{2}$$

$$\text{Tang. } \tfrac{1}{2}A = \text{cot. } (90^\circ - \tfrac{1}{2}A) = \text{cot. } \frac{180^\circ - A}{2}$$

$$\text{Cosec. } \tfrac{1}{2}A = \text{sec. } (90^\circ - \tfrac{1}{2}A) = \text{sec. } \frac{180^\circ - A}{2}$$

$$\text{Sec. } \tfrac{1}{2}A = \text{cosec.}(90^\circ - \tfrac{1}{2}A) = \text{cosec. } \frac{180^\circ - A}{2}$$

And as A, in trigonometry, must always be less than 180°, ½A will always be less than 90°, and consequently its sine, tangent, &c. will have the same sign as in the table (K. 100.)

GENERAL PROPERTIES OF THE SINES, TANGENTS, CHORDS, &c. OF SINGLE ARCS.

PROPOSITION VIII. (*Plate I. Fig.* 1.)

(P) *The chord of any arc is a mean proportional between the diameter and the versed sine of that arc.*

The angle *b*FB being in a semicircle, is a right angle (Euclid 31 *of* III.) And FG, by the definition of a sine, is perpendicular to *b*B, therefore the triangles *b*GF, and BGF, are similar to each other, and to the whole triangle *b*FB, (Euclid 8 *of* VI.) Hence,

*b*B : BF :: BF : BG; but *b*B is the diameter, BF the chord of the arc B*i*F, and BG is the versed sine of the arc B*i*F. Q. E. D.

(Q) COROL. *The sine of any arc is a mean proportional between the versed sine of that arc, and the versed sine of its supplement.*

For the ∠ F*b*G = ∠ BFG, consequently *b*G : GF :: GF : BG. Q. E. D.

PROPOSITION IX. (*Plate I. Fig.* 1.)

(R) *The square of the sine of any arc added to the square of its versed sine, is equal to the square of the chord of that arc; or to four times the square of the sine of half that arc.*

For $GF^2 + GB^2 = BF^2$ (Euclid 47 *of* I.)

Draw C*i* parallel to *b*F, and it will cut BF at right angles in Z; but if it cut it at right angles, it will bisect it (Euclid 3 *of* III.), therefore BZ = ZF, and BF = 2BZ, therefore $BF^2 = 4BZ^2$. Q. E. D.

PROPOSITION X. (*Plate I. Fig.* 1.)

(S) *The tangent of any arc is a mean proportional between the sum and difference of the radius and the secant; viz. rad. + sec. : tang. :: tang. : sec. − rad.*

Join FC, and produce it to *h* and to T, so as to meet the tangent BT of the arc BF. Then $hT \times FT = TB^2$ (Euclid 36 *of* III.) that is *h*T : TB :: TB : FT. (Euclid 17 *of* VI.) But *h*T = rad. + secant CT, and FT = secant CT − rad.; therefore rad. + sec. : tang. :: tang. : sec. − rad. Q. E. D.

PROPOSITION XI. (*Plate I. Fig.* 1.)

(T) *The right angled triangles* FGC, TBC, CAK, CEF, *are equiangular and similar.*

For FGC, TBC are right angles, and the angle FCB = $\angle$ TCB, therefore the remaining angle CFG = $\angle$ CTB.

The triangle CEF = triangle CGF, for the $\angle$ CFG = $\angle$ FCE (Euclid I. and 29), and $\angle$ FEC = $\angle$ CGF, being each of them right angles, and the side CF is common to both the triangles, therefore they are equal (Euclid 26 *of* I.) Again, AK is parallel to EF, by the definition of a tangent and sine; therefore the triangle CAK is equiangular with the triangle CEF; and consequently with CGF and CBT.

In the triangles CGF *and* CBT.

(U) CG : GF :: CB : BT.
cosine : sine :: radius : tangent.

(W) CF : FG :: CT : TB.
radius : sine :: secant : tangent.

(X) CB : CG :: CT : CF.
radius : cosine :: secant : radius.

In the triangle CEF = CGF *and* CKA.

(Y) CF : CE = GF :: CK : CA.
radius : sine :: co-secant : radius.

In the triangles CTB *and* CKA.

(Z) CB : BT :: AK : CA.
radius : tangent :: co-tangent : radius.

(A) AK : EF = CG :: CK : CF.
co-tangent : cosine :: co-secant : radius.

(B) By comparison.

Cosine : sine :: co-tangent : radius. (U and Z, above.)
Secant : tang. :: co-secant : radius. (W and Y, above.)
Secant : rad. :: co-secant : co-tang. (X and A, above.)

(C) *Tangent* × *co-tangent* = *radius square.* (Z, above.)

Therefore the tangent of any arc × its co-tangent = the tangent of any other arc × its co-tangent.

(D) *Sine* × *co-secant* = *radius square.* (Y, above.)

Therefore the sine of any arc × its co-secant = sine of any other arc × its co-secant.

(E) *Co-sine* × *secant* = *radius square.* (X, above.)

Therefore the co-sine of any arc × its secant = co-sine of any other arc × its secant.

(F) *Square radius* = *square sine* + *square co-sine.*

For, $CF^2 = GF^2 + CG^2$ (Euclid 47 *of* I.)

(G) *Square radius = square secant − square tangent = square co-secant − square co-tangent.*

For, $\text{CB}^2 = \text{CT}^2 - \text{TB}^2$ (Euclid 47 *of* I.), and $\text{AC}^2 = \text{CK}^2 - \text{AK}^2$.

(H) *Square radius = co-sine × secant* (X. 103.) = *sine × co-secant* (Y. 103.) = *tangent × co-tangent* (Z. 103.)

(I) $\text{Sine} = \dfrac{\text{tangent} \times \text{co-sine}}{\text{radius}}$ (U.103.) $= \dfrac{\text{tang.}}{\text{secant}} \times \text{rad}$(W.103.) $=$ $\dfrac{\text{square rad.}}{\text{co-sec.}}$ (Y. 103.) $= \dfrac{\text{cosine} \times \text{rad.}}{\text{co-tangent}}$ (B. 103.), &c.

(K) And, generally, if A = any arc*, *rad* = radius, *cos* = co-sine, *tang* = tangent, *cot* = cotangent, *sec* = secant, *cosec* = co-secant, *vers* = versed sine, *vers sup* = versed sine of the supplement, or superversed sine, the following formulæ will be easily deduced, where the sign × is represented by a point (.).

(L) I. $\textit{Sine}\ \text{A} = \sqrt{\text{rad}^2 - \cos^2 \text{A}} = \dfrac{\cos \text{A} \,.\, \text{tang A}}{\text{rad}} = \dfrac{\text{tang A} \,.\, \text{rad}}{\text{sec}}$ $= \dfrac{\text{rad} \,.\, \cos \text{A}}{\cot \text{A}} = \dfrac{\text{rad}^2}{\sqrt{\text{rad}^2 + \cot^2 \text{A}}} = \dfrac{\text{tang A} \,.\, \text{rad}}{\sqrt{\text{rad}^2 + \text{tang}^2\, \text{A}}} = \dfrac{\text{rad}^2}{\text{cosec A}} =$ $\dfrac{\sec \text{A} \,.\, \cos \text{A}}{\text{cosec A}} = \dfrac{\text{tang A} \,.\, \cot \text{A}}{\text{cosec A}} = \dfrac{\text{rad}}{\sec \text{A}} \sqrt{\sec^2 \text{A} - \text{rad}^2} =$ $\sqrt{\text{vers sup A} \,.\, \text{vers A}} = \sqrt{(2\ \text{rad} \,.\, \text{vers A}) - \text{vers}^2\, \text{A}}$.

(M) II. $\textit{Cos}\ \text{A} = \text{rad} - \text{vers A} = \text{vers sup A} - \text{rad} =$ $\sqrt{\text{rad}^2 - \text{sine}^2\, \text{A}} = \dfrac{\text{rad} \,.\, \text{sine A}}{\text{tang A}} = \dfrac{\text{rad}^2}{\sqrt{\text{rad}^2 + \text{tang}^2\, \text{A}}} = \dfrac{\text{rad}^2}{\sec \text{A}} =$ $\dfrac{\text{tang A} \,.\, \cot \text{A}}{\sec \text{A}} = \dfrac{\text{sine A} \,.\, \text{cosec A}}{\sec \text{A}} = \dfrac{\text{sine A} \,.\, \cot \text{A}}{\text{rad}} = \dfrac{\text{rad} \,.\, \cot \text{A}}{\text{cosec A}} =$ $\dfrac{\text{rad} \,.\, \cot \text{A}}{\sqrt{\text{rad}^2 + \cot^2 \text{A}}} = \dfrac{\text{rad}}{\text{cosec A}} \sqrt{\text{cosec}^2\, \text{A} - \text{rad}^2}$.

(N) III. $\textit{Tang}\ \text{A} = \dfrac{\text{rad} \,.\, \text{sine A}}{\cos \text{A}} = \dfrac{\text{rad} \,.\, \text{sine A}}{\sqrt{\text{rad}^2 - \text{sine}^2\, \text{A}}} = \dfrac{\text{rad}}{\cos \text{A}}$ $.\sqrt{\text{rad}^2 - \cos^2 \text{A}} = \dfrac{\text{rad}^2}{\cot \text{A}} = \dfrac{\text{rad}^2}{\sqrt{\text{cosec}^2\, \text{A} - \text{rad}^2}} = \dfrac{\cos \text{A} \,.\, \sec \text{A}}{\cot \text{A}} =$ $\sqrt{\sec^2 \text{A} - \text{rad}^2} = \dfrac{\text{sine A} \,.\, \text{cosec A}}{\cot \text{A}} = \dfrac{\text{rad} \,.\, \sec \text{A}}{\text{cosec A}} = \dfrac{\text{rad}^3 \,.\, \cos \text{A}}{\text{sine A} \,.\, \cot^2 \text{A}} =$ $\dfrac{\text{rad} \sqrt{(2\text{rad} \,.\, \text{vers A}) - \text{vers}^2 \text{A}}}{\text{rad} - \text{vers A}} = \dfrac{\text{rad} \sqrt{(2\text{rad} \,.\, \text{vers sup A}) - \text{vers sup}^2 \text{A}}}{\text{vers sup A} - \text{rad}}$.

(O) IV. $\textit{Cot}\ \text{A} = \sqrt{\text{cosec}^2\, \text{A} - \text{rad}^2} = \dfrac{\text{rad}^2}{\text{tang A}} = \dfrac{\text{sine A} \,.\, \text{cosec A}}{\text{tang A}}$

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* Emerson's Trigonometry, 2d edit. Prop. I. Scholium.

$$=\frac{\text{cos A.sec A}}{\text{tang A}}=\frac{\text{rad.cosec A}}{\text{sec A}}=\frac{\text{rad . cos A}}{\text{sine A}}=\frac{\text{rad}}{\text{sine A}}\sqrt{\text{rad}^2-\text{sine}^2\text{A}}$$

$$=\frac{\text{rad . cos A}}{\sqrt{\text{rad}^2-\text{cos}^2\text{ A}}}=\frac{\text{rad}^2}{\sqrt{\text{sec}^2\text{ A}-\text{rad}^2}}=\frac{\text{rad}^3\text{ . sine A}}{\text{cos A . tang}^2\text{ A}}=$$

$$\frac{(\text{rad}-\text{vers A})\text{ . rad}}{\sqrt{(2\text{ rad . vers A})-\text{vers}^2\text{A}}}=\frac{(\text{vers sup A}-\text{rad})\text{ . rad}}{\sqrt{(2\text{rad . vers sup A})-\text{vers sup}^2\text{ A}}}.$$

(P) V. *Sec* A $=\sqrt{\text{rad}^2+\text{tang}^2\text{ A}}=\dfrac{\text{rad}^2}{\text{cos A}}=\dfrac{\text{rad , tang A}}{\text{sine A .}}=$

$$\frac{\text{rad}^3}{\text{sine A . cot A}}=\frac{\text{rad . cosec A}}{\text{cot A}}=\frac{\text{tang A . cot A}}{\text{cos A}}=\frac{\text{sine A . cosec A}}{\text{cos A}}=$$

$$\frac{\text{rad}^2}{\sqrt{\text{rad}^2-\text{sine}^2\text{ A}}}=\frac{\text{rad}}{\text{cot A}}\sqrt{\text{rad}^2+\text{cot}^2\text{ A}}=\frac{\text{rad . cosec A}}{\sqrt{\text{cosec}^2\text{ A}-\text{rad}^2}}=$$

$$\frac{\text{tang A . cosec A}}{\text{rad}}=\frac{\text{rad}^2}{\text{rad}-\text{vers A}}=\frac{\text{rad}^2}{\text{vers sup A}-\text{rad}}.$$

(Q) VI. *Cosec* A $=\sqrt{\text{rad}^2+\text{cot}^2\text{A}}=\dfrac{\text{rad}^2}{\text{sine}}=\dfrac{\text{rad . sec A}}{\text{tang A}}=$

$$\frac{\text{rad . cot A}}{\text{cos A}}=\frac{\text{rad}^3}{\text{cos A . tang A}}=\frac{\text{tang A . cot A}}{\text{sine A}}=\frac{\text{cos A . sec A}}{\text{sine A}}=$$

$$\frac{\text{rad}^2}{\sqrt{\text{rad}^2-\text{cos}^2\text{ A}}}=\frac{\text{rad}}{\text{tang A}}\sqrt{\text{rad}^2+\text{tang}^2\text{ A}}=\frac{\text{rad . sec A}}{\sqrt{\text{sec}^2\text{ A}-\text{rad}^2}}=$$

$$\frac{\text{cot A . sec A}}{\text{rad}}=\frac{\text{rad}^2}{\sqrt{2\text{rad . vers A}-\text{vers}^2\text{A}}}.$$

(R) VII. *Vers* A $=2\text{ rad}-\text{vers sup A}=\text{rad}-\text{cos A}=\text{rad}-$

$$\sqrt{\text{rad}^2-\text{sine}^2\text{A}}=\text{rad}-\frac{\text{rad}^2}{\sqrt{\text{rad}^2+\text{tang}^2\text{A}}}=\text{rad}-\frac{\text{rad . cot A}}{\sqrt{\text{rad}^2+\text{cot}^2\text{ A}}}$$

$$=\text{rad}-\frac{\text{rad}^2}{\text{sec A}}=\text{rad}-\frac{\text{rad}}{\text{cosec A}}\sqrt{\text{cosec}^2\text{ A}-\text{rad}^2}.$$

(S) VIII. *Vers sup* A $=2\text{ rad}-\text{vers A}=\text{rad}+$

$$\sqrt{\text{rad}^2-\text{sine}^2\text{ A}}=\text{rad}+\text{cos A}=\text{rad}+\frac{\text{rad}^2}{\sqrt{\text{rad}^2+\text{tang}^2\text{ A}}}=\text{rad}+$$

$$\frac{\text{rad . cot A}}{\sqrt{\text{rad}^2+\text{cot}^2\text{A}}}=\text{rad}+\frac{\text{rad}}{\text{cosec A}}\sqrt{\text{cosec}^2\text{ A}-\text{rad}^2}=\frac{\text{sine A}+\text{tang A}}{\text{tang A}}$$

$$\text{rad}=\text{rad}+\frac{\text{sine A . cot A}}{\text{rad}}=\text{rad}+\frac{\text{rad}^2}{\text{sec A}}=\frac{\text{cot A}+\text{cosec A}}{\text{cosec A}}.\text{rad}.$$

(T) If the co-versed sine be wanted, it may be found by subtracting the sine from radius; that is, co-vers A $=\text{rad}-\text{sine A}$; also the chord $=\sqrt{\text{vers}^2\text{ A}+\text{sine}^2\text{ A}}=\sqrt{\overline{\text{rad}-\text{cos A}}|^2+(\text{rad}^2-\text{cos}^2\text{A})}=\sqrt{2\text{rad}\cdot(\text{rad}-\text{cos A})}$.

(U) Besides the preceding formulæ, others may be deduced, thus, $\text{sine}^2 \text{A} + \cos^2 \text{A} = \text{rad}^2$ (F. 103) $\text{rad}^2 + \text{tang}^2 \text{A} = \sec^2 \text{A}$, and $\text{rad}^2 + \cot^2 \text{A} = \text{cosec}^2 \text{A}$ (G. 104.) Now because $\text{tang A} = \frac{\text{rad . sine A}}{\cos \text{A}}$ (N. 104.) $\text{tang}^2 \text{A} = \frac{\text{rad}^2 . \text{sine}^2 \text{A}}{\cos^2 \text{A}}$, hence $\text{rad}^2 + \text{tang}^2 \text{A}$ $= \text{rad}^2 + \frac{\text{rad}^2 . \text{sine}^2 \text{A}}{\cos^2 \text{A}} = \frac{\text{rad}^2 . (\cos^2 \text{A} + \text{sine}^2 \text{A})}{\cos^2 \text{A}} = \frac{\text{rad}^4}{\cos^2 \text{A}}$. &c.

GENERAL PROPERTIES OF SINES, TANGENTS, &c. OF DOUBLE ARCS AND OF HALF ARCS.

PROPOSITION XII. (*Plate I. Fig.* 1.)

(W) *The right-angled triangles* *b*GF, BGF, *b*FB, *and* CZB, *are equiangular and similar; and* CZ, *the co-sine of the arc* B*i*, *is equal to the half of* *b*F, *the chord of the supplement of double the arc* B*i*.

For *b*GF, BGF, and *b*FB have already been shewn to be equiangular (P. 102.), and the triangles CZB, FGB, have the angle at B common to both of them; also the angle *b*FB, being an angle in a semicircle, is a right angle; and since CZ is parallel to *b*F, by construction (R. 102.), the angle CZB is likewise a right angle.

Now BZ : CZ :: BF : F*b*, but BZ=½BF (R. 102.), therefore CZ=½*b*F.

Hence the following proportions.

(X) CB : BZ (sine arc B*i*) :: BF (=2BZ) : BG
radius : sine of an arc :: double that sine : versed sine of double that arc.

(Y) CB : BZ :: *b*F (=2CZ) : GF
radius : sine of an arc :: double its cosine : sine double the arc.

(Z) CB : CZ :: *b*F (=2CZ) : *b*G
radius : cosine of an arc :: double its cosine : versed sine supplement of double the arc.

(A) CB : CZ :: BF (=2BZ) : GF
radius : cosine of an arc :: double the sine : sine of double the arc.

(B) BF (=2BZ) : BG :: *b*F (=2CZ) : GF
double the sine of an arc : versed sine double arc :: double cosine : sine of double the arc.

(C) BF (=2BZ) : GF :: *b*F (=2CZ) : *b*G
double the sine of an arc : sine double the arc :: double the cosine : versed sine of the supplement of double the arc.

(D) BZ : CZ :: GB : GF
sine of an arc : its cosine :: versed sine double arc : sine double arc.

(E) $\left\{\begin{array}{l} BZ : CZ :: GF : bG \\ \text{sine of an arc : its cosine :: sine double arc : versed} \\ \text{sine supplement of double arc.} \end{array}\right.$

(F) $\left\{\begin{array}{l} BG : GF :: GF : bG \\ \text{versed sine double arc : sine double arc :: sine double} \\ \text{arc : versed sine of the supplement of the double arc.} \end{array}\right.$

(G) By comparing these proportions, with the proportions Prop. XI. and casting out the equal terms, various other proportions may be formed. Thus (W. 103.) it is said

radius : sine :: secant : tangent, and (X. 106.) we have

radius : sine :: double sine : versed sine double arc, therefore secant : tangent :: double the sine : versed sine double the arc, &c. for others.

(H) COROL. I. *The rectangle of the radius and the sine of any arc, is equal to double the rectangle of the sine and cosine of half that arc.*

For (A. 106) $CB : CZ :: 2BZ : GF$

Therefore $CB \times GF = 2BZ \times CZ$.

(I) COROL. II. *The rectangle of radius and half the versed sine of an arc, is equal to the square of the sine of half that arc.*

For (X 106.) $CB : BZ :: BF : BG$

And $CB : BZ :: \frac{1}{2}BF : \frac{1}{2}BG$

That is $CB : BZ :: BZ : \frac{1}{2}BG$

Therefore $CB \times \frac{1}{2}BG = BZ^2$; now CB is the radius, BG the versed sine of BF, and BZ is the sine of B*i*, which is half the arc BF.

(K) COROL. III. *The rectangle of radius and half the versed sine of the supplement of an arc, is equal to the square of the cosine of half that arc.*

For (Z. 106) $CB : CZ :: bF : bG$

$CB : CZ :: \frac{1}{2}bF : \frac{1}{2}bG$

$CB : CZ :: CZ : \frac{1}{2}bG$

Therefore $CB \times \frac{1}{2}bG = CZ^2$. Where CB is the radius, *b*G the versed sine of the supplement of the arc BF, and CZ the cosine of half the arc BF = cosine of the arc B*i*.

(L) COROL. IV. *The diameter of the circle is to the versed sine of any arc, as the square of the radius is to the square of the sine of half that arc.*

For, (I. 107.) $CB \times \frac{1}{2}BG = BZ^2$, multiply by the radius CB,

then $CB^2 \times \frac{1}{2}BG = CB \times BZ^2$

therefore $CB : \frac{1}{2}BG :: CB^2 : BZ^2$

that is $2CB : BG :: CB^2 : BZ^2$

(M) COROL. V. *The rectangle of the sine of any arc, and of the cotangent of its half, is equal to double the square of the co-*

sine of half that arc. And the same is true, writing tangent and sine, for cotangent and cosine.

For, if A = any arc, sine of A $=\frac{\cos \text{A} . \text{rad}}{\cot \text{A}}$ (L.104.) and cos A $=\frac{\text{rad} . \text{sine A}}{\text{tang A}}$ (M. 104.)

And rad : cos A :: 2 sine A : sine 2A ; (A. 106.), hence rad : cos A :: $\frac{2 \cos \text{A} . \text{rad}}{\cot \text{A}}$: sine 2A ; therefore

sine 2A $=\frac{2 \cos^2 \text{A}}{\cot \text{A}}$; or sine 2A . cot A = 2cos² A, that is sine A . cot ½A = 2 cos² ½A.

Again, by substituting the value of the cosine of A in the second term of the first proportion, we shall obtain sine 2A . tang A = 2 sine² A, that is sine A . tang ½A = 2 sine² ½ A.

(N) Using the same notation as in K. 104. the following formulæ may be easily obtained, by simple algebraical reductions.*

$$\text{(O) I. } \textit{Sine } 2\text{A} = \frac{2 \cos \text{A} . \text{sine A}}{\text{rad}} = \frac{2 \text{ sine}^2 \text{ A}}{\text{tang A}} = \frac{2 \cos^2 \text{A}}{\cot \text{A}} = \frac{2 \text{ rad} . \text{sine A}}{\sec \text{A}} = \frac{2 \text{ rad} . \cos \text{A}}{\text{cosec A}} = \frac{2 \text{ rad}^2 . \text{tang A}}{\text{rad}^2 + \text{tang}^2 \text{ A}} = \frac{2 \text{ rad}^2 . \text{tang A}}{\sec^2 \text{A}} = \frac{2 \text{ rad}^2}{\cot \text{A} + \text{tang A}} = \frac{2 \text{ rad}^2 . \cot \text{A}}{\text{rad}^2 + \cot^2 \text{A}} = \frac{2 \text{ rad}^2 . \cot \text{A}}{\text{cosec}^2 \text{ A}}.$$

$$\text{(P) II. } \textit{Cos } 2\text{A} = \frac{\cos^2 \text{A} - \text{sine}^2 \text{ A}}{\text{rad}} = \frac{\text{rad}^2 - 2 \text{ sine}^2 \text{ A}}{\text{rad}} = \frac{2 \cos^2 \text{A} - \text{rad}^2}{\text{rad}} = \frac{\text{rad}^2 - \text{tang}^2 \text{ A}}{\text{rad}^2 + \text{tang}^2 \text{ A}} . \text{rad} = \frac{\cot \text{A} - \text{tang A}}{\cot \text{A} + \text{tang A}} . \text{rad} = \frac{\cot^2 \text{A} - \text{rad}^2}{\cot^2 \text{A} + \text{rad}^2} . \text{rad} = \frac{2\text{rad}^2 - \sec^2 \text{A}}{\sec^2 \text{A}} . \text{rad} = \frac{2\cos \text{A} - \sec \text{A}}{\sec \text{A}} . \text{rad} = \frac{\text{cosec}^2 \text{ A} - 2 \text{ rad}^2}{\text{cosec}^2 \text{ A}} . \text{rad} = \frac{\text{cosec A} - 2 \text{ sine A}}{\text{cosec A}} . \text{rad}.$$

$$\text{(Q) III. } \textit{Tang } 2\text{A} = \frac{2 \text{ rad}^2 . \text{tang A}}{\text{rad}^2 - \text{tang}^2 \text{ A}} = \frac{2 \text{ rad}^2}{\cot \text{A} - \text{tang A}} = \frac{2 \text{ rad} . \cos \text{A} . \text{sine A}}{\text{rad}^2 - 2 \text{ sine}^2 \text{A}} = \frac{2 \text{ rad} . \cos \text{A} . \text{sine A}}{2 \cos^2 \text{A} - \text{rad}^2} = \frac{2 \text{ rad}^2 . \cot \text{A}}{\cot^2 \text{A} - \text{rad}^2} = \frac{2 \text{ rad}^2 . \text{tang A}}{2 \text{ rad}^2 - \sec^2 \text{A}} = \frac{2 \text{ rad}^2 . \cot \text{A}}{\text{cosec}^2 \text{ A} - 2 \text{ rad}^2}$$

$$\text{(R) IV. } \textit{Cot } 2 \text{ A} = \frac{\text{rad}^2 - \text{tang}^2 \text{ A}}{2 \text{ tang A}} = \frac{\cot^2 \text{A} - \text{rad}^2}{2 \cot \text{A}} =$$

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* Emerson's Trigonometry, 2d edit. Prop. II. Scholium.

$$\frac{\mathrm{cot}\ A-\mathrm{tang}\ A}{2}=\frac{\mathrm{rad}^2-2\ \mathrm{sine}^2\ A}{2\ \mathrm{cos}\ A\ .\ \mathrm{sine}\ A}\ .\ \mathrm{rad}=\frac{2\ \mathrm{cos}^2\ A-\mathrm{rad}^2}{2\ \mathrm{cos}\ A\ .\ \mathrm{sine}\ A}\ .\ \mathrm{rad}=$$

$$\frac{2\ \mathrm{rad}^2-\mathrm{sec}^2\ A}{2\ \mathrm{tang}\ A}=\frac{\mathrm{cosec}^2\ A-2\ \mathrm{rad}^2}{2\ \mathrm{cot}\ A}.$$

(S) V. *Sec* $2\ A=\dfrac{\mathrm{rad}\ .\ \mathrm{sec}^2\ A}{2\ \mathrm{rad}^2-\mathrm{sec}^2\ A}=\dfrac{\mathrm{sec}\ A\ .\ \mathrm{rad}}{2\ \mathrm{cos}\ A-\mathrm{sec}\ A}=$

$$\frac{\mathrm{rad}^3}{2\ \mathrm{cos}^2\ A-\mathrm{rad}^2}=\frac{\mathrm{rad}^3}{\mathrm{rad}^2-2\ \mathrm{sine}^2\ A}=\frac{\mathrm{rad}^2+\mathrm{tang}^2\ A}{\mathrm{rad}^2-\mathrm{tang}^2\ A}.\ \mathrm{rad}=$$

$$\frac{\mathrm{cot}\ A+\mathrm{tang}\ A}{\mathrm{cot}\ A-\mathrm{tang}\ A}.\ \mathrm{rad}=\frac{\mathrm{cot}^2\ A+\mathrm{rad}^2}{\mathrm{cot}^2\ A-\mathrm{rad}^2}\ .\ \mathrm{rad}=\frac{\mathrm{rad}\ .\ \mathrm{cosec}^2\ A}{\mathrm{cosec}^2\ A-2\ \mathrm{rad}^2}.$$

(T) VI. *Cosec* $2\ A=\dfrac{\mathrm{sec}\ A\ .\ \mathrm{cosec}\ A}{2\ \mathrm{rad}}=\dfrac{\mathrm{rad}\ .\ \mathrm{sec}\ A}{2\ \mathrm{sine}\ A}=$

$$\frac{\mathrm{rad}\ .\ \mathrm{cosec}\ A}{2\ \mathrm{cos}\ A}=\frac{\mathrm{rad}^2}{2\ \mathrm{cos}\ A\ .\ \mathrm{sine}\ A}=\frac{\mathrm{rad}^2+\mathrm{tang}^2\ A}{2\ \mathrm{tang}\ A}=\frac{\mathrm{sec}^2\ A}{2\ \mathrm{tang}\ A}=$$

$$\frac{\mathrm{cot}\ A+\mathrm{tang}\ A}{2}=\frac{\mathrm{rad}^2+\mathrm{cot}^2\ A}{2\ \mathrm{cot}\ A}=\frac{\mathrm{sec}^2\ A}{2\ \mathrm{tang}\ A}=\frac{\mathrm{cosec}^2\ A}{2\ \mathrm{cot}\ A}.$$

(U) VII. *Vers* $2\ A=\dfrac{2\ \mathrm{sine}^2\ A}{\mathrm{rad}}=\dfrac{2\ \mathrm{rad}^2-2\ \mathrm{cos}^2\ A}{\mathrm{rad}}=$

$$\frac{2\ \mathrm{vers}\ A.\ \mathrm{vers\ sup}\ A}{\mathrm{rad}}=\frac{2\mathrm{vers}\ A\ .\ (2\mathrm{rad}-\mathrm{vers}\ A)}{\mathrm{rad}}=\frac{2\mathrm{rad}\ .\ \mathrm{tang}^2\ A}{\mathrm{rad}^2+\mathrm{tang}^2 A}=$$

$$\frac{2\ \mathrm{rad}\ .\ \mathrm{tang}^2\ A}{\mathrm{sec}^2\ A}=\frac{2\ \mathrm{rad}\ .\ \mathrm{tang}\ A}{\mathrm{cot}\ A+\mathrm{tang}\ A}=\frac{2\ \mathrm{rad}^3}{\mathrm{rad}^2+\mathrm{cot}^2\ A}=\frac{2\ \mathrm{rad}^3}{\mathrm{cosec}^2\ A}=$$

$$\frac{2\ \mathrm{rad}\ .\ \mathrm{sine}\ A}{\mathrm{cosec}\ A}=\frac{\mathrm{sec}^2\ A-\mathrm{rad}^2}{\mathrm{sec}^2\ A}.\ 2\ \mathrm{rad}=\frac{\mathrm{sec}\ A-\mathrm{cos}\ A}{\mathrm{sec}\ A}.\ \mathrm{rad}.$$

(W) VIII. *Vers sup* $2\ A=\dfrac{2\ \mathrm{rad}^2-2\ \mathrm{sine}^2\ A}{\mathrm{rad}}=\dfrac{2\ \mathrm{cos}^2\ A}{\mathrm{rad}}=$

$$\frac{2\ \mathrm{rad}^2}{\mathrm{rad}^2+\mathrm{tang}^2\ A}.\ \mathrm{rad}=\frac{2\ \mathrm{cot}^2\ A}{\mathrm{rad}^2+\mathrm{cot}^2\ A}.\ \mathrm{rad}=\frac{2\ \mathrm{rad}\ .\ \mathrm{cot}\ A}{\mathrm{tang}\ A+\mathrm{cot}\ A}=\frac{2\mathrm{rad}^3}{\mathrm{sec}^2\ A}$$

$$=\frac{2\mathrm{cos}A.\mathrm{rad}}{\mathrm{sec}\ A}=\frac{2\ \mathrm{cosec}^2 A-2\ \mathrm{rad}^2}{\mathrm{cosec}^2\ A}\ .\ \mathrm{rad}=\frac{2\mathrm{cosec}\ A-2\mathrm{sine}A}{\mathrm{cosec}\ A}\ .\ \mathrm{rad}$$

$$=\frac{2\left(\mathrm{verse\ sup}\ A-\mathrm{rad}\right)^2}{\mathrm{rad}}=\frac{2\left(\mathrm{rad}-\mathrm{vers}\ A\right)^2}{\mathrm{rad}}.$$

(X) The coversed sine may be expressed in terms of the rest by substituting its value in any of the above forms.

Covers $2\ A=\mathrm{rad}-\mathrm{sine}\ 2\ A=\mathrm{rad}-\dfrac{2\ \mathrm{cos}\ A\ .\ \mathrm{sine}\ A}{\mathrm{rad}}=$ $\dfrac{\mathrm{rad}^2-(2\ \mathrm{cos}\ A\ \mathrm{sine}\ .\ A)}{\mathrm{rad}}$, and the chord of $2\ A=2\ \mathrm{sine}\ A$.

(Y) Also if $\frac{1}{2}A$ be substituted for A in each of the foregoing expressions, we shall have,

(Z) I. *Sine* $\text{A}=\frac{2\cos\frac{1}{2}\text{A}\,.\,\text{sine}\,\frac{1}{2}\text{A}}{\text{rad}}=\frac{2\,\text{sine}^2\,\frac{1}{2}\text{A}}{\text{tang}\,\frac{1}{2}\text{A}}$, &c. exactly as in O. 108.

(A) II. *Cos* $\text{A}=\frac{\cos^2\frac{1}{2}\text{A}-\text{sine}^2\,\frac{1}{2}\text{A}}{\text{rad}}=\frac{\text{rad}^2-2\,\text{sine}^2\,\frac{1}{2}\text{A}}{\text{rad}}$, &c. (*see* P. 108.)

(B) III. *Tang* $\text{A}=\frac{2\,\text{rad}^2\,.\,\text{tang}\,\frac{1}{2}\text{A}}{\text{rad}^2-\text{tang}^2\,\frac{1}{2}\text{A}}=\frac{2\,\text{rad}^2}{\cot\frac{1}{2}\text{A}-\text{tang}\,\frac{1}{2}\text{A}}$, &c. (*see* Q. 108.)

(C) IV. *Cot* $\text{A}=\frac{\text{rad}^2-\text{tang}^2\,\frac{1}{2}\text{A}}{2\,\text{tang}\,\frac{1}{2}\text{A}}=\frac{\cot^2\frac{1}{2}\text{A}-\text{rad}^2}{2\cot\frac{1}{2}\text{A}}$, &c. (*see* R. 108.)

(D) V. *Sec* $\text{A}=\frac{\text{rad}\,.\,\sec^2\frac{1}{2}\text{A}}{2\,\text{rad}^2-\sec^2\frac{1}{2}\text{A}}=\frac{\sec\frac{1}{2}\text{A}\,.\,\text{rad}}{2\cos\frac{1}{2}\text{A}-\sec\frac{1}{2}\text{A}}$, &c. (*see* S. 109.)

(E) VI. *Cosec* $\text{A}=\frac{\sec\frac{1}{2}\text{A}\,.\,\text{cosec}\,\frac{1}{2}\text{A}}{2\,\text{rad}}=\frac{\text{rad}\,.\,\sec\frac{1}{2}\text{A}}{2\,\text{sine}\,\frac{1}{2}\text{A}}$, &c. (*see* T. 109.)

(F) VII. *Vers* $\text{A}=\frac{2\,\text{sine}^2\,\frac{1}{2}\text{A}}{\text{rad}}=\frac{2\,\text{rad}^2-2\cos^2\frac{1}{2}\text{A}}{\text{rad}}$, &c. (*see* U. 109.)

(G) VIII. *Vers sup* $\text{A}=\frac{2\text{rad}^2-2\,\text{sine}^2\,\frac{1}{2}\text{A}}{\text{rad}}=\frac{2\cos^2\frac{1}{2}\text{A}}{\text{rad}}$, &c. (*see* W. 109.)

(H) By finding the values of sine $\frac{1}{2}$A, cos $\frac{1}{2}$A, tang $\frac{1}{2}$A, &c. from the most convenient of the foregoing equations, the sine, cosine, &c. of the half arc will be obtained in terms of the sine, cosine, &c. of the whole arc, by easy algebraic reductions.

(I) I. *Sine* $\frac{1}{2}\text{A}=\text{rad}\sqrt{\frac{\text{rad}-\cos\text{A}}{2\,\text{rad}}}=\text{rad}\sqrt{\frac{\sec\text{A}-\text{rad}}{2\sec\text{A}}}=\sqrt{\frac{\text{rad}^2-(\text{rad}\,.\,\cos\text{A})}{2}}=\frac{1}{2}\sqrt{\text{rad}^2+(\text{rad}\,.\,\text{sine}\,\text{A})}-\frac{1}{2}\sqrt{\text{rad}^2-\text{rad}\,.\,\text{sine}\,\text{A}}=\sqrt{\frac{\text{rad.vers}\,\text{A}}{2}}=\text{rad}\sqrt{\frac{2\,\text{rad}-\text{vers}\,\text{sup}\,\text{A}}{2\,\text{rad}}}=\frac{1}{2}$ chord A.

(K) II. *Cos* $\frac{1}{2}\text{A}=\text{rad}\sqrt{\frac{\text{rad}+\cos\text{A}}{2\,\text{rad}}}=\text{rad}\sqrt{\frac{\sec\text{A}+\text{rad}}{2\sec\text{A}}}=\sqrt{\frac{\text{rad}^2+(\text{rad}\,.\,\cos\text{A})}{2}}=\frac{1}{2}\sqrt{\text{rad}^2+(\text{rad}\,.\,\text{sine}\,\text{A})}+\frac{1}{2}\sqrt{\text{rad}^2-(\text{rad}\,.\,\text{sine}\,\text{A})}=\sqrt{\frac{2\text{rad}^2-(\text{rad}\,.\,\text{vers}\,\text{A})}{2}}=\sqrt{\frac{\text{rad}\,.\,\text{vers}\,\text{sup}\,\text{A}}{2}}$.

$$\text{(L) III. } \textit{Tang } \tfrac{1}{2}\text{A} = \text{rad}\sqrt{\frac{\text{rad} - \cos \text{A}}{\text{rad} + \cos \text{A}}} = \text{rad}\sqrt{\frac{\sec \text{A} - \text{rad}}{\sec \text{A} + \text{rad}}}$$

$$= \text{cosec A} - \cot \text{A} = \frac{\text{rad . sine A}}{\text{rad} + \cos \text{A}} = \frac{\text{rad}^2 - (\text{rad . }\cos \text{A})}{\text{sine A}} = \text{rad}\,.$$

$$.\sqrt{\frac{\text{vers A}}{2\text{ rad} - \text{vers A}}} = \text{rad}\sqrt{\frac{2\text{ rad} - \text{vers sup A}}{\text{vers sup A}}}\,.$$

$$\text{(M) IV. } \textit{Cot } \tfrac{1}{2}\text{A} = \text{rad}\sqrt{\frac{\text{rad} + \cos \text{A}}{\text{rad} - \cos \text{A}}} = \text{rad}\sqrt{\frac{\sec \text{A} + \text{rad}}{\sec \text{A} - \text{rad}}}$$

$$= \cot \text{A} + \text{cosec A} = \frac{\text{rad . sine A}}{\text{rad} - \cos \text{A}} = \frac{\text{rad}^2 + (\text{rad . }\cos \text{A})}{\text{sine A}} =$$

$$\sqrt{\frac{2\text{ rad}^2 - (\text{rad vers A})}{\text{vers A}}} = \text{rad}\sqrt{\frac{\text{vers sup A}}{2\text{ rad} - \text{vers sup A}}}\,.$$

$$\text{(N) V. } \textit{Sec}\tfrac{1}{2}\text{A} = \text{rad}\sqrt{\frac{2\text{ rad}}{\text{rad} + \cos \text{A}}} = \text{rad}\sqrt{\frac{2\sec \text{A}}{\text{rad} + \sec \text{A}}} = \text{rad}$$

$$.\sqrt{\frac{2\text{ tang A}}{\text{sine A} + \text{tang A}}} = \text{rad}\sqrt{\frac{2\text{ rad}}{2\text{ rad} - \text{vers A}}} = \text{rad}\sqrt{\frac{2\text{ rad}}{\text{vers sup A}}}\,.$$

$$\text{(O) VI. } \textit{Cosec}\tfrac{1}{2}\text{A} = \text{rad}\sqrt{\frac{2\text{ rad}}{\text{rad} - \cos \text{A}}} = \text{rad}\sqrt{\frac{2\sec \text{A}}{\sec \text{A} - \text{rad}}}$$

$$= \text{rad}\sqrt{\frac{2\text{ tang A}}{\text{tang A} - \text{sine A}}} = \text{rad}\sqrt{\frac{2\text{ rad}}{\text{vers A}}} = \text{rad}\sqrt{\frac{2\text{ rad}}{2\text{rad} - \text{vers sup A}}}\,.$$

And in the same manner the versed sines, coversed sines, chords, &c. of the half arcs may be found.

GENERAL PROPERTIES OF SINES, TANGENTS, &c. OF THE SUMS, AND OF THE DIFFERENCES OF ARCS.

PROPOSITION XIII. *(Plate I. Fig. 2.)*

(P) *The sum of the sines of two arcs is to their difference, as the tangent of half the sum of those arcs is to the tangent of half their difference.*

Let BA and BO be the two arcs; draw the diameter BX, and OD and AG perpendicular to it. Produce OD to meet the circumference in F, and draw FHN parallel to the diameter; join AO and produce it to N, draw C*n*E perpendicular to AO, and at E draw EIK perpendicular to C*n*E, meeting CO and CB (produced) in I and K.

Because OF is perpendicular to BC it is bisected in D (Euclid 3 *of* III.): hence PH is bisected in G; therefore AH is th of the sines AG and OD, and AP their difference. The and OF are bisected in E and B (Euclid 30 *of* III.), then is the sum of the arcs BO and BA, and AO is their difference; BE is the half sum, and OE the half difference; but EK and EI are the tangents of the arcs BE and OE.

PH being bisected in G, ON will be bisected in M; now since AO is bisected in *n*, and ON in M, *n*M will be the half of AN, therefore

AN : AO :: *n*M : *no* and
AN : AO :: AH : AP per similar triangles.
Hence *n*M : *no* :: AH : AP
but *n*M : *no* :: EK : EI per similar triangles;
therefore AH : AP :: EK : EI
viz. sine AB + sine OB : sine AB − sine OB ::
Tang $\frac{1}{2}$ (BA + BO) : tang $\frac{1}{2}$ (BA − BO). Q.E.D.

(Q) Hence, if the two arcs BA and BO be represented by A and B, we shall have, sine A + sine B : sine A − sine B :: tang $\frac{1}{2}$ (A + B) : tang $\frac{1}{2}$ (A − B),

or, $$\frac{\text{sine A} + \text{sine B}}{\text{sine A} - \text{sine B}} = \frac{\text{tang } \frac{1}{2} (\text{A} + \text{B})}{\text{tang } \frac{1}{2} (\text{A} - \text{B})}.$$

PROPOSITION XIV. (*Plate I. Fig.* 2.)

(R) *The sum of the cosines of two arcs, is to their difference; as the co-tangent of half the sum of those arcs is to the tangent of half their difference.*

Let BA and BO be the two arcs, as in the preceding proposition, BE their half sum, and OE their half difference.

Draw *n*Q parallel to AG, then because AO is bisected in *n*, GD (= PO) will be bisected in Q, viz. GQ = QD.

Hence 2CQ = CD + CG the sum of the cosines
and 2GQ = CD − CG the difference of the cosines.
By similar triangles
nt (= CQ) : *ni* (= GQ) :: *nr* : *n*A
or 2CQ : 2GQ :: *nr* : *n*A
But *nr* : *n*A :: ES : ET

therefore 2CQ : 2GQ :: ES : ET, where ES is the tangent of the complement of the arc BE, or its cotangent; and ET = EI, is the tangent of the arc OE.

Consequently CD + CG : CD − CG :: cot arc BE : tang arc OE. Q. E. D.

(S) Let the two arcs BA and BO be represented by A and B;

then $$\frac{\cos \text{A} + \cos \text{B}}{\cos \text{B} - \cos \text{A}} = \frac{\cot \frac{1}{2} (\text{A} + \text{B})}{\text{tang } \frac{1}{2} (\text{A} - \text{B})}.$$

PROPOSITION XV. (*Plate I. Fig.* 2.)

(T) *The sum of the tangents of two arcs, is to their difference* *as the sine of the sum of the arcs is to the sine of their difference.*

The same construction remaining as in Props. XIII. and XIV.
Let BE and OE be the two arcs.
Because AO is bisected in *n*, IT will be bisected in E,
Wherefore TK is equal to the sum of the tangents,
EI and EK, and IK is their difference.
And since the arc AO is bisected in E, the arc
BA is equal to the sum of the arcs BE and OE, and
BO is equal to their difference; and because of the parallel
lines TK, AM; TK : IK :: AM : OM, and
AM : OM :: AG : PG = DO, therefore
TK : IK :: AG : OD. Q.E.D.

(U) If the two arcs BE and OE be represented by A and B

$$\text{then}\ \frac{\text{tang A} + \text{tang B}}{\text{tang B} - \text{tang A}} = \frac{\text{sine (A+B)}}{\text{sine (B}-\text{A)}}.$$

PROPOSITION XVI. (*Plate I. Fig.* 3.)

(W) *The difference between the rectangle of the sines, and the rectangle of the cosines, of two arcs, is equal to the rectangle of the radius and the cosine of the sum of these arcs.*

Let BO and OA be the two arcs, OD and A*n* their sines, CD and C*n* their cosines; then AG will be the sine of the sum of the arcs, and CG will be the cosine.

Draw *ni* parallel to CB, and *n*Q parallel to OD; then the triangles COD, C*n*Q, and A*ni*, are equiangular and similar.

The ∠A*n*C = ∠*in*Q, being each of them a right angle, and if the common ∠*in*C be taken away, there will remain the ∠A*ni* = ∠C*n*Q.

Then if the arc BO be represented by A, and the arc AO by B, we shall have

CO : C*n* :: CD : CQ, *viz.* rad : cos B :: cos A : CQ = $\frac{\text{cos B . cos A}}{\text{rad}}$

CO : OD :: A*n* : *ni* = GQ, *viz.* rad : sine A :: sine B : GQ = $\frac{\text{sine A . sine B}}{\text{rad}}$.

But CQ − GQ = CG the cosine of the arc BA, = cos (A+B)

$$\text{therefore}\ \frac{\text{cos B . cos A}}{\text{rad}} - \frac{\text{sine A . sine B}}{\text{rad}} = \text{cos (A+B)}$$

that is (cos A . cos B) − (sine A . sine B) = cos (A+B) . rad
Q.E.D.

(X) The same construction remaining, we shall have

CO : C*n* :: OD : *n*Q, *viz.* rad : cos B :: sine A : *n*Q = $\frac{\text{cos B . sine A}}{\text{rad}}$

CO : CD :: A*n* : A*i*, *viz.* rad : cos A :: sine B : A*i* $= \frac{\text{cos A . sine B}}{\text{rad}}$

But *n*Q + A*i* = AG the sine of the arc BA, = sine (A + B)

$$\text{therefore } \frac{\text{cos B . sine A}}{\text{rad}} + \frac{\text{cos A . sine B}}{\text{rad}} = \text{sine (A + B)}$$

that is (cos B . sine A) + (cos A . sine B) = sine (A + B) . rad.

PROPOSITION XVII. (*Plate I. Fig.* 3.)

(Y) *The sum of the rectangle of the sines, and the rectangle of the cosines, of two arcs, is equal to the rectangle of the radius, and the cosine of the difference between these arcs.*

Let BO and OA be the two arcs, make OR = AO, then the chord AR will be bisected in *n*, and BR will be the difference between the arcs BO and AO; through R draw RP parallel to GB, and RH parallel to AG, then will AG be the sine of the sum of the arcs BO and AO, and RH the sine of their difference. By similar triangles A*n* : AR :: A*i* : AP, but A*n* is the half of AR, therefore A*i* is the half of AP, that is A*i* = *i*P.

CQ + *ni* = CQ + GQ = CQ + QH = CH. But by the preceding proposition, $\text{CQ + GQ} = \frac{\text{cos B . cos A}}{\text{rad}} + \frac{\text{sine A . sine B}}{\text{rad}}$, therefore

$$\frac{\text{cos B . cos A}}{\text{rad}} + \frac{\text{sine A . sine B}}{\text{rad}} = \text{cos (A} - \text{B)}, \text{ hence}$$

(cos B . cos A) + (sine A . sine B) = cos (A − B) . rad. Q.E.D.

(Z) *n*Q − A*i* = G*i* − P*i* = PG = RH. But *n*Q − A*i* =

$$\frac{\text{cos B . sine A}}{\text{rad}} - \frac{\text{cos A . sine B}}{\text{rad}}; \text{ therefore}$$

(cos B . sine A) − (cos A . sine B) = sine (A − B) . rad.

PROPOSITION XVIII. (*Plate I. Fig.* 3.)

(A) *The square of the radius, is to the square of the radius diminished by the rectangle of the tangents of two arcs; as the tangent of the sum of these arcs, is to the sum of their tangents.*

The same construction remaining, let BO and AO be the arcs. Draw the tangent B*t* and CO and CA to meet it in T and *t*; through A draw AI perpendicular to C*t*, and through T draw ET parallel to AI: when the arc AO is greater than the arc BO, the line ET will fall within the circle, but that will not affect the conclusion of the problem.

The triangles CAI and CET are equiangular and similar; also the triangles *t*BC and *t*ET are equiangular and similar; for the triangles CBT and *t*ET are right angled at B and E, and have the angle at *t* common.

$tT\ (=tB-TB) : tE\ (=tC-EC) :: tC : tB.$

Hence $tB^2-\overline{tB\times TB}=tC^2-\overline{tC\times EC}$

or, $tC\times EC=tC^2-tB^2+\overline{tB\times TB}$

but $tC^2-tB^2=CB^2$ (Euclid 47 *of* I.)

therefore $tC\times EC=CB^2+\overline{tB\times TB}$

By similar triangles $\left\{\begin{array}{l} tT : ET :: tC : CB \\ ET : AI :: EC : AC=CB \end{array}\right.$

Hence $tT : AI :: tC\times EC : CB^2$

or $tB-TB : AI :: CB^2+\overline{tB\times TB} : CB^2$

therefore $\overline{tB\times CB^2}-\overline{TB\times CB^2}=AI\times CB^2+\overline{AI\times tB\times TB}$

or, $\overline{tB\times CB^2}-\overline{AI\times tB\times TB}=\overline{AI+TB}\times CB^2$

hence $CB^2 : CB^2-\overline{AI\times TB} :: tB : AI+BT.$ Q. E. D.

(B) *The square of the radius, is to the square of the radius increased by the rectangle of the tangents of two arcs; as the tangent of the difference between these arcs, is to the difference between their tangents.*

This is proved in the last proportion but one, by inversion; in the above investigation.

(C) If the two arcs BO and AO be represented by A and B,

$$\text{then}\ \frac{\text{rad}^2}{\text{rad}^2-(\text{tang A.tang B})}=\frac{\text{tang (A+B)}}{\text{tang A}+\text{tang B}}$$

$$\text{and}\ \frac{\text{rad}^2}{\text{rad}^2+(\text{tang A.tang B})}=\frac{\text{tang (A}-\text{B)}}{\text{tang A}-\text{tang B}}.$$

(D) From the propositions already given, a great variety of formulæ for the sums and differences of arcs may be deduced, some of the most important of which will here be pointed out.

By Props. XVI. and XVII. it is shewn,

$$1.\ \text{Sine (A+B)}=\frac{(\text{sine A.cos B})+(\text{cos A.sine B})}{\text{rad}}$$

$$2.\ \text{Cos (A+B)}=\frac{(\text{cos A . cos B})-(\text{sine A.sine B})}{\text{rad}}$$

$$3.\ \text{Sine (A}-\text{B)}=\frac{(\text{sine A . cos B})-(\text{cos A.sine B})}{\text{rad}}$$

$$4.\ \text{Cos (A}-\text{B)}=\frac{(\text{cos A.cos B})+(\text{sine A.sine B})}{\text{rad}}$$

(E) By adding the first and third equations together, and dividing by 2.

$$1.\ \frac{\text{Sine A . cos B}}{\text{rad}}=\tfrac{1}{2}\ \text{sine (A+B)}+\tfrac{1}{2}\ \text{sine (A}-\text{B)}^*$$

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[*] Élémens de Géométrie, par A. M. Legendre, sixième édition, p. 348. et seq.

By subtracting the third equation from the first, and dividing by 2.

$$2.\quad \frac{\cos A\,.\,\text{sine } B}{\text{rad}} = \tfrac{1}{2}\,\text{sine}\,(A+B) - \tfrac{1}{2}\,\text{sine}\,(A-B)$$

By adding the second and third equations together, and dividing by 2.

$$3.\quad \frac{\cos A\,.\,\cos B}{\text{rad}} = \tfrac{1}{2}\cos\,(A-B) + \tfrac{1}{2}\cos\,(A+B)$$

By subtracting the second equation from the third, and dividing by 2.

$$4.\quad \frac{\text{sine } A\,.\,\text{sine } B}{\text{rad}} = \tfrac{1}{2}\cos\,(A-B) - \tfrac{1}{2}\cos\,(A+B)$$

(F) If in the formulæ (E. 115.) there be substituted P for $A+B$, and Q for $A-B$, then $A=\frac{1}{2}(P+Q)$, $B=\frac{1}{2}(P-Q)$, and we shall obtain

$$1.\quad \text{Sine } P \times \text{sine } Q = \frac{2}{\text{rad}}\,.\,\text{sine } \tfrac{1}{2}(P+Q)\,.\,\cos \tfrac{1}{2}(P-Q)$$

$$2.\quad \text{Sine } P - \text{sine } Q = \frac{2}{\text{rad}}\,.\,\text{sine } \tfrac{1}{2}(P-Q)\,.\,\cos \tfrac{1}{2}(P+Q)$$

$$3.\quad \text{Cos } P + \cos Q = \frac{2}{\text{rad}}\,.\,\cos \tfrac{1}{2}(P+Q)\,.\,\cos \tfrac{1}{2}(P-Q)$$

$$4.\quad \text{Cos } Q - \cos P = \frac{2}{\text{rad}}\,.\,\text{sine } \tfrac{1}{2}(P+Q)\,.\,\text{sine } \tfrac{1}{2}(P-Q)$$

(G) And because $\frac{\text{sine } A}{\cos A} = \frac{\text{tang } A}{\text{rad}}$ (U. 103.) $= \frac{\text{rad}}{\cot A}$ (Z. 103.) we shall obtain by division and reduction.

$$1.\quad \frac{\text{sine } P + \text{sine } Q}{\text{sine } P - \text{sine } Q} = \frac{\text{sine } \frac{1}{2}(P+Q).\cos \frac{1}{2}(P-Q)}{\cos \frac{1}{2}(P+Q).\text{sine } \frac{1}{2}(P-Q)} = \frac{\text{tang } \frac{1}{2}(P+Q)}{\text{tang } \frac{1}{2}(P-Q)}$$

$$2.\quad \frac{\text{sine } P + \text{sine } Q}{\cos P + \cos Q} = \frac{\text{sine } \frac{1}{2}(P+Q)}{\cos \frac{1}{2}(P+Q)} = \frac{\text{tang } \frac{1}{2}(P+Q)}{\text{rad}}$$

$$3.\quad \frac{\text{sine } P + \text{sine } Q}{\cos Q + \cos P} = \frac{\cos \frac{1}{2}(P-Q)}{\text{sine } \frac{1}{2}(P-Q)} = \frac{\cot \frac{1}{2}(P-Q)}{\text{rad}}$$

$$4.\quad \frac{\text{sine } P - \text{sine } Q}{\cos P + \cos Q} = \frac{\text{sine } \frac{1}{2}(P-Q)}{\cos \frac{1}{2}(P-Q)} = \frac{\text{tang } \frac{1}{2}(P-Q)}{\text{rad}}$$

$$5.\quad \frac{\text{sine } P - \text{sine } Q}{\cos Q - \cos P} = \frac{\cot \frac{1}{2}(P+Q)}{\text{sine } \frac{1}{2}(P+Q)} = \frac{\cot \frac{1}{2}(P+Q)}{\text{rad}}$$

$$6.\quad \frac{\cos P + \cos Q}{\cos Q - \cos P} = \frac{\cos \frac{1}{2}(P+Q).\cos \frac{1}{2}(P-Q)}{\text{sine } \frac{1}{2}(P+Q).\text{sine } \frac{1}{2}(P-Q)} = \frac{\cot \frac{1}{2}(P+Q)}{\text{tang } \frac{1}{2}(P-Q)}$$

$$7.\quad \frac{\text{sine } (P+Q)}{\text{sine } P + \text{sine } Q} = \frac{2\,\text{sine } \frac{1}{2}(P+Q).\cos \frac{1}{2}(P+Q)}{2\,\text{sine } \frac{1}{2}(P+Q).\cos \frac{1}{2}(P-Q)} = \frac{\cos \frac{1}{2}(P+Q)}{\cos \frac{1}{2}(P-Q)}$$

8. $\frac{\text{sine}\,(\text{P}+\text{Q})}{\text{sine}\,\text{P}-\text{sine}\,\text{Q}}=\frac{2\,\text{sine}\,\frac{1}{2}(\text{P}+\text{Q})\,.\,\cos\frac{1}{2}(\text{P}+\text{Q}}{2\,\text{sine}\,\frac{1}{2}(\text{P}-\text{Q})\,.\,\cos\frac{1}{2}(\text{P}+\text{Q})}=\frac{\text{sine}\,\frac{1}{2}(\text{P}+\text{Q})}{\text{sine}\,\frac{1}{2}(\text{P}-\text{Q})}$

Now if two fractions be equal to each other, the numerator of the one is to its denominator, as the numerator of the other is to its denominator. The first equation is therefore the same as Prop. XIII. (or Q. 112.) The sixth is the same as Prop. XIV. (or S. 112.)

(H) If we suppose $\text{B}=\text{A}$, we shall have, from the third equation, (E. 116.) $\cos^2\text{A}=\frac{1}{2}\text{rad}^2+(\frac{1}{2}\,\text{rad}\,.\,\cos 2\,\text{A})$; and from the fourth equation of the same article.

$$\text{Sine}^2\text{A}=\tfrac{1}{2}\,\text{rad}^2-(\tfrac{1}{2}\,\text{rad}\,.\,\cos 2\,\text{A})$$

(I) But when $\text{B}=\text{A}$, $\text{Q}=0$ (F. 116.) and $\cos 0^\circ=\text{rad}$ (D. 99.)

hence 1. $\text{rad}+\cos\text{P}=\frac{2\cos^2\frac{1}{2}\text{P}}{\text{rad}}$. From the third equation (F. 116.)

2. $\text{rad}-\cos\text{P}=\frac{2\,\text{sine}^2\,\frac{1}{2}\text{P}}{\text{rad}}$. From the fourth equation (F. 116.)

3. $\text{sine}\,\text{P}=\frac{2\,\text{sine}\,\frac{1}{2}\text{P}\,.\,\cos\frac{1}{2}\text{P}.}{\text{rad}}$. From the first equation (F. 116.)

4. $\frac{\text{sine}\,\text{P}}{\text{rad}+\cos\text{P}}=\frac{\text{tang}\,\frac{1}{2}\text{P}}{\text{rad}}$. From the fourth equation (G. 116.) $=\frac{\cot\frac{1}{2}\text{P}}{\text{rad}}$ (Z. 103.)

5. $\frac{\text{sine}\,\text{P}}{\text{rad}-\cos\text{P}}=\frac{\cot\frac{1}{2}\text{P}}{\text{rad}}=\frac{\text{rad}}{\text{tang}\,\frac{1}{2}\text{P}}$. From the third equation (G. 116.)

6. $\frac{\text{rad}+\cos\text{P}}{\text{rad}-\cos\text{P}}=\frac{\cot\frac{1}{2}\text{P}}{\text{rad}^2}=\frac{\text{rad}}{\text{tang}^2\,\frac{1}{2}\text{P}}$. From the sixth equation (G. 116 and Z. 103.)

7. $\frac{\text{rad}-\cos\text{P}}{\text{rad}+\cos\text{P}}=\frac{\text{tang}^2\,\frac{1}{2}\text{P}}{\text{rad}^2}=\frac{\text{rad}^2}{\cot\frac{1}{2}\text{P}}$. From the sixth equation (G. 116.) &c.

(K) Because $\text{tang}\,(\text{A}+\text{B})=\frac{\text{rad}\,.\,\text{sine}\,(\text{A}+\text{B})}{\cos\,(\text{A}+\text{B})}$ (N. 104.) we shall have from the first and second equations (D. 115.) $\text{tang}\,(\text{A}+\text{B})=\frac{(\text{sine}\,\text{A}\,.\,\cos\text{B})+(\text{sine}\,\text{B}\,.\,\cos\text{A})}{(\cos\text{A}\,.\,\cos\text{B})-(\text{sine}\,\text{B}\,.\,\text{sine}\,\text{A})}\,.\,\text{rad}$, and since $\text{sine}\,\text{A}=\frac{\cos\text{A}\,.\,\text{tang}\,\text{A}}{\text{rad}}$, and $\text{sine}\,\text{B}=\frac{\cos\text{B}\,.\,\text{tang}\,\text{B}}{\text{rad}}$ (L. 104.) by substituting these values and dividing the numerator and denominator by $\cos A \,.\, \cos B$, we obtain $\text{tang}\,(A+B)=\frac{\text{tang}\,A+\text{tang}\,B}{\text{rad}^2-(\text{tang}\,A\,.\,\text{tang}\,B)}\,.\,\text{rad}^2$, but this has been already obtained by another process (C. 115.)

$$\text{tang}\,(A-B)=\frac{\text{tang}\,A-\text{tang}\,B}{\text{rad}^2+(\text{tang}\,A.\,\text{tang}\,B)}\,.\,\text{rad}^2 \quad (\text{C. }115.)$$

$$\cot\,(A+B)=\frac{\text{rad}^2-(\text{tang}\,A\,.\,\text{tang}\,B)}{\text{tang}\,A+\text{tang}\,B}$$ (from above, and L. 104.)

$$\cot\,(A+B)=\frac{\text{rad}^2+(\text{tang}\,A\,.\,\text{tang}\,B)}{\text{tang}\,A-\text{tang}\,B}$$ (from above, and L. 104.)

GENERAL PROPERTIES OF SINES, TANGENTS, &c. OF ARCS, IN ARITHMETICAL PROGRESSION.

PROPOSITION XIX. (*Plate I. Fig.* 2.)

Let the three arcs BO, BE, and BA, be in arithmetical progression; viz. let AO be bisected in E, then AE = OE is half the difference between the arcs BO and BA and therefore BE is half the sum of the arcs BO and BA (C. 35.)

But $BO + BA = BO + (BO + OE + AE) = BO + (BO + 2OE) = 2BO + 2OE = 2BE$, that this the sum of the extreme arcs, BO and BA, is equal to double the mean arc BE.

Draw the diameter ACL, and join OL; draw BZ parallel to OA, and join XZ; also from L, draw LW perpendicular to OW.

Then the following triangles are equiangular and similar, viz. CBR, VBZ, XZV, AOP, and OLW. For CBR and VBZ are right angled at R and V, and have the angle at B common. XZV is right angled at V, and the angle at Z is equal to the angle at B, for XZB is a right angle, being contained in a semicircle, and VBZ is the complement of VZB to a right angle, and so is XZV.

The triangle AOP having the sides AO and OP parallel to the two sides ZB and BV of the triangle VBZ, is equiangular with it; the angle AOL is a right angle, being contained in a semi-circle, therefore WOL, the complement of AOP, is equal to the angle OAP: hence the angle OLW is equal to the angle AOP.

The following proportions, &c. are naturally derived from these similar triangles; but first we must observe that $AG + OD = AH = WL$ the sum of the sines of the extreme arcs; for AH is parallel to WL, and the angles WLC and HAC being alternate angles are equal (Euclid 29 *of* I.), therefore the right angled triangles CGA, CYL having one side $AC = LC$, and the angles equal, are equal in all respects; hence $LY = AG$, and $WY = OD$, consequently $LW = AG + OD$. Now $CG + CD$ is the sum of the cosines of the extreme arcs, but on account of the equality of the triangles LYC and AGC, YC=CG, therefore YC+CD=YD=WO, is the sum of the cosines of the extreme arcs.

(L) CB : BR :: BZ : BV
radius : sine of the mean arc :: double the sine of the mean arc : versed sine of double the mean arc.

(M) CB : BR :: ZX (=2RC) : ZV
radius : sine of the mean arc :: double the cosine of the mean arc : sine of twice the mean arc.

(N) CB : BR :: AO (=2*on*) : OP (=CD−CG)
radius : sine mean arc :: double the sine of the common difference between the arcs : difference between the cosines, or versed sines of the extreme arcs.

(O) CB : BR :: OL (=2C*n*) : LW
radius : sine mean arc :: double the cosine of the common difference between the arcs : the sum of the sines of the extreme arcs.

(P) CB : CR :: BZ (=2BR) : ZV
radius : cosine of the mean arc :: double the sine of the mean arc : sine of double the mean arc.

(Q) CB : CR :: ZX (=2CR) : XV
radius : cosine mean arc :: double the cosine of the mean arc : versed sine of the supplement of double the mean arc.

(R) CB : CR :: AO (=2*on*) : AP (=AG−OD)
radius : cosine of mean arc :: double the sine of the common difference between the arcs : difference between the sines of the extreme arcs.

(S) CB : CR :: OL (=2C*n*) : WO (=CG+CD)
radius : cosine of the mean arc :: double the cosine of the common difference between the arcs : sum of the cosines of the extreme arcs.

(T) BR : CR :: BV : ZV
sine of the mean arc : cosine of the mean arc :: versed sine of double the mean arc : sine of double the mean arc.

(U) BR : CR :: ZV : XV
sine of the mean arc : cosine of the mean arc :: sine of double the mean arc : versed sine of the supplement of double the mean arc.

(W) BR : CR :: PO (=CD−CG) : AP (=AG−QD)
sine of the mean arc : cosine of the mean arc :: difference between the cosines (=difference between the versed sines) of the extreme arcs : the difference between the sines of the extreme arcs.

(X) BR : CR :: LW (= AG + OD) : WO (= CG + CD)
sine of the mean arc : cosine of the mean arc :: the sum of the sines of the extreme arcs : the sum of the cosines of the extreme arcs.

(Y) By comparing the above proportions, various other proportions may be formed. Thus BZ : BV :: ZX : ZV by comparing (L. 119. with M.) that is double the sine of the mean arc : versed sine of double the mean arc :: double the cosine of the mean arc : sine of double the mean arc, &c. for the rest.

(Z) By what has been already shewn, *two* arcs may be compared, and their sums and differences formed into a proportion. The arcs being in arithmetical progression, the mean arc BE is half the sum of the extreme arcs BO and BA. Therefore, in all the above proportions, and their variations, wherever the words *mean arc* occur, read *half the sum of the arcs.* Also the sine *on*, of the arc EO, the common difference between the arcs, is the sine of half the difference between the arcs BO and BA; hence in the above proportions for *common difference* between the arcs, read *half the difference* between the arcs. Thus for example (N. 119.) will run thus; radius : sine of half the sum of the arcs :: double the sine of half the difference of the arcs : difference between the cosines of the arcs.

(A) Hence it follows that the difference between the cosines of two arcs × radius = 2 × sine of half the difference × sine of half the sum of these arcs.

But this has been obtained in another manner, see the fourth equation, F. 116.

Indeed from this proposition, the greater part of the formulæ noted in Prop. XVIII. may be deduced, and it will be no bad exercise for the young student to trace out the coincidence.

PROPOSITION XX. (*Plate I. Fig. 2.*)

(B) *If three arcs be in arithmetical progression, the sum of the sines of the extreme arcs, is to their difference, as the tangent of the mean arc is to the tangent of the common difference between the arcs.*

The same construction remaining as in the preceding proposition, AG + OD = sum of the sines of the extreme arcs BA and BO, and AP = AG − PG = AG − OD = the difference of the sines of the extreme arcs. But AG + OD = AH, because GH = DF = OD = PG.

By Prop. XIII. AH : AP :: EK : EI.

Now EK is the tangent of the mean arc BE, and EI is the tangent of EO, the common difference between the arcs. Hence the proposition is evident. Q. E. D.

(C) *Viz.* If the extreme arcs be represented by A and B, of which A is greater than B, then

$$\frac{\text{sine A}+\text{sine B}}{\text{sine A}-\text{sine B}}=\frac{\text{tang }\frac{1}{2}(\text{A}+\text{B})}{\text{tang }\frac{1}{2}(\text{A}-\text{B})}\quad\text{(See Z. 120.)}$$

PROPOSITION XXI. (*Plate I. Fig. 2.*)

(D) *If three arcs be in arithmetical progression, radius is to double the cosine of the middle arc, as the sine of the common difference of the arcs, is to the difference between the sines of the extreme arcs.*

Let the three arcs be BO, BE, and BA, as in the proceding propositions, and $OE=AE$ the common difference between the arcs. Then An is the sine of AE, and $AP=$difference between the sines of BA and BO (B. 120.)

The triangles Ain, cnQ, are equiangular and similar, for Anc and inQ are right angles; if therefore inc be taken away from both, there will remain the $\angle Ani=\angle cnQ$; but the triangle cnQ is equiangular with the triangle CEm.

Therefore $CE : cm :: An : Ai$; but $Ai=iP$, for
$An : AO :: Ai : AP$, and since $An=\frac{1}{2}AO$, $Ai=\frac{1}{2}AP$.

Hence $CE : cm :: An : iP$, and doubling the consequents,
$CE : 2cm :: An : AP$. Q.E.D.

(E) *Viz.* If A and B represent the extreme arcs, A being the greater.

$$\frac{\text{rad}}{2\cos\frac{1}{2}(\text{A}+\text{B})}=\frac{\text{sine }\frac{1}{2}(\text{A}-\text{B})}{\text{sine A}-\text{sine B}}.\quad\text{(See Z. 120.)}$$

PROPOSITION XXII. (*Plate I. Fig. 2.*)

(F) *If three arcs be in arithmetical progression, the rectangle of radius, and the sum of the sines of the extreme arcs, is equal to the rectangle of double the cosine of the common difference of the arcs, and the sine of the mean arcs.*

The same construction remaining as in the foregoing propositions, $CE : cn :: Em : nQ=\dfrac{cn\times Em}{CE}$.

But $AG+OD=AH$ (B. 120.)$=2nQ$.

For $PG=GH$ (B. 120.) and $Ai=iP$ (D. 121.); now $Ai+GH=iP+PG=nQ$, therefore $Ai+iP+PG+GH=AH=2nQ$.

Hence $AG+od=\dfrac{2cn\times Em}{CE}$, that is

$$(AG+OD)\,.\,CE=2cn\,.\,Em.\quad\text{Q.E.D.}$$

(G) Let the extreme arcs be represented by A and B, then the mean arc will be $\frac{1}{2}$(A+B), and the common difference of the arcs $\frac{1}{2}$(A−B). Hence, rad. (sine A+sine B) =

$2\cos\frac{1}{2}(A-B)\,.\,\text{sine}\,(A+B)$ and if rad $=1$, $\text{sine}\,A+\text{sine}\,B=2\cos\frac{1}{2}(A-B)\,.\,\text{sine}\,(A+B)$.

Consequently. *If the sine of the mean of three equidifferent arcs be multiplied by twice the cosine of the common difference, and the sine of either of the extreme arcs be deducted from the product, the remainder will be the sine of the other extreme arc; the radius being 1.*

OF THE SINES, COSINES, TANGENTS, &C. OF THE MULTIPLES OF ARCS.*

(H) $\text{Sine}\,(A+)\,B=\dfrac{(\text{sine}\,A\,.\cos B)+(\text{sine}\,B\,.\cos A)}{\text{rad}}$ (D. 115.)

Let $B=2A$, and rad $=1$, then we shall have

$\text{Sine}\,3A=(\text{sine}\,A\,.\cos 2A)+(\text{sine}\,2A\,.\cos A)$.

But $\text{sine}\,A=\dfrac{\text{sine}\,2A}{2\cos A}$ (O. 108.) and $\cos 2A=2\cos^2 A-1$ (P. 108.)

Therefore $\text{sine}\,A\,.\cos 2A=\dfrac{\text{sine}\,2A}{2\cos A}.(2\cos^2 A-1)=\text{sine}\,2A.\cos A-\dfrac{\text{sine}\,2A}{2\cos A}=(\text{sine}\,2A\,.\cos A)-\text{sine}\,A$, and substituting this in the first equation.

$\text{Sine}\,3A=(\text{sine}\,2A\,.\cos A)-\text{sine}\,A+(\text{sine}\,2A\,.\cos A)$.

Viz. $\text{sine}\,3A=2\,(\text{sine}\,2A\,.\cos A)-\text{sine}\,A$.

And by making $B=3A$, and following the same method,

$\text{Sine}\,4A=(2\cos A\,.\,\text{sine}\,3A)-\text{sine}\,2A$. Hence we obtain

(1) † $\text{Sine}\;A=\text{sine}\,A$

$\text{Sine}\,2A=2\cos A\,.\,\text{sine}\,A$

$\text{Sine}\,3A=(2\cos A\,.\,\text{sine}\,2A)-\text{sine}\,A$

$\text{Sine}\,4A=(2\cos A\,.\,\text{sine}\,3A)-\text{sine}\,2A$

$\text{Sine}\,5A=(2\cos A\,.\,\text{sine}\,4A)-\text{sine}\,3A$

$\text{Sine}\,6A=(2\cos A\,.\,\text{sine}\,5A)-\text{sine}\,4A$, &c.

Where the law of continuation is evident.

OR,

(K) By substituting the value of $\cos A=\sqrt{1-\text{sine}^2 A}$, &c. as in the single arcs (M. 104.), sine 2A found above, &c.

‡ $\text{Sine}\;A=\text{sine}\,A$.

$\text{Sine}\,2A=2\,\text{sine}\,A\sqrt{1-\text{sine}^2 A}$

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[*] See Euler's Introductio ad Analysin Infinitorum; Traité de Trigonométrie, par M. Cagnoli; Crakelt's translation of Mauduit's Trigonometry; Baron Masere's Trigonometry; Emerson, &c.

[†] Cagnoli, page 27. Legendre, page 358.

[‡] Crakelt's translation of Mauduit, page 15.

$\text{Sine } 3A = 3 \text{ sine } A - 4 \text{ sine}^3 A.$

$\text{Sine } 4A = (4 \text{ sine } A - 8 \text{ sine}^3 A)\sqrt{1 - \text{sine}^2 A}$

$\text{Sine } 5A = 5 \text{ sine } A - 20 \text{ sine}^3 A + 16 \text{ sine}^5 A$

$\text{Sine } 6A = (6 \text{ sine } A - 32 \text{ sine}^3 A + 32 \text{ sine}^5 A)\sqrt{1 - \text{sine}^2 A}$, &c.

OR,

(L) $\text{Sine } A = \sqrt{1 - \cos^2 A}$

$\text{Sine } 2A = 2 \cos A \sqrt{1 - \cos^2 A}$

$\text{Sine } 3A = (4 \cos^2 A - 1)\sqrt{1 - \cos^2 A}$

$\text{Sine } 4A = (8 \cos^3 A - 4 \cos A)\sqrt{1 - \cos^2 A}$

$\text{Sine } 5A = (16 \cos^4 A - 12 \cos^2 A + 1)\sqrt{1 - \cos^2 A}$

$\text{Sine } 6A = (32 \cos^5 A - 32 \cos^3 A + 6 \cos A)\sqrt{1 - \cos^2 A}$, &c.

(M) $\text{Cosine } (A+B) = \dfrac{(\cos B . \cos A) - (\text{sine } A . \text{sine } B)}{\text{rad}}$ (D.115.)

Now if $B = 2A$, and radius $= 1$ as before, we shall obtain
$\text{Cosine } 3A = (\cos 2A . \cos A) - (\text{sine } A . \text{sine } 2A)$
But $\text{sine } A . \text{sine } 2A = \text{sine}^2 A 2 . \cos A$ (O. 108.) $=$
$(1 - \cos 2A) . \cos A$. (P.108.) $= \cos A - (\cos A . \cos 2A)$ therefore
$\text{Cos } 3A = (\cos 2A . \cos A) - \cos A + (\cos A . \cos 2A) = (2 \cos A . \cos 2A) - \cos A.$

And by making $B = 3A$, and pursuing the same method, we shall find $\cos 4A = 2 \cos A . \cos 3A - \cos 2A$.

(N)* $\text{Cos } A = \cos A$

$\text{Cos } 2A = 2 \cos A . \cos A - 1$

$\text{Cos } 3A = 2 \cos A . \cos 2A - \cos A$

$\text{Cos } 4A = 2 \cos A . \cos 3A - \cos 2A$

$\text{Cos } 5A = 2 \cos A . \cos 4A - \cos 3A$

$\text{Cos } 6A = 2 \cos A . \cos 5A - \cos 4A$, &c.

OR,

(O) By substituting the value of $\cos A = \sqrt{1 - \text{sine}^2 A}$, $\cos 2A$, found above, &c.

†$\text{Cos } A = \sqrt{1 - \text{sine}^2 A}$

$\text{Cos } 2A = 1 - 2 \text{ sine}^2 A$

$\text{Cos } 3A = (1 - 4 \text{ sine}^2 A)\sqrt{1 - \text{sine}^2 A}$

$\text{Cos } 4A = 8 \text{ sine}^4 A - 8 \text{ sine}^2 A + 1$

$\text{Cos } 5A = (16 \text{ sine}^4 A - 12 \text{ sine}^2 A + 1)\sqrt{1 - \text{sine}^2 A}$

$\text{Cos } 6A = 1 - 18 \text{ sine}^2 A + 48 \text{ sine}^4 A - 32 \text{ sine}^6 A$, &c.

OR,

(P) $\text{Cos } A = \cos A$

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* Cagnoli, page 27. Legendre, page 358.

† Crakelt's translation of Mauduit, page 17.

$\text{Cos } 2A = 2\cos^2 A - 1$

$\text{Cos } 3A = 4\cos^3 A - 3\cos A$

$\text{Cos } 4A = 8\cos^4 A - 8\cos^2 A + 1$

$\text{Cos } 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

$\text{Cos } 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$, &c.

(Q) $\text{Tang }(A+B) = \dfrac{\text{tang } A \cdot \text{tang } B}{\text{rad}^2 - (\text{tang } A . \text{tang } B)}$ (C. 115. or K. 118.)

Now if $B = 2A$, and $\text{rad} = 1$; $\text{tang } 2A = \dfrac{2 \text{ tang } A}{1 - \text{tang}^2 A}$ by substitution, and $\text{tang } 3A = \dfrac{3 \text{ tang } A - \text{tang}^3 A}{1 - 3 \text{ tang}^2 A}$. Hence,

* $\text{Tang } A = \text{tang } A$

$\text{Tang } 2A = \dfrac{2 \text{ tang } A}{1 - \text{tang}^2 A}$

$\text{Tang } 3A = \dfrac{3 \text{ tang } A - \text{tang}^3 A}{1 - 3 \text{ tang}^2 A}$

$\text{Tang } 4A = \dfrac{4 \text{ tang } A - 4 \text{ tang}^3 A}{1 - 6 \text{ tang}^2 A + \text{tang}^4 A}$

$\text{Tang } 5A = \dfrac{5 \text{ tang } A - 10 \text{ tang}^3 A + \text{tang}^5 A}{1 - 10 \text{ tang}^2 A + 5 \text{ tang}^4 A}$, &c.

(R) And since $\text{tang } A = \dfrac{\text{rad}^2}{\cot A}$; if $\dfrac{1}{\cot A}$ be substituted for tang A, we shall obtain

$\text{Cot } A = \cot A$

$\text{Cot } 2A = \dfrac{\cot^2 A - 1}{2 \cot A}$

$\text{Cot } 3A = \dfrac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}$

$\text{Cot } 4A = \dfrac{\cot^4 A - 6 \cot^2 A + 1}{4 \cot^3 A - 4 \cot A}$

$\text{Cot } 5A = \dfrac{\cot^5 A - 10 \cot^3 A + 5 \cot A}{5 \cot^4 A - 10 \cot^2 A + 1}$, &c.

Also if $\dfrac{1}{\sec A}$ be substituted for cos A (N. 123.) the secant of the multiple of any arc may be obtained.

OF THE SINES AND COSINES OF THE POWERS OF ARCS.†

(S) $\text{Cos } 2A = 1 - 2 \text{ sine}^2 A$ (O. 123.)

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* Cagnoli, page 29. Crakelt's translation of Mauduit, page 34.

† Cagnoli, page 30. Crakelt's translation of Mauduit, page 20.

Therefore $2 \text{ sine}^2 A = 1 - \cos 2A$
$\text{Sine } 3A = 3 \text{ sine } A - 4 \text{ sine}^3 A$ (K. 123.)
Hence $4 \text{ sine}^3 A = 3 \text{ sine } A - \text{sine } 3A$
$\text{Cos } 4A = 8 \text{ sine}^4 A - 8 \text{ sine}^2 A + 1$ (O. 123.)
Consequently $8 \text{ sine}^4 A = \cos 4A + 8 \text{ sine}^2 A - 1 =$ $\text{Cos } 4A - 4 \cos 2A + 3$, by substituting for $8 \text{ sine}^2 A$ its value, $4 - 4 \cos 2A$, obtained from the equation $2 \text{ sine } A = 1 - \cos 2A$, and by following the same method, we shall obtain

$\text{Sine } A = \text{sine } A$
$\text{Sine}^2 A = \frac{1}{2} - \frac{1}{2} \cos 2A$
$\text{Sine}^3 A = \frac{3}{4} \text{ sine } A - \frac{1}{4} \text{ sine } 3A$
$\text{Sine}^4 A = \frac{3}{8} - \frac{4}{8} \cos 2A + \frac{1}{8} \cos 4A$
$\text{Sine}^5 A = \frac{10}{16} \text{ sine } A - \frac{5}{16} \text{ sine } 3A + \frac{1}{16} \text{ sine } 5A$
$\text{Sine}^6 A = \frac{10}{32} - \frac{15}{32} \cos 2A + \frac{6}{32} \cos 4A - \frac{1}{32} \cos 6A$
$\text{Sine}^7 A = \frac{35}{64} \text{ sine } A - \frac{21}{64} \text{ sine } 3A + \frac{7}{64} \text{ sine } 5A - \frac{1}{64} \text{ sine } 7A$, &c.

The law of continuation is obvious, for the odd powers are expressed in terms of the sines, and the even powers in terms of the cosines of the multiples of A; and the signs are alternately + and −.

The numerators of the co-efficients (reckoning from the right hand towards the left), are the co-efficients of a binomial whose power is the same as that of the sine of A; except in the even powers, where the term in which A is not found, has the numerator of its co-efficient only one half of the corresponding co-efficient of the binomial, and the denominators are 2 involved to the power of the sine − 1.

(T) To deduce formulæ for the successive powers of the cosine of any arc, we must apply P. 124. in the same manner as above.

$\text{Cos } A = \cos A$
$\text{Cos}^2 A = \frac{1}{2} + \frac{1}{2} \cos 2A$
$\text{Cos}^3 A = \frac{3}{4} \cos A + \frac{1}{4} \cos 3A$
$\text{Cos}^4 A = \frac{3}{8} + \frac{4}{8} \cos 2A + \frac{1}{8} \cos 4A$
$\text{Cos}^5 A = \frac{10}{16} \cos A + \frac{5}{16} \cos 3A + \frac{1}{16} \cos 5A$
$\text{Cos}^6 A = \frac{10}{32} + \frac{15}{32} \cos 2A + \frac{6}{32} \cos 4A + \frac{1}{32} \cos 6A$
$\text{Cos}^7 A = \frac{35}{64} \cos A + \frac{21}{64} \cos 3A + \frac{7}{64} \cos 5A + \frac{1}{64} \cos 7A$, &c.

The law of continuation is the same as S. 125, only all the terms here are positive.

(U) The sine and cosine of any arc, or the sine and cosine of any multiple of that arc, may also be derived by substituting the imaginary quantities $x\sqrt{-1}$ and $-x\sqrt{-1}$, successively for

z, in the exponential expression $e^z = 1 + \frac{z}{1} + \frac{z^2}{1\cdot2}\ \frac{z^3}{1\cdot2\cdot3} + \frac{z^4}{1\cdot2\cdot3\cdot4} +$, &c.

Where e is the number whose hyperbolic logarithm is 1.* The addition of the two new equations, obtained by substitution, if divided by 2, will give a series expressing the sine; and the subtraction, if divided by $2\sqrt{-1}$, will give a series for the cosine of any arc.

THE DETERMINATION OF THE VALUE OF THE SINE AND OF THE COSINE, &c. OF ANY ARC, IN TERMS OF THAT ARC.

PROPOSITION XXIII. (*Plate IV. Fig.* 1.)

To determine the increment of an arc, in terms of the increment of the sine, tangent, secant, &c.; and thence to deduce several useful formulæ.

Let Am be any arc; Pm its sine, CP its cosine, AT its tangent, CT its secant, &c. Take the arc mo indefinitely small, draw onv parallel to Pm, and mn parallel to AC; also, from the centre C, with the radius CT, describe the arc ST.

Then mo is the increment of the arc Am; mn is the decrement of the cosine, or the increment of the versed sine; on is the increment of the sine; Tt the increment of the tangent, and St the increment of the secant.

The triangles mno, and CPm, are equiangular and similar; for the arc mo, being extremely small, may be considered as a straight line. Likewise the triangles CAT and TSt are equiangular and similar, for the lines Ct and CT are supposed nearly to coincide, so that TS may be considered as a straight line, the ∠ TSt a right angle, and the ∠ StT = ∠ CTA. Lastly, the sectors COm and CST are similar. Hence we deduce the following proportions:

(W) $Cm : CP :: mo : on$, hence $mo = (Cm.on) \div CP$

(X) $Cm : CT :: mo : TS$, or, $TS = (CT.mo) \div Cm$

(Y) $CA = Cm : CT :: TS : Tt$, hence $mo = (Cm^2.Tt) \div CT^2$

(Z) $CA : AT :: TS : St$, hence $mo = (Cm^2.St \div AT.CT$

(A) $Pm : Cm :: mn : mo$, hence $mo = (Cm.mn) \div Pm$

(B) Now it is shewn by writers on fluxions, that the limiting ratio of the cotemporary increments or decrements of any two quantities, will be the ratio of the fluxions of those quantities. If, therefore, we put z = the arc AM, *om* will be represented by $\dot{z}$, and if AT $= t$, P*m* $= x$, AP $= v$, and CT $= s$, then T*t* will be represented by $\dot{t}$, *no* by $\dot{x}$, VP (= *nm*) by $\dot{v}$, and *st* by $\dot{s}$, consequently

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* Élements de Géométrie par A. M. Legendre, 6th edit. page 354. Vince's Trigonometry, pro. 27. and 28. page 79. and 80. 2d edit.

$$\text{I.}\quad \dot{z} = \frac{r\dot{x}}{\text{CP}} = \frac{r\dot{x}}{\sqrt{r^2 - x^2}} \text{ (W. 126. and M. 104.).}$$

$$\text{II.}\quad \dot{z} = \frac{r^2\dot{t}}{\text{CT}^2} = \frac{r^2\dot{t}}{r^2 + t^2} \text{ (Y. 126. and P. 105.)}$$

$$\text{III.}\quad \dot{z} = \frac{r^2\dot{s}}{\text{AT . CT}} = \frac{r^2\dot{s}}{s\sqrt{s^2 - r^2}} \text{ (Z. 126. and N. 104.)}$$

$$\text{IV.}\quad \dot{z} = \frac{r\dot{v}}{\text{P}m} = \frac{r\dot{v}}{\sqrt{2rv - v^2}} \text{ (A. 126. and L. 111.)}$$

(C) If these formulæ be expanded, and the fluents of each term be taken, we shall obtain the common series for the arc, in terms of its sine, tangent, &c.*

I. Thus $\dot{z} = \frac{r\dot{x}}{\sqrt{r^2 - x^2}} = r\dot{x}\left(\frac{1}{r} + \frac{x^2}{2r^3} + \frac{3x^4}{2\cdot4\cdot r^5} + \frac{3\cdot5\cdot x^6}{2\cdot4\cdot6\cdot r^7} + \frac{3\cdot5\cdot7\cdot x^8}{2\cdot4\cdot6\cdot8\cdot r^9}, \&c.\right) = \dot{x} + \frac{x^2\dot{x}}{2r^2} + \frac{3x^4\dot{x}}{2\cdot4\cdot r^4} + \frac{3\cdot5\cdot x^6\dot{x}}{2\cdot4\cdot6r^6} + \frac{3\cdot5\cdot7\cdot x^8\dot{x}}{2\cdot4\cdot6\cdot8\cdot r^8}, \&c.;$ the fluent of which is $z = x + \frac{x^3}{2\cdot3\cdot r^2} + \frac{3x^5}{2\cdot4\cdot5\cdot r^4} + \frac{3\cdot5\cdot x^7}{2\cdot4\cdot6\cdot7\cdot r^6} + \frac{3\cdot5\cdot7\cdot x^9}{2\cdot4\cdot6\cdot8\cdot9\cdot r^8}$, &c. and by reversion of the series $x = z - \frac{z^3}{2\cdot3\cdot r^2} + \frac{z^5}{2\cdot3\cdot4\cdot5\cdot r^4} - \frac{z^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot r^6}$, &c.

II. $\dot{z} = \frac{r^2\dot{t}}{r^2 + t^2} = r^2\dot{t}\left(\frac{1}{r^2} - \frac{t^2}{r^4} + \frac{t^4}{r^6} - \frac{t^6}{r^8}, \&c.\right) = \dot{t} - \frac{t^2\dot{t}}{r^2} - \frac{t^4\dot{t}}{r^4} + \frac{t^6\dot{t}}{r^6}$, &c. the fluent of which is $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6}$, &c. and by reverting the series $t = z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} + \frac{17z^7}{315r^6} + \frac{62z^9}{2835r^8}$, &c.

In a similar manner the rest may be found.

But $\sqrt{r^2 - x^2}$ = cosine z; rad − cos z = versed sine z; $\frac{r^2}{\text{tang } z}$ = cot z; $\frac{r^2}{\cos z}$ = sec z, $\frac{r^2}{\text{sine } z}$ = cosec z.

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* Vide Baron Masere's Trigonometry, page 494, &c.

(D) Hence, if A = any arc, and r = radius, then

$$A = \text{sine } A + \frac{\text{sine}^3 A}{2 \cdot 3 r^2} + \frac{3 \text{ sine}^5 A}{2 \cdot 4 \cdot 5 r^4} + \frac{3 \cdot 5 \cdot \text{sine}^7 A}{2 \cdot 4 \cdot 6 \cdot 7 \cdot r^6} + \frac{3 \cdot 5 \cdot 7 \cdot \text{sine}^9 A}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9 r^8}, \text{ \&c.}$$

$$\text{Sine } A = A - \frac{A^3}{2 \cdot 3 \cdot r^2} + \frac{A^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot r} - \frac{A^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot r^6} + \frac{A^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 r^8}, \text{ \&c.}$$

The same series will apply to the chord of any arc by substituting the diameter d, or $2r$ instead of r.*

(E) If $90° - A$ be substituted for A, then we shall obtain

$$90° - A = \text{sine } (90° - A) + \frac{\text{sine}^3 (90° - A)}{2 \cdot 3 \cdot r^2} + \frac{3 \text{ sine}^5 (90° - A)}{2 \cdot 4 . 5 \cdot r^4}, \text{ \&c.};$$

hence $A = 90° - \cos A - \dfrac{\cos^3 A}{2 \cdot 3 \cdot r^2} - \dfrac{3 \cos^5 A}{2 \cdot 4 \cdot 5 \cdot r^4} - \dfrac{3 \cdot 5 \cos^7 A}{2 \cdot 4 \cdot 6 \cdot 7 r^6}$, &c.; or,

$$A = \frac{r - \cos A}{1} + \frac{r^3 - \cos^3 A}{2 \cdot 3 \cdot r^2} + \frac{3}{2 \cdot 4 \cdot 5} \cdot \frac{r^5 - \cos^5 A}{r^4} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7.} \frac{r^7 - \cos^7 A}{r^6}, \text{ \&c.}$$

$$\text{Cos } A = r - \frac{A^2}{2r} + \frac{A^4}{2 \cdot 3 \cdot 4 \cdot r^3} - \frac{A^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot r^5} + \frac{A^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot r^7}, \text{ \&c.}$$

(F) $A = \text{tang } A - \dfrac{\text{tang}^3 A}{3 \cdot r^2} + \dfrac{\text{tang}^5 A}{5 r^4} - \dfrac{\text{tang}^7 A}{7 r^6} + \dfrac{\text{tang}^9 A}{9 r^8}$, &c. (C. 127.)

Hence $\text{tang } A = A + \dfrac{A^3}{3r^2} + \dfrac{2A^5}{15r^4} + \dfrac{17A^7}{315r^6} + \dfrac{62A^9}{2835r} + \dfrac{1382A^{11}}{155925r^{10}} + \dfrac{218844 A^{13}}{6081075r^{12}}$, &c.†

$$A = \frac{r^2}{\cot A} - \frac{r^4}{3 \cot^3 A} + \frac{r^6}{5 \cot^5 A} - \frac{r^8}{7 \cot^7 A} + \frac{r^{10}}{9 \cot^9 A}, \text{ \&c.}$$

$$\text{Cot } A = \frac{r^2}{A} - \frac{A}{3} - \frac{A^3}{45r^2} - \frac{2A^5}{945r^4} - \frac{A^7}{4725r^6} - \frac{2A^9}{93555r^8} - \frac{1382A^{11}}{638512875r^{10}}, \text{ \&c.}$$

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[*] Viz. chord $A = A - \dfrac{A^3}{2 \cdot 3 \cdot 4 r^2}$, hence the difference between an arc and its chord, the radius being 1, is $\dfrac{A^3}{24}$, the 24th part of the cube of the length of the arc nearly. See note, Chap. IV. (G. 83.)

[†] Emerson's Trigonometry, 2d edit. page 32, &c.

$$\text{(G) } \mathrm{A} = 90^\circ - \frac{r^2}{\sec \mathrm{A}} - \frac{r^4}{2 \cdot 3 \cdot \sec^3 \mathrm{A}} - \frac{3r^6}{2 \cdot 4 \cdot 5 \cdot \sec^5 \mathrm{A}} - \frac{3 \cdot 5r^8}{2 \cdot 4 \cdot 6 \cdot 7 \cdot \sec^7 \mathrm{A}}$$

&c. or,

$$\mathrm{A} = \frac{r^2}{\operatorname{cosec} \mathrm{A}} + \frac{r^4}{2 \cdot 3 \cdot \operatorname{cosec}^3 \mathrm{A}} + \frac{3r^6}{2 \cdot 4 \cdot 5 \cdot \operatorname{cosec}^5 \mathrm{A}} + \frac{3 \cdot 5r^8}{2 \cdot 4 \cdot 6 \cdot 7 \cdot \operatorname{cosec}^7 \mathrm{A}},$$

&c.,

$$\operatorname{Sec} \mathrm{A} = r + \frac{\mathrm{A}^2}{2r} + \frac{5\mathrm{A}^4}{24r^3} + \frac{61\mathrm{A}^6}{720r^5} + \frac{277\mathrm{A}^8}{8064r^7} + \frac{50521\mathrm{A}^{10}}{3628800r^9} + \frac{540553\mathrm{A}^{12}}{95800320r^{11}}, \text{ \&c.}$$

$$\operatorname{Cosec} \mathrm{A} = \frac{r^2}{\mathrm{A}} + \frac{\mathrm{A}}{6} + \frac{7\mathrm{A}^3}{360r^2} + \frac{31\mathrm{A}^5}{15120r^4} + \frac{127\mathrm{A}^7}{604800r^6} + \frac{73\mathrm{A}^9}{3421440r^8}, \text{ \&c.}$$

$$\text{(H) } \mathrm{A} = \sqrt{2r \cdot \operatorname{vers} \mathrm{A}} \left(1 + \frac{\operatorname{vers} \mathrm{A}}{3 \cdot 2^2 \cdot r} + \frac{3 \operatorname{vers}^2 \mathrm{A}}{2^3 \cdot 4 \cdot 5 \cdot r^2} + \frac{3 \cdot 5 \operatorname{vers}^3 \mathrm{A}}{2^4 \cdot 4 \cdot 6 \cdot 7 \cdot r^3}, \right.$$

&c.

$$\operatorname{Vers} \mathrm{A} = \frac{\mathrm{A}^2}{2r} - \frac{\mathrm{A}^4}{2 \cdot 3 \cdot 4 . r^3} - \frac{\mathrm{A}^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 . r^5} - \frac{\mathrm{A}^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot r^7}, \text{ \&c.}$$

$$\operatorname{Covers} \mathrm{A} = r - \mathrm{A} + \frac{\mathrm{A}^3}{2 \cdot 3 \cdot r^2} - \frac{\mathrm{A}^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot r^4} + \frac{\mathrm{A}^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot r^6} - \frac{\mathrm{A}^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot r^8}, \text{ \&c.}$$

THE CONSTRUCTION OF A TABLE OF SINES, &c.

(I) There are various methods of constructing a table of sines, but the following, though not the least laborious, is the most simple.

$$\text{Sine } 30^\circ = \tfrac{1}{2} \text{ radius} = \cos 60^\circ \text{ (K. 31.)}$$

$$\text{Sine } 15^\circ = \tfrac{1}{2} \text{ chord } 30^\circ = \frac{\text{rad}}{2} \sqrt{2 - \sqrt{3}}$$

$$\text{Sine } 7\tfrac{1}{2}^\circ = \tfrac{1}{2} \text{ chord } 15^\circ = \frac{\text{rad}}{2} \sqrt{\frac{4 - \sqrt{2} - \sqrt{6}}{2}}, \text{ \&c.}$$

These sines are found thus. From the square of the radius subtract the square of the sine, the square root of the difference is the cosine (M. 104). The radius diminished by the cosine leaves the versed sine (R. 105); and $\sqrt{\text{vers}^2 + \text{sine}^2} = \text{chord}$ (T. 105), the half of which is the sine of the next arc, &c.

From the sine of 7°.30′, find the sine of 3°.45′, a continually, till the sines are as the arcs, which will b the end of the twelfth division from 30°: that is, at the twelfth division, the arc will be 52″.44‴.3⁗.45

sine ·0002556634; but at the end of the 11th division the arc is $1'.45''.28'''.7''''.30'''''$, and its sine ·0005113269; therefore the sine of the 12th division is just half the sine of the eleventh, in the same manner as the twelfth arc is half the eleventh. Now in indefinitely small arcs, the arcs will be to each other as their corresponding sines, hence $52''.44'''.3''''.45'''''$. : $1'$ :: ·0002556634, &c. : ·0002908882 the sine of one minute.

The cosine of $1' =$ the square-root of the radius 1, diminished by the square of the sine of $1'$, viz. cosine $1' =$ ·9999999577.

And it is shewn (G. 122.) that

$2 \times \cos.1' \times \text{sine}1' - \text{sine}0' = \cdot0005817764 = \text{sine}2'$, or $\cos.89^\circ.58'$
$2 \times \cos.1' \times \text{sine}2' - \text{sine}1' = \cdot0008726645 = \text{sine}3'$, or $\cos.89^\circ.57'$
$2 \times \cos.1' \times \text{sine}3' - \text{sine}2' = \cdot0011635526 = \text{sine}4'$, or $\cos.89^\circ.56'$
$2 \times \cos.1' \times \text{sine}4' - \text{sine}3' = \cdot0014544406 = \text{sine}5'$, &c. &c.

This method may likewise be extended to the cosines, by beginning at the other end of the arc, for the cosine is only the sine of the complemental arc.

$2 \times \cos.1' \times \cos.1' - \cos.0' = \cdot9999998308 = \cos.2'$, or $\text{sine}89^\circ.58'$
$2 \times \cos.1' \times \cos.2' - \cos.1' = \cdot9999996192 = \cos.3'$, or $\text{sine}89^\circ.57'$
$2 \times \cos.1' \times \cos.3' - \cos.2' = \cdot9999993231 = \cos.4'$, or $\text{sine}89^\circ.56'$
$2 \times \cos.1' \times \cos.4' - \cos.3' = \cdot9999989423 = \cos.5'$, or $\text{sine}89^\circ.55'$

Proceed thus to find the sine and cosine for every minute of the arc as far as 30°.

(K) Then, if A and B be any two arcs, $\frac{\text{sine}\,A \,.\, \cos B}{\text{rad}} = \frac{1}{2}\text{sine}\,(A+B) + \frac{1}{2}\text{sine}\,(A-B)$ E. 115.; let $A = 30^\circ$, then sine $A = \frac{1}{2}$ radius (K. 31.); and $\cos B = \text{sine}(30^\circ + B) + \text{sine}(30^\circ - B)$; hence, $\text{sine}\,(30^\circ + B) = \cos B - \text{sine}\,(30^\circ - B)$.

If, therefore, $B = 1'.2'.3'.4'$, &c. successively, we shall have

Sine $30^\circ.1' = \cos 1' - \text{sine } 29^\circ.59'$
Sine $30^\circ.2' = \cos 2' - \text{sine } 29^\circ.58'$
Sine $30^\circ.3' = \cos 3' - \text{sine } 29^\circ.57'$, &c.

And in this manner find all the sines, and thence all the cosines (M. 104). as far as 45°. By these means all the sines and cosines from 0° to 90° will be obtained; for $\text{sine}\,(45^\circ + A) = \cos\,(45^\circ - A)$ and $\cos\,(45^\circ + A) = \text{sine}\,(45^\circ - A)$.

(L) The sines and cosines being constructed.

Cosine : sine :: radius : tangent (U. 103); hence the tangents are found.

Tangents : radius :: radius : co-tangent (Z. 103); hence the co-tangents are found.

Cosine : radius :: radius : secant (X. 103); hence the secants are found.

Sine : radius :: radius : co-secant (Y. 103); hence the co-secants are found.

The versed sines are found by subtracting the cosine from radius; or adding the cosine to the radius, if the arc be greater than a quadrant.

(M) ARTIFICIAL, or LOGARITHMICAL, SINES, &c. are only the logarithms of the natural sines, &c.

The natural sines are generally calculated to the radius 1, and being in all cases, except when the arc is 90°, less than radius or 1, they must of course be decimals.

Now the logarithm of a whole number and the logarithm of a decimal is the same, only the index or whole number prefixed to the former is affirmative, and that to the latter negative. Hence if logarithmical sines, &c. were immediately formed from natural sines which are calculated to the radius 1, their indices would all be negative. To avoid this, the logarithmical radius instead of being taken 1, as in the natural sines, is generally considered as ten thousand millions.

(N) *To find the logarithmical sine of* 1′ *to* 7 *places of figures, without the index.*

Rad. 1 : rad. 10,000,000,000 :: ·0002908882 (the natural sine of 1′, the radius being 1) : 2908882 (the natural sine of 1′, the radius being 10,000,000,000.) The logarithm of 2908882 =6.4637261 the logarithmical sine of 1′. Hence the indices to the logarithmical sines from 0′ to 4′, will be 6; from 4′ to 35′ the indices will be 7; from 35′ to 5°.45′ the indices will be 8; from thence upwards to 89 they will be 9; and at 90° the index will be 10=the logarithm of the radius 10,000,000,000.

(O) *Hence if we take the natural sine of any arc from a table of natural sines (where the radius is unity), and multiply it by* 10,000,000,000*; the logarithm of the product will give the logarithm sine of that arc to as many places of figures as the natural sines are carried to.* It may be proper to inform the learner, that this method will not be exactly true for the first five degrees; because the natural sines in the tables are not carried to a sufficient number of places.

(P) In logarithms the operation of multiplication is performed by addition, and division by subtraction. The logarithm sines being constructed, the tangents, &c. are formed thus:

$$\overline{\text{Sine}+10}-\text{cosine}=\text{tangent (U. 103.)}$$
$$20-\text{tangent}=\text{co-tangent (Z. 103.)}$$
$$20-\text{cosine}=\text{secant (X. 103.)}$$
$$20-\text{sine}=\text{co-secant (Y. 103.)}$$

If you double the logarithmical sine of half an arc, and subtract 9·6989700 *from the product; the remainder will be the logarithmical versed sine of that arc.*

For radius × ½ versed sine=square sine ½ arc (I. 107.) or, ½ radius × versed sine=square sine ½ arc, or, log. 5,000,000,000

$+\text{log. versed sine} = 2 \times \text{log. sine } \tfrac{1}{2} \text{ arc}$, or, $9.6989700 + \text{log. versed sine} = 2 \times \text{log. sine } \tfrac{1}{2} \text{ arc}$. Therefore, $\text{log. versed sine} = 2 \times \text{log. sine } \tfrac{1}{2} \text{ arc} - 9{\cdot}6989700$.

(Q) The following logarithmical formulæ for the sine, cosine, &c. of any arc, may in some cases be useful, *viz.*

$\text{Sine} = \text{cos} + \text{tang} - 10 = \text{tang} + 10 - \text{sec} = \text{cos} + 10 - \text{cot} = 20 - \text{cosec}.$

$\text{Cos} = \text{sine} + 10 - \text{tang} = 20 - \text{sec} = \text{sine} + \text{cot} - 10 = \text{cot} + 10 - \text{cosec}.$

$\text{Tang} = \text{sine} + 10 - \text{cos} = 20 - \text{cot}.$

$\text{Cot} = \text{cos} + 10 - \text{sine} = 20 - \text{tang}.$

$\text{Sec} = \text{tang} + 10 - \text{sine} = 30 - \text{sine} - \text{cot} = \text{cosec} + 10 - \text{cot} = 20 - \text{cos}.$

$\text{Cosec} = \text{cot} + 10 - \text{cos} = 30 - \text{cos} - \text{tang} = \text{sec} + 10 - \text{tang} = 20 - \text{sine}.$

THE END OF PLANE TRIGONOMETRY.

BOOK III.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS, &c. OF SPHERICAL ANGLES, ARCS, AND TRIANGLES.

(A) SPHERICAL TRIGONOMETRY treats on the properties of spherical triangles, or the position and magnitudes of arcs of circles described upon the surface of a sphere or globe.

(B) A sphere, or globe, is a round solid body, each part of its surface being equidistant from its centre. This centre is common to every circle described on the surface of the sphere, wherein spherical trigonometry is concerned, and such circles are called great circles.

(C) All great circles divide the globe into two equal parts, and consequently bisect each other at the distance of a semicircle or 180°. Hence two arcs cannot enclose a space, unless they are both semi-circles.

(D) A spherical angle is the inclination of the planes of two great circles to each other, which circles intersect or meet each other on the surface of the sphere, in a point called the angular point. The inclination of these planes must always be measured on the arc of a great circle, 90° from the angular point.

(E) A spherical triangle is formed on the surface of the sphere by the intersection of three great circles, and consists of three sides and three angles; any three of which parts being given the rest may be found.

(F) All the sides of a spherical triangle are arcs of equal circles, and the angles of a spherical triangle are measured by arcs of circles, having the same radii as the sides. Hence the sides and angles of spherical triangles are always expressed in degrees, and parts of degrees.

(G) If one angle of a triangle be 90°, it is called a right-angled triangle.—If one side be 90°, a quadrantal triangle. —If no angle or side be 90°, it is called an oblique-angled triangle.

(H) The pole of a circle is a point on the surface of the sphere equidistant from every part of that circle of which it is the pole. Consequently every circle has two poles diametrically opposite to each other, and the arc of a circle comprehended between each of these poles, and the circumference of such a circle, is a quadrant. No two circles can have the same, or a common pole. If a straight line be drawn from the pole of any circle to the centre of the sphere, it will cut the diameter of that circle at right angles.

(I) All great circles passing through the pole of another great circle, cut that circle at right-angles; and if two circles cut each other at right-angles, in the poles of a third circle, the four points of intersection with that third circle, will be the four poles of the cutting circles; viz. the two opposite points will be the poles of that circle which is described between them.

(K) Sides and angles of spherical triangles are said to be of the same *species*, *kind*, or *affection*, when by comparing any two sides, any two angles, or an angle and a side together, you discover each to be greater or less than a right-angle, or equal to a right-angle.

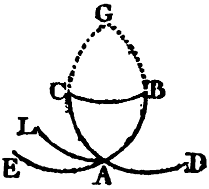
But when by comparing a side with a side, an angle with an angle, or a side with an angle, you discover one to be less and another greater than a right-angle; such sides and angles are said to be of different species.

(L) Spherical triangles are equilateral, isosceles, or scalene, according as they have three equal sides, two equal sides, or three unequal sides.

N.B. In any of the following propositions, wherever the word circle, or arc of a circle occurs, it must always be understood to be a great circle, or the arc of a great circle. And, that all circles concerned in spherical trigonometry are equal to each other.

PROPOSITION I.

(M) *If one great circle intersect another great circle in any point* A, *all the angles described from the point* A, *on the same side of any arc* CAD, *or* EAB, *are equal to two right-angles; and all the angles made about the point* A, *are equal to four right-angles.*



Demonstration. Let CA be perpendicular to EB, then will the angles EAC and CAB be each of them a right-angle. If AL be drawn, then LAB and EAL are equal to two right-angles, for the angle CAB is increased by the angle LAC, and CAE is diminished by the same, therefore $LAB + EAL = EAC + CAB$.

In the same manner it may be proved that the angles EAD and DAB are together equal to two right-angles, for if AC be perpendicular to EB, AD will likewise be perpendicular to it, therefore all the angles about the point A are equal to four right-angles. Q. E. D.

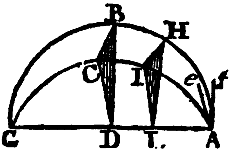
(N) COROLLARY I. *If two arcs of circles intersect each other, the vertical or opposite angles will be equal.*

For the angles EAC and CAB are equal to two right-angles, also CAB and BAD are equal to two-right angles; from each of these equals take the angle CAB, then EAC is equal to BAD. In the same manner it may be shown that the angle CAB is equal to the angle EAD.

(O) COROLLARY II. *All the exterior and interior angles of a spherical triangle are together equal to six right-angles.*

For the interior angle CAB and exterior angle BAD are equal to two right-angles; likewise ACB and BCG are equal to two right-angles; and ABC and CBG are also equal to two right-angles.

PROPOSITION II.



(P) *If a great circle* BC *be described meeting two great circles* ABG *and* ACG, *which pass through the pole* A *of the circle* BC; *the angle* CDB *at the centre of the sphere, upon the circumference* BC, *is the same with the spherical angle* BAC, *and the arc* BC *is called the measure of the spherical angle* BAC.

DEMONSTRATION. Since A is the pole of BC, AB and AC are quadrants (H. 133.) and the angles ABC and ACB are right-angles (I. 134.)

Let D be the centre of the sphere, and join DC and DB; then because the arcs AC and AB are each of them quadrants, the angles ADC and ADB are right-angles (H. 133.): therefore the angle CDB is the inclination of the planes of the circles ABG and ACG to each other, and consequently, (D. 133.) it is the measure of the angle CAB. Q. E. D.

(Q) COROLLARY. *If two circles cut each other in two points, the angles at these points are equal to each other, and to the distance between the poles of these circles.* Viz. the angle BAC is equal to the angle BGC, for the arc BC is the measure of them both; and the distance between the point B and the pole of ABG is a quadrant, also the distance between C and the pole of ACG is a quadrant, (H. 133.) therefore the difference of their distances will be BC.

PROPOSITION III.

(R) *If to the point of intersection* A *of two great circles, two tangents* Ae *and* Af *be drawn; the angle* eAf *will be the measure of the spherical angle* BAC.

For AG is the diameter of the sphere, and ADC and ADB are right-angles, but *f*AD and *e*AD are also right-angles, being tangents to the circles ABG and ACG; therefore *f*A and *e*A are parallel to DB and DC, hence the angle *e*A*f* is equal to the angle CDB; but CDB is the measure of the spherical angle BAC, therefore *e*A*f* is the measure thereof.

In the same manner it may be shewn that the angle ILH is equal to CDB, for it is likewise the inclination of the planes ABG and ACG; hence all spherical angles must be measured on the arc of a great circle 90° distant from the angular point, for CB, and not IH, is the measure of the angle BAC.

PROPOSITION IV.

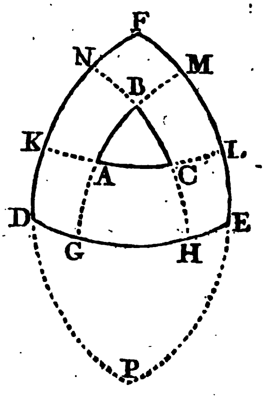
(S) *If from the angular points* A, B, C, *of a spherical triangle* ABC *as poles, there be described three arcs of circles* EF, DE, *and* DF, *forming a new spherical triangle* DFE; *each side of this new triangle is the supplement of the measure of the angle at its pole,* viz. *the side* FE *is the supplement of the measure of the angle* A; DE *of the angle* B; *and* DF *of the angle* C.—*Likewise each angle of this new triangle is the supplement of the measure of that side of the original triangle* ABC, *to which it is opposite;* viz. *the angle* D *is the supplement of the measure of* BC, *the angle* F *of* AC, *and the angle* E *is the supplement of the measure of the side* AB.

DEMONSTRATION. Since B is the pole of the circle DGHE, every part of this circle is 90° distant from B (H. 133.); therefore the distance between B and E is a quadrant, and BH and BG are quadrants.

Since A is the pole of the circle ELMF, every part of this circle is 90° distant from A (H. 133.), therefore the distance between A and E is a quadrant, and AM and AL are quadrants.

Since A and B have each been proved to be 90° distant from E, and AM and BG quadrants, being the side AB produced to M and G; E is the pole of the circle GABM.

In the same manner it may be proved that F is the pole of KALC, and that D is the pole of NBCH.



Therefore EM, EG; FL, FK; DN and DH are quadrants, likewise AM, AL; BH, BG; CN and CK are quadrants; consequently (D. 133.) LM is the measure of the angle A, and NH the measure of the angle D.

Now EM and FL are together equal to a semi-circle, or FL, LE, and LM are together equal to a semi-circle; but FL and LE are equal to FE, therefore FE and LM are together equal to a semi-circle; that is they are supplements of h other; but LM is the measure of the angle A, therefore the measure of the angle A in the triangle ABC, and the measure of the side EE in the triangle DFE, are supplements of each other.

In the same manner it may be proved that DE is the supplement of the measure of the angle B, and that DF and the angle C are supplements of each other.

It remains now to prove that the angles in the new triangle DFE, and the sides of the original triangle ABC, are supplements of each other.

Since CN and BH are quadrants, they are together equal to a semi-circle, or CN, CH, and BC, are together equal to a semi-circle; but NC and CH are equal to NH, therefore NH and BC together make a semi-circle; now NH is the measure of the angle D; therefore the angle D in the triangle DFE is the supplement of the side BC in the triangle ABC.

In the same manner it may be shown, that the angle F is the supplement of the measure of AC, and the angle E of AB. Q. E. D.

(T) COROLLARY I. *The three angles of any spherical triangle* ABC *are together greater than two right angles, and less than six.*

For,

The angle A and the side FE are equal to two right-angles. The angle B and the side DE are equal to two right-angles. The angle C and the side DF are equal to two right-angles. Therefore the three angles A, B, C, together with the three sides FE, DE, and DF, are equal to six right-angles; but the three sides FE, DE, and DF, are together *less* than four right-angles (Z. 138), therefore the three angles A, B, C, are together greater than two right-angles.

Every spherical angle is less than two right-angles (M. 134.); therefore the sum of any three spherical angles is less than six right-angles.

(U) COROLLARY II. *The sides of spherical triangles may be changed into angles, and the contrary.*

Thus if three angles of a triangle are given to find the sides; subtract each of the angles from 180°, and the three remainders will be the three sides of a new triangle, with which sides find the angles of the new triangle; then subtract each of these angles from 180°, and the three remainders will be the respective sides of the original triangle, whose angles were given. And the contrary when the sides are given to find the angles.

(W) COROLLARY III. *When the three angles* (A, B, C,) *of a triangle* (ABC) *are given to find a side* (AC), *take the angle* (B) *opposite to the side required from* 180°, *and use the remainder and the other two angles* (A *and* C) *as sides in a new tria* (DPE): *In this new triangle find the angle* (P) *opposite that*

(DE) *where the supplement is used. Subtract this angle from 180°, and the remainder will be the side* (AC) *required.*

For if FD and FE be continued till they meet in P; DP and EP being the supplements of FD and FE are equal to the angles C and A; and the side DE is the supplement of the angle B opposite to AC the side required.

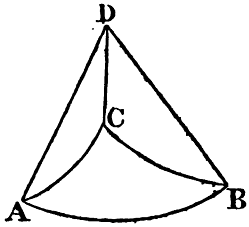
Now find the angle P which is equal to the angle F (Q. 135.) but the supplement of the angle F is equal to the side AC, therefore the supplement of the angle P is equal to the side AC. If you find the angle PDE it will be equal to the side BC, and PED will be equal to the side AB. For PDE and EDF are supplements of each other, and EDF and BC are supplements of each other, therefore the angle PDE is equal to the side BC.

(X) COROLLARY IV. *A quadrantal triangle may be changed into a right-angled triangle, by calling the supplement of the angle opposite to the quadrantal side, the hypothenuse, and the other angles the legs.*

For if AC be a quadrant, the angle at P is a right angle, being the supplement of AC (W. 137.), and DE is the hypothenuse, being the supplement of the angle B (S. 136.)

PROPOSITION V.

(Y) *Any two sides of a spherical triangle are together greater than the third side.*



DEMONSTRATION. Let ABC be a spherical triangle, any two sides AC and BC, are together greater than AB. Let D be the centre of the sphere, and join DA, DC, DB.

The solid angle at D is contained by the three plane angles BDC, BDA, CDA, any two of which BDC, CDA, are greater than third BDA (*Keith's Geom.* 1 *of* X, *or* EUCLID 20 *of* XI.)

Now the arc BC is the measure of the angle BDC, the arc AC of CDA, and the arc AB of BDA.

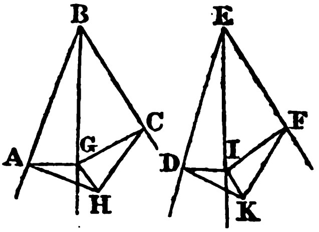
Therefore AC and BC together are greater than AB. Q. E. D.

(Z) COROL. The three sides of a spherical triangle are together less than the circumference of a circle.

For every solid angle is contained by plane angles which are together less than four right angles (*Keith's Geom.* 2 *of* X.) and the three sides of the spherical triangle are the measures of a solid angle, and therefore are less than four quadrants, or the circumference of a circle.

PROPOSITION VI. LEMMA.

(A) *If two solid angles be composed of three plane angles, such, that the plane angles of the one are equal to the plane* *angles of the other, each to each; the planes in which these equal angles are situated, shall have the same inclination to each other.*



DEMONSTRATION. Let B and E be two solid angles, and let the plane angle ABC be equal to the plane angle DEF; ABG to DEI, and GBC to IEF; the angle formed by the planes ABC and ABG is equal to the angle formed by the planes DEF and DEI.

For, suppose the straight line BG to be elevated above the plane ABC, then from any point G in that line let fall the perpendicular GH upon the plane ABC (*Keith's Geom.* 11 *of* IX.) in which plane draw HA and HC perpendiculars to AB and BC, and join AG, GC.

Make EI = BG, and from the point I draw IK perpendicular to the plane DEF, in which plane draw KD and KF at right angles to ED and EF, and join ID, IF.

Then because GH is perpendicular to the plane ABC, and that HA is at right angles to AB a straight line in that plane, the angle BAG is a right angle (*Keith's Geom.* 4 *of* IX. *Corol.* 3.): for the same reason EDI is a right angle, therefore the triangle ABG is equal to the triangle DEI.

In the same manner it may be shown that the triangle GBC is equal to the triangle IEF, therefore the quadrilateral figure ABCH, situated in the plane ABC, is equal to the quadrilateral figure DEFK, situated in the plane DEF.

For, if the angle ABC be applied to the angle DEF, AB, BC will coincide with DE and EF; and because all right angles are equal to each other, AH, which is at right angles to AB, will fall upon DK, which is at right angles to ED, and for the same reason CH will fall upon FK, therefore the point H will fall upon the point K.

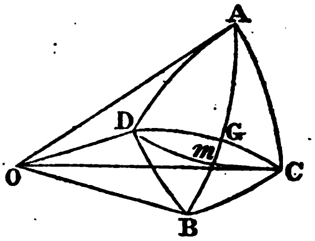
Now because the triangles AHG and DKI are right-angled at H and K, and the hypothenuse AG is equal to the hypothenuse DI, these triangles are equal (*Keith's Geom.* 14 *of* III. *note* 11.), therefore the angle HAG is equal to the angle KDI.

But the angle HAG is the inclination of the two planes ABC, ABG; and the angle KDI is the inclination of the two planes DEF, DEI; therefore these planes have equal inclinations to each other. Q. E. D.

SCHOLIUM. If the angle HAG be obtuse, viz. if the point H, and the straight line BC fall on contrary sides of AB; th angle KDI will be equally obtuse.

PROPOSITION VII.

(B) *In any two spherical triangles, if the three sides of the one be equal to the three sides of the other, each to each, the angles which are opposite to the equal sides will be equal.*



Let ABC be any triangle on the surface of the sphere.— From A as a pole, with the distance AC, describe the small circle CmD; from B as a pole, with the distance BC, describe the small circle CGD, crossing CmD in the point D. From the points A and B draw the great circles AD and BD, then AC = AD and BC = BD, by construction, also AB is common to the two triangles ABC and ABD, therefore the three sides of the one are equal to the three sides of the other, each to each.

The angles which are opposite to the equal sides in each triangle are equal. For,

If O be the centre of the sphere, the solid angle formed at the point O by the three planes AOB, AOD, BOD, will be equal to the solid angle at O formed by the three planes AOB, AOC, BOC.

Because the sides of the triangles ABC and ABD are equal each to each, by construction; and that these sides are the respective measures of the plane angles which constitute the solid angles at the centre of the sphere (Prop. v. Y. 138.); it follows, that the plane angles which form one of these solid angles, are equal to the plane angles which form the other solid angle, each to each.

But the planes in which such angles are situated have the same inclination to each other (Prop. VI. A. 138.) therefore the angles of the spherical triangle ABC are equal to those of the spherical triangle ABD, viz. $\angle \mathrm{DAB} = \angle \mathrm{CAB}$, $\angle \mathrm{DBA} = \angle \mathrm{ABC}$, and $\angle \mathrm{ADB} = \angle \mathrm{ACB}$.

SCHOLIUM. The equal angles ACB and ADB being situated on contrary sides of the centre of the sphere, the triangle ABC will not coincide with the triangle ADB, unless we conceive the convexity of the one to be applied to the concavity of the other.

If the equal angles ACB and ADB were situated on the same side of the centre of the sphere, the triangle ABC when applied to the triangle ADB would coincide with it.

Triangles which are equilateral, or isosceles, will coincide when applied to each other.

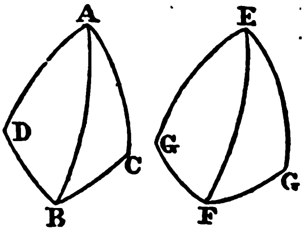
PROPOSITION VIII.

(C) *In two triangles, if the three angles of the one be equal to the three angles of the other, each to each, these triangles are equal in all respects.*

DEMONSTRATION. The supplements of the three angles in one triangle are equal to the supplements of the three angles in the other: and each of these three supplements form a new spherical triangle, whose sides and angles are respectively equal (Prop. IV. and VII.); and the supplements of the equal angles in the new spherical triangle, are equal to the sides of the original triangle whose angles were given; but the supplements of the angles of the new triangle being equal, the sides of the original triangle are equal; and if the sides be equal, the triangles are equal in all respects. (Prop. VII.) Q. E. D.

PROPOSITION IX.

(D) *If there be two sides and the included angle in one spherical triangle, equal to two sides, and the included angle in another, each to each; these two triangles are equal in all respects.*



DEMONSTRATION. Let ABC and EFG be two triangles on the surface of the sphere, and let the side AB = EF, AC = EG, and the angle BAC = ∠ FEG. The triangle ABC is equal to the triangle EFG.

First, let the triangles be on the same side of the centre of the sphere. Then if the triangle EFG be applied to the triangle ABC, EF will coincide with AB and EG with AC (like plane triangles in EUCLID 4 *of* I.)

The point F coinciding with B, and G with C, the arc FG must coincide with BC, since only *one*[*] great circle can be drawn through two given points on the surface of the sphere.

Hence, the three sides of the one triangle being equal to the three sides of the other, each to each, the triangles are equal. (Prop. VII.)

Secondly, let the triangles be situated on contrary sides of the centre of the sphere; then a triangle ADB may be constituted contiguous to ABC, and equal to EGF (as in Prop. VII.), and consequently EGF will coincide with ADB. Therefore the side FG = BC, the angle EFG = ∠ ABC, and ∠ EGF = ∠ ACB. Q. E. D.

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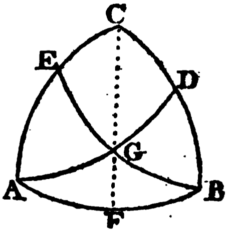
[*] For the centre of every great circle is the centre of the sphere, and three given points determine the position of a plane. (*Keith's Geom.* 2 *of* IX. *Corol.*)

(E) COROLLARY. *If a side and the two adjacent angles in one spherical triangle, be equal to a side and the two adjacent angles in another, each to each; these triangles are equal in all respects.*

For by taking the supplements of each of the two angles, and of the sides contained between them, two new triangles will be formed which will be equal, by the proposition, and therefore their supplements, viz. the original triangles will be equal.

PROPOSITION X.



(F) *The angles at the base of an isosceles spherical triangle are equal to each other.*

DEMONSTRATION. Let ABC be an isosceles triangle having the side AC equal to the side BC.

Make CE equal to CD, and draw EB and DA; now AC being equal to BC by hypothesis, and EC equal to CD by construction, and the angle ACB common to both the triangles ACD and BCE, these two triangles are equal in all respects (D. 141.); therefore AD is equal to EB.

Again, since AC is equal to BC, and EC equal to CD, the remaining parts AE and DB are equal, hence the triangles BAE and ABD are equal in all respects; for the two sides AE and EB are equal to the two sides BD and DA each to each, and the base AB is common to both, therefore (B. 140.) the angle BAE is equal to the angle ABD. Q. E. D.

(G) COROLLARY I. *In any spherical triangle if two of the angles be equal, the sides opposite to them will be equal.*

Make BD equal to AE, and draw EB and AD, then the triangle BAE is equal to the triangle ABD (D. 141.); therefore AD is equal to BE, and the angle AEB equal to the angle ADB, the angle DAB equal to the angle EBA; now the angle BEC is equal to the angle ADC, being each of them supplements of the equal angles AEB and BDA, therefore the triangle ADC is equal to the triangle BEC (E. 142.); consequently AE + EC is equal to ED + DC. viz. equal sides are opposite to equal angles.

(H) COROLLARY II. *Hence every equilateral triangle is likewise equiangular, and the contrary.*

(I) COROLLARY III. *A line drawn from the vertex of an isosceles triangle to the middle of the base, is perpendicular to the base.*

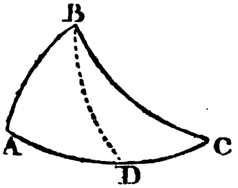
For the two sides FB and BC are equal to the two sides FA and AC, and the angle FBC is equal to the angle FAC, therefore the angle CFB is equal to the angle CFA (D. 141.); but CFA and CFB are equal to two right angles (M. 134.), therefore CF is perpendicular to AB.

(K) COROLLARY IV. *A perpendicular drawn from the vertex of an isosceles triangle upon the base, always bisects the base, and the vertical angle, except when the two sides are quadrants, in which case there may be an indefinite number of perpendiculars.*

For the triangles FBC and FAC are equal in all respects. (D. 141.) But if AC and BC are quadrants, that is, if C is the pole of the base, then an infinite number of lines may be drawn from the point C upon the base, to cut it at right angles. (H. and I. 134.)

PROPOSITION XI.

(L) *In any spherical triangle* (ABC) *the greatest side* (AC) *is opposite to the greatest angle* (B), *and the least to the least.*

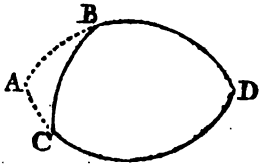


DEMONSTRATION. Make the angle DBA equal to the angle BAD, then AD is equal to BD. (G. 142.) Now AD + DC is equal to AC, but AD + DC or BD + DC is greater than BC (Y. 138.), therefore AC is greater than BC.

Lastly, DC is less than AD or AB, and it may be shewn, as above, that the angle DBC is less than the angle DCB. Q. E. D.

PROPOSITION XII.

(M) *If one side* (DC) *of a spherical triangle* (DCB) *be produced, and if the sum of the other sides* (CB + BD) *be greater, equal to, or less than a semi-circle; the internal angle* (D) *at the base* (DC) *is accordingly greater, equal to, or less than the outward and opposite angle* (BCA) *adjacent to the side produced.*



DEMONSTRATION. Produce the sides DB and DC till they meet in A. Then the arcs ABD and ACD are each of them a semi-circle (C. 133.), and the angle at D is equal to the angle at A. (Q. 135.)

1. If CB + BD be greater than 180°, BC will be greater than AB, and the angle CAB, or, which is the same thing, the angle D will be greater than the angle BCA. (L. 143.)

2. If CB + BD be equal to 180°, then BA and BC will be equal, and the angle CAB will be equal to the angle BCA, or the angle D will be equal to the angle BCA. (F. 142.)

3. If CB + BD be less than 180°, BC will be less than AB, and the angle CAB, or the angle D will be less than the angle ACB. (L. 143.) Q. E. D.

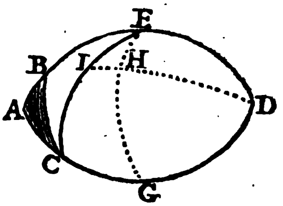
(N) Corollary. *In any spherical triangle, if the sum of any two sides be greater, equal to, or less than a semi-circle; the sum of the opposite angles will be greater, equal to, or less than a semi-circle.*

For, according as CB + BD is greater, equal to, or less than a semi-circle, the angle D opposite to the side BC is greater, equal to, or less than the angle BCA; but the angle BCA and the angle BCD, opposite to the side DB, are always equal to two right-angles (M. 134.); therefore whenever D is greater than BCA, the sum of the angles BCD and BDC is greater than two right-angles; *et contra.*

PROPOSITION XIII.

(O) A right-angled spherical triangle may have either,

1. *One right angle, and two acute angles.*
2. *One right angle, and two obtuse angles.*
3. *One obtuse angle, and two right angles.*
4. *One acute angle, and two right angles.* OR,
5. *Three right angles.*



Demonstration. Let the angles A and D be right angles; and the arcs AE and ED quadrants; then E will be the pole of the circle ACGD, and EC and EG will be quadrants (H. 133.) Since AE and DE are quadrants, A or D will be the pole of EG; therefore AG and GD are quadrants.

1. Because AE and EC are quadrants, the angles EAC and ACE are right angles (I. 134.); therefore the angle ACB is less than a right angle. And since AC is less than AG, the angle ABC is less than the angle AEG (L. 143.); but AEG is a right-angle; therefore the right-angled triangle CAB has two acute angles and one right-angle.

2. The triangle BDC right-angled at D, has two obtuse angles and one right angle. For the angles DBC and DCB are supplements of the two acute angles ABC and ACB, and therefore must be obtuse.

3. The triangle CED has one obtuse angle, and two right angles. For, since GD and ED are quadrants, the angles DEG and DGE are right angles, therefore DEC is an obtuse angle. But EC is a quadrant, and E is the pole of CGD, therefore ECD is a right angle. (H. 133.)

4. The triangle AEC has already been shewn to have two right angles; and AEG is a right angle, but AEC is less than a right angle.

5. The right-angled spherical triangle GAE has three right angles, for A is a right angle by the proposition, and A is the pole of the circle EG, therefore AEG and AGE are right angles. (H. 133.) Q. E. D.

(P) COROLLARY I. In any right-angled triangle, *if the two legs be of the same species, viz. both acute, or both obtuse, the hypothenuse will be less than a quadrant, and the contrary.*

For BC the hypothenuse, or side opposite to the right angles A and D, is common to the triangle BAC having two acute angles, and to the triangle BDC having two obtuse angles.

(Q) COROLLARY II. In any right-angled spherical triangle, *if the legs be of different species, the hypothenuse will be greater than a quadrant, and the contrary: but one of the legs, at least, will be a quadrant, if the hypothenuse be a quadrant.*

For, in the triangle ICD, right-angled at C, CI is less than a quadrant and CD greater, from what has already been demonstrated; but since D is the pole of the circle EHG, DH is a quadrant, therefore DI the hypothenuse is greater than a quadrant. And when CI becomes equal to CE, DI will become equal to DE; but DE is a quadrant, therefore if the hypothenuse be a quadrant, one of the legs, at least, will be a quadrant. But both the legs will be quadrants when all the angles are right angles, as in the triangle AEG.

(R) COROLLARY III. *The sides containing the right angle of any triangle, are always of the same species as their opposite angles, and the angles are of the same species as their opposite sides.*

For in the triangle CAB where the two angles are acute, the opposite sides AB and AC are acute; and in the triangle CDB where the two angles are obtuse, the opposite sides BD and CD are obtuse.

(S) COROLLARY IV. *If the hypothenuse be less than* 90°, *the legs are of the same species as their adjacent angles. If the hypothenuse be greater than* 90°, *the legs and their adjacent angles are of different species.*

This is evident from the triangles BAC, CDB, and DCI.

(T) COROLLARY V. *According as the hypothenuse, and one side (or its opposite angle) are of the same, or different species: the other side (and its opposite angle) will be less or greater than a quadrant.*

This is also evident, from the triangle DCI.

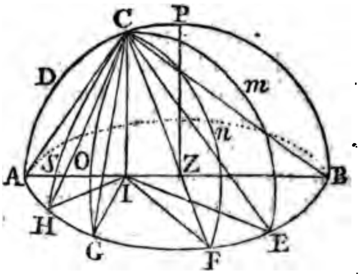
PROPOSITION XIV.

(U) *If any number of arcs of circles* CPB, CME, CNF, CDA, CSH, COG, *be drawn from one and the same point* C, *on the surface of the sphere, to the circumference of a great circle* AHGFEB; *the greatest arc will be that which passes through the pole* P.

of the great circle AHGFEB; *and the least arc will be the supplement* ADC *of the greatest* CPB. *And that arc which is the nearest to the greatest will be greater than any one more remote; likewise, that arc which is nearest to the least arc, will be less than any one more remote from it.*

DEMONSTRATION. Let Z be the centre of the sphere, or of the great circle AHGFEB, and AB its diameter; from C, the point where all the circles intersect each other, draw CI perpendicular to AB, and join IE, IF, IG and IH.

Then of all the straight lines which can be drawn from the point I to the circumference of the circle AHGFEB, that line IB, is the greatest which passes through the centre Z, and its opposite line IA is the least; that line IE, which is nearest to IB, is greater than the line IF which is more remote; and the line IH which is nearest to IA, is shorter than the line IG which is more remote. (Euclid 7 *of* III.)

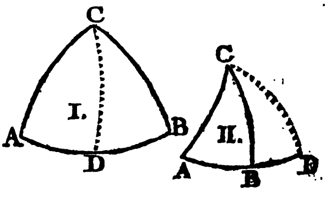


Let AC, HC, GC, FC, &c. be joined by straight lines; then will CIB, CIE, CIF, CIG, CIH and CIA be all right-angled plane triangles, having the perpendicular CI common to each; therefore that triangle CIB, which has the greatest base IB, will have the greatest hypothenuse BC; and that triangle CIA which has the least base IA, will have the least hypothenuse AC, &c.; of the others, EC will be greater than FC, and HC less than GC.

But these several hypothenuses are all chords of equal circles, and the greater chord cuts off the greater arc, or circumference, therefore CPB is the greatest arc, and ADC the least; CmE is greater than CnF, and CSH is less than COG. Q. E. D.

PROPOSITION XV.

(W) *If the angles at the base of an oblique-angled spherical triangle be of the same species, viz. both acute, or both obtuse, a perpendicular from the vertical angle upon the base will fall within the triangle; if they be of different species, it will fall without.*



DEMONSTRATION. In the right-angled triangle CDB, CD is of the same species as CBD (R.145.), and in the right-angled triangle CDA, CD is of the same species as CAD, therefore the perpendicular in the first figure falls between the angles A and B.

In the second figure, CBD and CBA are supplements of each other (M. 134.), and the perpendicular CD falls opposite to both CBD and CAD being of the same species, therefore it cannot fall opposite to CBA at the same time, hence it must necessarily fall without the triangle. Q. E. D.

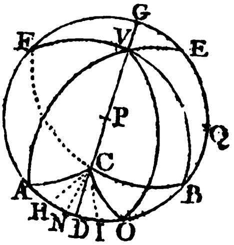
(X) COROLLARY. *In an isosceles triangle, the equal angles are of the same species as their adjacent, or opposite sides.*

For then AD = DB (K. 143.) and these segments are always acute, since their sum can never be equal to a semi-circle. (C. 133.)

But according as DB and DC, or AD and DC, are of the same, or different species, BC or its equal AC, is acute or obtuse; or since the segments are always acute, BC and AC will be of the same species as DC, but DC is of the same species as A and B, therefore the sides are of the same species as their adjacent or opposite base angles.

PROPOSITION XVI.

(Y) *If the* LEAST *two sides of a spherical triangle be of the same species, the perpendicular drawn from their included angle upon the third side, will fall within the triangle.—Or, if the third side be less than either of the others, they being of the same species, but greater than their supplements, the perpendicular will fall within the triangle, and the less segment of the base and less vertical angle will be adjacent to the greater side, and the greater segment, and greater vertical angle will join the less side, provided the sum of the sides be greater than* 180°.



DEMONSTRATION. Let ABC be the triangle, CD a perpendicular upon its base AB; make BN equal to the hypothenuse BC, and AI equal to the hypothenuse AC, then BCN and ACI are isosceles triangles; the angle BNC is of the same species as BC, and the angle AIC is of the same species as AC. (X. 147.). But AC and BC are of the same species, by the hypothesis, therefore the angles BNC and AIC are of the same species; consequently CD falls between them (W. 146.) and of course within the triangle ABC.

SECONDLY. In the triangle AVB, where AV and BV are each greater than AB, the least two sides of the supplemental triangle FVE, which by the hypothesis are each less than AB or FE, are of the same species, and consequently VG falls between them, therefore VD falls between AV and BV. But AV is greater than BV, and AD is less than DB; consequently AVD is less than BVD. Q. E. D.

(Z) Corollary I. *The perpendicular let fall on the base of an oblique spherical triangle is either less or greater than each side.*

For since CD is perpendicular to AB, it must pass through P the pole of AB (I. 134.); and if C be below the pole P, that is if CD be less than a quadrant, it will be the least of all the lines that can be drawn from the point C to the circumference of the circle ABEF, and CG will be the greatest. (U. 145.)

If C the vertex of the triangle ABC fall above the pole P, suppose in V, then VD will be the perpendicular, and of all the arcs that can be drawn from V upon the arc AB, VD is the greatest (U. 145.), therefore the perpendicular is either less or greater than each side; when it becomes equal to one of the sides, the triangle will be right angled, as BDC.

If the perpendicular fall without the triangle, as in BOC, or BOV, the demonstration will hold equally true.

(A) Corollary II. *When the perpendicular falls without the triangle, it may be either less or greater than a quadrant; or less or greater than each side.*

In the triangle BOC, either CD or CG may be esteemed the perpendicular upon the base OB continued; the former CD is less than a quadrant, and less than either CO or CB, and the latter CG is greater.

(B) Corollary III. *When the perpendicular falls without the triangle, the less perpendicular is next the less side, and the greater perpendicular is next the greater side; and either of them may be considered as the proper perpendicular or the base produced.*

In the triangle BOC, CD the less perpendicular is next the less side CO, and CG the greater perpendicular is next the greater side BC, and either of them may be used for determining the several parts of the triangle BOC, but the one is sometimes more convenient than the other.

(C) Corollary IV. *In any triangle* ABC *or* BOC, *the perpendicular* CD *falling within the triangle* ABC *upon the base* AB, *or* NEAREST *perpendicular* CD *falling without the triangle* BOC, *is always of the same species as half the sum of the two sides of the triangle; and if half the sum of the two sides of the triangle be acute, this perpendicular falls nearest the less side, if obtuse, it falls nearest the greater side.*

In the triangle ABC the perpendicular DC is less than a quadrant, and each side AC and BC is less than a quadrant, therefore half their sum is less than a quadrant, and the perpendicular CD falls nearest to AC.

In the triangle BOC, CD is less than a quadrant, and each of the sides CO and CB is less than a quadrant, therefore half their sum is less than a quadrant, and the perpendicular falls nearest to CO.

And, in the triangles ABV and BOV, where the perpendicular VD is greater than a quadrant, each of the sides AV, BV, and OV is greater than a quadrant; consequently half their sum must be greater than a quadrant, and the perpendicular VD falls nearest to AV in the former triangle, and to OV in the latter.

(D) COROLLARY V. *If an oblique spherical triangle have two acute sides and one obtuse, a perpendicular drawn on the longest side will always fall within the triangle* (Y. 147.)—*But if it has two obtuse sides and one acute, a perpendicular drawn to fall on the longest side may sometimes fall within, and sometimes without the triangle.*

Take any right-angled triangle BDC, and produce its sides to a semi-circle; now because DB and DC are each less than a quadrant, the hypothenuse BC will be less than a quadrant (Q. 145.), consequently its supplement CF will be greater than a quadrant. The angles B and C being acute or each less than a right angle, BC is greater than DB (L. 143.), therefore DF is greater than CF.

Make FA equal to FC, then a great circle as HC may be drawn between A and D; now in the triangle FHC, FH is greater than FC; FC, HC, are of different species, and the perpendicular CD falls without the triangle FHC.

In the triangle FOC, OF is greater than FC; FC and OC are of different species, and the perpendicular CD falls within the triangle FOC.

Therefore there can be no *general* rule for drawing a perpendicular when the three sides of a triangle are given, especially if two of those sides be obtuse.

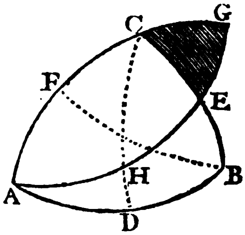
(E) COROLLARY VI. *If the two obtuse sides of an oblique-angled triangle be* EQUAL, *a perpendicular drawn from their included angle will fall within the triangle.*

For then the triangle is isosceles, and the angles at the base are equal to each other. (F. 142.)

PROPOSITION XVII.

(F) In any spherical triangle,

1. *If the three angles be each acute, each side will be less than a quadrant.*
2. *If the three angles be each right angles, each side will be a quadrant.*
3. *If the three angles be each obtuse, each side will be greater than a quadrant.*



DEMONSTRATION. *First*, Let all the

angles A, B, and C of the triangle ABC be acute, then a perpendicular as CD drawn from any angle will fall within the triangle.

In the right-angled triangle ADC, the angles A and ACD are acute, and therefore (P. and R. 145.) AC is less than a quadrant. By the same manner of reasoning from the triangle BDC, BC is less than a quadrant; and because the angles A and ABF are acute, AB is less than a quadrant. (P. and R. 145.)

Second. This part is demonstrated (O. 144.)

Third. If the three angles be obtuse, each side of a supplemental triangle will be less than 90° (Prop. VI.); if the three sides are less than 90°, the three angles will be acute, by this proposition, and the supplements of these three angles must necessarily be obtuse; but (U. 137.) they will be the sides of the original triangle, the angles of which were all obtuse. Q. E. D.

(G) COROLLARY I. *In any spherical triangle having two obtuse angles and one acute, the sides are of the same species as their opposite angles.*

Imagine the triangle AEC to have an acute angle at A, and two obtuse ones at E and C; then the triangle CEG has three acute angles, viz. the angle G is equal to the angle A, (Q. 135.) and the acute angles at E and C are supplements of the obtuse angles in the triangle AEC (M. 184.):

But when the three angles are acute, each side of the triangle CEG is less than a quadrant, therefore AE and AC, opposite to the obtuse angles C and E, are greater than quadrants, and CE opposite to the acute angle A, is less than a quadrant.

(H) COROLLARY II. *If a spherical triangle have two sides each less, and one greater than a quadrant; the angles will be of the same species as their opposite sides.*

In the triangle AHC, let AH and HC be each of them less than a quadrant, and AC greater; the supplemental triangle to it (U. 137.) will have two obtuse angles and one acute, and consequently the sides thereof, by this proposition, are of the same species as their opposite angles: but the supplements of these sides are the *angles* of the original triangle AHC, therefore it has two acute angles HAC, ACH, and one obtuse, viz. AHC.

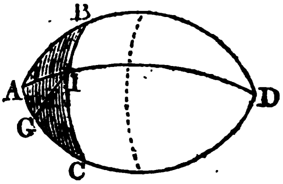
(I) COROLLARY III. The reverse of this proposition is true in all its parts. VIZ. *If the three sides be each less than a quadrant, each angle will be acute; if the three sides are each quadrants, the angles will be right-angles; and if the three sides are each greater than quadrants, the three angles will be each obtuse.*

These follow from considering the supplemental triangles. (U. 137.)

PROPOSITION XVIII.

(K) *If two sides of a spherical triangle be of the same species* *and the angle included acute, the third side is less than a quadrant; but if the two sides be of different species, and the included angle obtuse, the third side is greater than a quadrant.* OR,

When two angles are of the same species, and the included side greater than a quadrant, the third angle is obtuse; but if the two angles be of different species and the included side less than a quadrant, the third angle is acute.



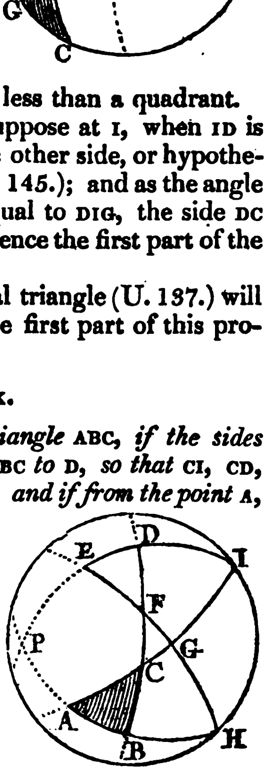
DEMONSTRATION. Let the triangles BAC and BDC be right-angled at A and D, then will BC be less than a quadrant (Q. 145.); but as the arc ABD approaches nearer to ACD, it will diminish the angles A and D, and CI will consequently be less than CB, which has been proved to be less than a quadrant.

In the triangle CID right-angled, suppose at I, when ID is greater and IC less than a quadrant, the other side, or hypothenuse, CD is greater than a quadrant (Q. 145.); and as the angle DIC increases, viz. when it becomes equal to DIG, the side DC increases till it becomes equal to DG; hence the first part of the proposition is evident. And,

In the second part, the supplemental triangle (U. 137.) will have exactly the same properties as the first part of this proposition. Q. E. D.

PROPOSITION XIX.

(L) *In any right-angled spherical triangle* ABC, *if the sides be produced, viz.* AC *to* I, AB *to* H, *and* BC *to* D, *so that* CI, CD, *and* AH *may be quadrants, or each* 90°; *and if from the point* A, *as a pole, the great circle* HGFE *be described, and from the point* C, *as a pole,* IDE *be drawn of such a length that* GE *may be a quadrant; then shall the triangles* CGF *and* EDF *have their respective sides and angles either equal to those of the triangle* ABC, *or they will be complements of each other; and eight right-angled spherical triangles will be formed, having (every two of them) equal angles at their bases.*



DEMONSTRATION. Since A is the pole of the circle HGFE, AG and AH are quadrants and perpendiculars to FGH (H. 133.): and the arcs FGH, FCB, being each of them perpendicular to ABH, are each of them quadrants, and F is the pole of ABH: therefore the triangle CGF is right-angled at G; CG is the complement of AC, CF of BC; BH, the complement of AB, is the measure of the angle F (D. 133.); and GH, the complement of FG, is the measure of the angle A.

Secondly. The arc GE being a quadrant, and perpendicular to CGI, E is therefore the pole of CGI, and EI is a quadrant. The arcs CFD and CGI being quadrants, for C is the pole of IDE, are perpendicular to IDE, therefore the triangle EDF is right-angled at D; ED is the complement of ID, and ID measures the angle ACB equal to DCI (N. 135.); DF is equal to BC; the hypothenuse EF is equal to the angle A, for it is equal to GH the measure of A, and for the same reason FED is equal to AC, and EFD is equal to the complement of AB. *Lastly*,

The right-angled triangles ABC, AHG, have the angle A common; FGC, FHB have the angle F common; CGF, CID have the angle C common; and EDF, EIG have the angle E common. Q. E. D.

CHAPTER II.

THE STEREOGRAPHIC PROJECTION OF THE SPHERE.

(M) 1. *Definition. The Sterographic projection of the sphere*, is such a representation of the various parts of its surface, on the plane of one of its great circles, as would be formed by lines drawn from the pole of that great circle, to every point of the circle, to be projected, viz.

Conceive a point E (Plate IV. Fig. 8.) situated any where on the surface of a globe, and at the same time a plane CD, to stand at right angles to an imaginary line EO, connecting the centre of the globe and the point E. Then, if an indefinite number of lines be supposed to be drawn from the point E, to every point of all the circles described upon the spherical surface, they will trace out upon the cutting plane CD a stereographical projection of the sphere.

(N) 2. The *plane* on which the points, lines, and circles of the sphere are represented, is called the *plane of projection*; and the point from which all the lines are drawn through the several parts of the circles of the sphere, to this plane, is called the *projecting point*.

(O) 3. *The primitive circle* is situated in the plane of projection; and the projecting point, on the sphere, is one of the poles of this circle; but on the plane of projection, the poles of the primitive are in its centre.

(P) 4. A circle, the plane of which is parallel to the plane of the primitive, is called *a parallel circle*, and is represented on the plane of projection, by a circle parallel to, and comprehended within, the primitive.

(Q) 5. A circle, whose plane is perpendicular to the plane of the primitive, is called a *right circle;* because, passing through the projecting point, its circumference becomes a straight line on the plane of projection.

(R) 6. A circle whose plane is oblique to the plane of the primitive, is called an *oblique circle.*

(S) 7. Lines drawn from the projecting point to every part of the circumference of a circle to be projected, will form the convex surface of a cone, whose vertex is the projecting point. Thus if AB be a diameter of a circle to be projected, and E the projecting point, then AEB will be a cone, of which E is the vertex.

(T) 8. All writers on conic-sections have demonstrated, that if a cone be cut by a plane *ab* parallel to the base AB, the section will be a circle; hence it follows, that all small circles parallel to the plane of projection will become circles in the projection. And the radii of all projected small circles, parallel to the primitive, will be equal to the semi-tangent of their distances from the remotest pole; thus $ao = ob$ is the tangent of the angle aEO, or the semi-tangent of the arc Ae where e is the remotest pole.

(U) 9. It is also shewn by writers on conic-sections that if a scalene, or oblique cone, be cut by a plane not parallel to the base, but in such a manner that the cone cut from the base towards the vertex be equi-angular with the original cone, the section shall be a circle.

Thus (Plate IV. Fig. 9.) if the cone ABE be so cut by a plane CD, that the angle CDE be equal to the angle BAE, and DCE equal to ABE, then the section will be a circle, having CD for its diameter. And this is what is termed cutting a cone in a *sub-contrary position.*

Emerson's Conic Sections, Book I. Prop. 89th.

PROPOSITION I. *(Plate IV. Fig. 10.)*

(W) *Every circle of the sphere which does not pass through the poles of the primitive, is projected into a circle.*

Let DC be the diameter of a circle to be projected on the plane of the primitive FB, from the point E. Lines from the point E to the circumference of that circle form a cone, whose triangular section is CED.

Now the extremity C, of the diameter CD, will be projected into the point A, and the other extremity D into the point B in the plane of the primitive; therefore AB is the projected diameter.

Through D draw D*d* parallel to FB, that is, perpendicular to E*e*, then will the arc E*d* be equal to the arc ED; *and since an angle at the circumference of a circle is measured by half the arc on which it stands, (Keith's* Euclid 167 *of* VII). the angle ECD will be equal to the angle *d*DE, or, which is the same thing, equal to the angle ABE, because AB is parallel to *d*D.

Hence the cone ECD is equiangular with the cone EBA, and it is cut by the line AB in a *subcontrary position*, therefore (U. 153.) the section is a circle.

PROPOSITION II. *(Plate IV. Fig.* 11.*)*

(X) *The distance of the centre of any oblique great circle from the centre of the primitive, is equal to the tangent of the angle formed on the sphere, by that circle and the primitive; and the radius of the projected circle is equal to the secant of the same angle.*

Let E be the projecting point, FG the diameter of the primitive EF*e*G, and CD the diameter of a circle to be projected. The point C will appear at A, and D at B; therefore, by the 1st proposition, AB is the projected diameter. Bisect AB in *p*, and *p* will be the centre of the projected circle *e*AEB.

Now CED = CEB, being an angle in a semi-circle, is a right angle; and EPA and EPB are likewise right angles; hence the triangles AEB, APE and EPB are equiangular and similar. The angle *e*PC is double of the angle *e*EC, the one being at the centre and the other at the circumference of the circle, and for the same reason A*p*E is double of ABE; but the angle *e*EC has been shewn to be equal to ABE, therefore *e*PC = P*p*E, and CPF = PE*p*: and it is plain that P*p* is the tangent of PE*p* to the radius of the sphere, therefore it is equal to the tangent of the angle CPF, which is the spherical angle formed by the circle CD and the primitive FG.

It is very obvious that A*p* = E*p*, the radius of the projected circle, is the secant of the angle PE*p* or CPF.

PROPOSITION III. *(Plate IV. Fig.* 11.*)*

(Y) *The distance of the pole of any oblique great circle from the centre of the primitive, is equal to the semi-tangent of the angle which that circle makes on the sphere with the primitive.*

Let E be the projecting point, FG the diameter of the primitive, and CD the diameter of the circle to be projected.

Make *e*M = CF, then CM will be a quadrant; draw MN, which will be the axis of CD; and M, N, its poles. If EM be joined, the pole M will appear at *m* in the primitive; but P*m* is the tangent of the angle PE*m*, or the semi-tangent of *e*PM, or of its equal CPF.

Likewise a line drawn from E through N, the other pole will cut PF, produced beyond F, in the exterior projected pole. Now PF, produced to meet EN, is evidently the tangent of the angle NE*e*, or the semi-tangent of NP*e*, which is the supplement of the arc *e*M or of the arc CF.

(Z) COROLLARY. *It is also obvious that* (AP) *the distance of the projected circle from the centre of the primitive, is the semi-tangent of the complement of* (CF) *the circle's inclination to the primitive.*

For AP is the tangent of the angle AEP, which is the half of CP*e*, the complement of FPC.

PROPOSITION IV. *(Plate IV. Fig. 12.)*

(A) *The distance of the centre of a small circle (perpendicular to the primitive) from the centre of the primitive, is equal to the secant of its distance from its own pole; and the radius thereof is equal to the tangent of that distance.*

Let CD be the diameter of a small circle cutting the primitive at right angles, or, which amounts to the same thing, whose poles G and F lie in the circumference of the plane of projection. Now AB is the diameter of its representative on the plane of projection FG; bisect AB in *p*, and describe the small circle ACBD; draw *p*C, CP, and *e*CB; then I say *p*C the radius of the small circle, is the tangent of the arc GC, the distance of the small circle from its pole G; and P*p* is equal to the secant of the same arc.

The angle PEA is equal to the angle CBA, because EC*e* being in a semi-circle is a right angle, consequently ECB is also a right angle; the angle EPA is a right angle, and PAE=CAB, therefore the remaining angle PEA=CBA,=*p*CB, for *p*C=*p*B; but the angle PCA is equal to PEA, because PC is equal to PE, hence PCA=*p*CB, therefore the angle ACB is equal to the angle PC*p*, but ACB, is a right angle, therefore PC*p* is a right angle, and *p*C a tangent to the point C, or arc GC; hence P*p* is the secant thereof.

PROPOSITION V. *(Plate IV. Fig. 13.)*

(B) *The tangent of any arc of a great circle on the surface of the sphere, drawn from any point towards the plane of projection, will be projected into an equal straight line;—the secant* *of the same arc will be represented by a line equal to itself: and any angle comprehended between two great circles, passing through the same point, on the surface of the sphere, is equal to the angle comprehended between their representatives on the plane of projection.*

If the sphere be cut in halves by a plane ABCD passing through its centre, and extending to an indefinite distance on all sides, in an horizontal direction; and if from any point whatever (except that point exactly perpendicular to the plane) any number of tangents be drawn towards the plane, they will all fall upon it. For, if this plane be supposed to coincide with the plane of projection, or, which is the same thing, to be perpendicular to the projecting point E, it is evident that the *secant* PB, of any arc KG, will fall wholly within the plane, and be projected into a line equal to itself; but the secant and tangent of any arc will terminate in the same point; therefore one extremity of the tangent must necessarily fall upon the plane.

Let G be a point on the sphere, GB and GC tangents to that point, meeting the plane in B and C, and let E be the projecting point; then it is evident that GB will be projected into FB, and GC into FC; and I say that GB=FB, GC=FC, and the angle CGB, the measure of the spherical angle IGK (R.135.), is equal to the angle CFB formed on the plane of projection.

For, let HG be drawn parallel to AK, and let EP*e* be drawn through the centre P, then OG=OH (Euclid 3 *of* III.), and the angle EGH is equal to the angle EHG. But the angle EGB is equal to the angle EHG (Euclid 32 *of* III.) therefore the angle EGB or FGB is equal to the angle EGH or FGH.

But HG is parallel to AK, and the angles FGH and GFK are equal to each other, being alternate angles (Euclid 29 *of* I.), therefore the angle FGB is equal to the angle GFB, and consequently (Euclid 6 *of* I.) BG is equal to BF.

Exactly in the same manner it may be proved that CG=CF.

Now the triangles CGB and CFB have the two sides GB and GC, equal to the two sides FB and FC, and they stand on the same base BC, therefore (Euclid 8 *of* I.) the angle CGB is equal to the angle CFB, which was to be demonstrated.

PROPOSITION VI. *(Plate IV. Fig. 11.)*

(C) *The projected extremities of the diameter of any circle on the sphere, inclined to the plane of projection, are in that diameter of the primitive which is perpendicular to the projecting point: and distant from the centre of the primitive the semi-tangents of the circle's nearest and greatest distance from that pole of projection, opposite to the projecting point.*

For the diameter CD of a circle on the sphere inclined to the plane of projection in an angle CPF, is projected into the line AB from the projecting point at E. But AP is the semi-tangent of *ec* the circle's nearest distance from the pole *e*, and PB is the semi-tangent of *e*D the circle's greatest distance from the pole *e*. Hence the diameter AB is the sum of these semi-tangents, and P the centre is half the sum.

(D) COROLLARY I. *The projected diameter of a small circle inclined to the plane of projections and encompassing the pole* (e) *of the plane of projection, is equal to the sum of the semi-tangents of that circle's greatest and least distance from the pole of projection.*

For such a small circle may be conceived to be parallel to the great circle, and therefore its projected diameter will fall on both sides of the pole P of the primitive.

(E) COROLLARY II. *The projected diameter of a small circle inclined to the plane of projection, and situated wholly on one side of the pole* (e), *is equal to the difference of the semi-tangents of that circle's greatest and least distance from the pole* (e) *of projection.*

For a small circle on the sphere whose diameter is MO, will be projected into *mp*; P*p* is the semi-tangent of *e*O, the circle's greatest distance from the pole; P*m* the semi-tangent of *e*M, the circle's least distance from the pole; and the difference between these semi-tangents is *mp*.

(F) COROLLARY III. *The points where the projected diameter of a great circle terminates, are distant from the centre of the primitive, the tangent and co-tangent of half the complement of the circle's inclination to the primitive.*

For C*e* is the complement of FC the circle's inclination to the primitive, and AP is the tangent of the half of C*e*, or the angle AEP. And, because AEB is a right angle, PEB is the complement thereof; but PB is the tangent of PEB, that is, the tangent of the complement of AEP, or tho co-tangent of ½ C*e*.

PROPOSITON VII. (*Plate V. Fig.* 5.)

(G) *In any projected great circle, inclined to the primitive, the radius of the projected circle, is to the radius of the primitive, as the distance of the projected pole from the centre of the projected circle, is to the distance of the projected pole from the centre of the primitive.*

Let EA*e*BC be the projection of a great circle cutting the primitive ENF*e*G in E and *e*, and the line FG produced, in A and C; and let P be the centre of the primitive, and O the centre of the projected circle; *p* the internal, and *r* the external pole of the projected circle. I say

EO : EP :: *p*O : *p*P; or, EO : EP :: *r*O : *r*P.

Join EO, E*p*, and E*r*; then the angle PEO is the measure of the circle's inclination to the primitive: for PO is the tangent of PEO to the radius of the sphere, and it is likewise the tangent of the circle's inclination to the primitive. (X. 154.)

The angle PE*p* is the measure of half the circle's inclination to the primitive (Y. 154.), therefore E*p* bisects the angle PEO.

And, because E*p* is perpendicular to E*r* (for *p* and *r* are the two poles), the angle *r*E*p* is equal to the angle *p*EK, being each of them a right angle; also PE*p* and *p*EO have been shewn to be equal, therefore the angle *r*EP is equal to the angle OEK.

Consequently E*p* makes equal angles internally, and *r*EK equal angles externally with the sides EP and EO of the triangle PEO. Therefore (by Dr. *Simson's* or *Keith's* Euclid 3 *of* VI. and Proposition A.)

EO : EP :: *p*O : *p*P; and, EO : EP :: *r*O : *r*P.

(H) Corollary I. *The line* *r*O, *comprehended between the centre of the projected circle and its external pole, is harmonically** *divided by the internal pole of the projected circle, and the centre of the primitive.*

For EO : EP :: *p*O : *p*P
And EO : EP :: *r*O : *r*P
Therefore, *r*O : *r*P :: *p*O : *p*P.

(I) Corollary II. *Every projected great circle, making an angle with the primitive, is greater than the primitive.*

For the radius EO of the projected great circle EA*e*BC, is the secant of that circle's inclination to the primitive (X. 154.); and therefore is always greater than EP, the radius of the primitive.

PROPOSITION VIII. *(Plate V. Fig. 6.)*

(K) *If through the internal projected pole of a great circle, and the extremities of an arc of that projected circle, straight lines be drawn to cut the primitive; the intercepted arcs of the projected circle and the primitive will be similar.*

Let EF*e*G represent the primitive circle, and ELA*ehf* the projection of a great circle inclined to the primitive; P the pole of the primitive, and *p* the internal pole of the projected great circle; draw *p*A, *p*L, *p*K, and *p*G, then will the arc GK be similar to the arc AL.

For, it is obvious that if a straight line be drawn from a point within a circle to meet the circumference, the angle formed at the circumference between this line and the radius, will be less than a right angle: therefore in the triangles PK*p*, *p*LO, the angles PK*p* and *p*LO are each less than a right angle. The angle P*p*K is equal to the angle L*p*O (Euclid 15 *of* I.), and the sides about the remaining angles at P and O are proportional. For OL : PK : : O*p* : P*p* (G. 157.) Hence PK : P*p* : : OL : O*p*, therefore the triangles *p*PK and *p*OL are equiangular and similar (Euclid 7 *of* VI.), and consequently the angle KP*p* is equal to the angle *p*OL.

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* When a straight line is divided into three such parts, that *the whole is to the first part, as the third part is to the second*; it is said to be divided in harmonical ratio.

But equal angles at the centres of circles are subtended by similar circumferences, therefore the arc GK is similar to the arc AL; and whatever part of the circumference of the projected circle AL is, the same part of the circumference of the primitive circle will GK be. For OL : PK : : the arc AL : the arc GK.

(L) COROLLARY I. *If straight lines be drawn from the projected pole of a great circle (towards the pole of the primitive) to cut the projected great circle and the primitive; the arc of the primitive intercepted between these lines, will be the measure of an arc on the sphere, represented by that part of the projected great circle intercepted between these lines.*

Let *p*F, *p*H, and *p*E be drawn; FH will be the measure of a part of a great circle on the sphere represented by AL; and FE will be the measure of a part of a great circle on the sphere represented by AE; for, join OE and P*m*, then OL : PK : : arc AL : arc GK (K. 158.), and OE : P*m* : : arc AE : arc G*m*. (K. 158.) But OE = OL and P*m* = PK, therefore arc AL : arc GK : : arc AE : arc G*m*.

Hence, whatever part of the great circle EA*ehf* is represented by AE, the same part of the primitive G*m* must represent; but AE is the projection of a quadrant of a circle on the sphere, having the same radius as the primitive, therefore it must be measured by FE, and not G*m*; for the same reason AL must be measured by FH and not by GK.

(M) COROLLARY II. *Hence it follows, that all great circles inclined to the primitive, have equal arcs on the sphere, represented by unequal arcs on the plane of projection.*

PROPOSITION IX. *Problem.* (*Plate IV. Fig.* 14.)

(N) *To find the pole of any great circle.*

I. The pole of the primitive *a*EBD is the centre P. (O. 152.)

II. Let *a*B be a right circle. Through P the centre of the primitive draw DE perpendicular to *a*B, then D and E are the poles of *a*B.

III. Let CA*e* be an oblique circle. Through the centr primitive draw CP*e*, and *m*P*n* at right angles to it, cu oblique circle in O; draw *ev* through akes a of 90 degrees; join *ew*, then *p* is the

Or, the semi-tangent* of the complement of PO set from P to p, will give the pole. (Y. 154.)

The pole of a small circle is the same with the pole of its parallel great circle.

(O) The truth of this proposition is shewn at Y. 154.; for on, the measure of the angle C, is the complement of PO, and the semi-tangent of on is applied from P to p.

(P) PROPOSITION X. *Problem.* *(Plate IV. Fig. 14.)*

Through any given point in the circumference of the primitive circle, to describe a great circle making any given angle with the primitive.

Let EaDB be the primitive circle, and C the given point.

Through C draw CPe, and mPn at right angles to it; make the angle PCr equal to the given angle, and draw Cr; with r as a centre, and distance rC, describe the circle CAe, then ACB will be equal to the given angle.

Or, Make Pr equal to the tangent of the given angle to the radius CP; or make Cr the secant thereof.

Or thus: Set off the semi-tangent of the complement of the given angle from P to O, and through the three points COe describe a circle.

(Q) These constructions are obvious from X. 154. and Z. 155.

(R) PROPOSITION XI. *Problem.* *(Plate IV. Fig. 15.)*

Through a given point A *in a right circle* AB, *to describe a great circle* CAe, *making an angle at* A, *equal to a given number of degrees.*

Draw DE at right angles to aB; find the centre S of a circle which will pass through the three points DAE. Through S draw dsm parallel to DE; with the centre A, and a radius equal to PB, describe an arc ww, set off bo from a scale of chords equal to the complement of the given angle A, and draw Aod; then d is the centre of the circle CAe required.

(S) The truth of this construction may be shewn thus: S is the centre of the circle DAE, and since all great circles cut each other at a semi-circle's distance (C. 133.), all circles passing through A must necessarily pass through the point G, and therefore their centres must be somewhere in the line dm. Now dAe is a right angle, therefore SAe=CAB (N. 135.) taken from dAe, leaves dAS the complement of the angle CAB, agreeably to the construction.

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* The common plain scale generally contains a line of semi-tangents, marked S. T. See Book I. Chap. IV. page 16.

(T) PROPOSITION XII. *Problem.* (*Plate IV. Fig.* 16.)

Through a given point C, *within the primitive, to draw a great circle* AC*e making a given angle* BAC *with the primitive.*

From the centre P of the primitive, with the tangent of the given angle as a radius, describe an arc; and from the point C, with the secant of the same angle as a radius, cross it in O. The point O is the centre, and OC is the radius of the circle AC*e* required.

(U) The truth of this construction is shewn at X. 154.

PROPOSITION XIII. *Problem.* (*Plate IV. Fig.* 17.)

(W) *Through two given points within the primitive, to draw a great circle.*

Let O and *m* be the two points. Through one of the points, as O, and the centre of the primitive, draw AB of an unlimited length towards B, and draw CD at right angles to AB. Join CO, and produce it till it cuts the primitive in *v*, through *v* draw the diameter *vw*, meeting the circumference in *w*, and from C through *w*, draw C*w*E, meeting AB in E. Then a circle described through the three points O, *m*, E, will be the great circle required.

(X) The angle *v*C*w* being in a semi-circle is a right angle (Euclid 31 *of* III.), therefore the distances PO, and PE, measured on the line of semi-tangents are together equal to 180 degrees, consequently the point E is diametrically opposite to O, and O*m*E is a great circle.

PROPOSITION XIV. *Problem.* (*Plate IV. Fig.* 18.)

(Y) *Through a given point, in any projected great circle, to draw another great circle perpendicular to the given one.*

General Rule. Find the pole of the given circle (N. 159.) and through that pole and the given point draw a great circle (W. 161.) For (I. 134.) if one great circle pass through the pole of another, it cuts it at right angles.

Or, I. The pole of the primitive is in the centre, therefore a diameter of the primitive always cuts its circumference at right angles.

II. *To draw an oblique circle perpendicular to an oblique circle.*

Let BAD be the given oblique circle and A the given point, find *p* the pole of the oblique circle BAD (N. 159.), and through the points *p*, and A, draw the great circle ECA*p*. (W. 161.)

III. *To draw an oblique circle perpendicular to a right circle.*

Let HG be the right circle and A the given point, draw DB at right angles to HG, then B, and D, are the poles of HG;

through the three points B, A, D, draw the circle BAD, and it will cut HG at right angles.

PROPOSITION XV. *Problem. (Plate IV. Fig.* 19, 20, *and* 21.*)*

(Z) *About any given point, as the pole of a great circle, to describe a small circle at a given distance from that pole. Or, at a proposed distance from a given great circle to describe a parallel circle.*

I. If the small circle be parallel to the primitive. With the semi-tangent of its distance from the pole P, (Plate IV. Fig. 19.) or radius PC, describe the circle CDE.

II. If the small circle be parallel to a right circle. Let the given right circle be AB, whose poles are C and D. (Plate IV. Fig. 20.) Set off the chord of the small circle's distance from its pole, from C to *n* and *m*; or the chord of its distance from the right circle AB, from B to *n*, and from A to *m*; and draw A*m*F and A*on*, where these lines intersect the axis PF, in *o* and F, will give the two extremities of the diameter of the circle *mon* to be projected, the middle point E will be the centre.

Or, Having found the points *m* and *n*, draw P*n*, and *n*E at right angles to it, and it will cut the axis PC produced in E, the centre of the parallel circle *mon*. (A. 155.)

Or, Find the three points *m*, *o*, and *n*, as above, and draw a circle through them.

III. If the small circle be parallel to an oblique circle. Let the oblique circle be DAB. (Plate IV. Fig. 21.) Find *p* the pole of the oblique circle DAB, and through *p* draw D*p*E; make E*v* and E*w* each equal to the proposed distance from the pole, and draw D*w* and D*v*, cutting the axis GF produced in *o* and F, the two extremities of the diameter of the circle *mon* to be projected, the middle point C will be the centre.

Or, Find *p* the pole of the oblique circle DAB (N. 159.), and measure its distance P*p* from the centre of the primitive by a scale of semi-tangents. Then add and subtract this distance to and from the complement of the parallel circle's distance from the oblique circle; set off the sum, by a scale of semi-tangents, from P to F, and the difference from P to *o*; the middle point between *o* and F, (*viz.* C) is the centre. (D. 157.)

PROPOSITION XVI. *Problem. (Plate V. Fig.* 1.*)*

(A) *About any given point, as the pole of a great circle, to describe a great circle in a given primitive circle.*

I. If the given point be (P) in the centre of the primitive, the primitive is the great circle required.

II. If the given point be (C) in the circumference of the primitive, through C, draw a diameter CP*e*, and another *b*PB at right angles to it. Then, *b*PB is the great circle required.

III. If the given point be (*p*), neither in the centre nor in the circumference of the primitive. Through *p*, and the centre of the primitive, draw a straight line *r*PB, and cross it at right angles with the diameter *e*C, through *p* draw *epw*, make *wm* equal to *w*C, and through *m* draw *emr*, then with *r* as a centre and radius *re*, describe the required great circle CA*e*.

(B) This problem is easily deduced from X. 154.

PROPOSITION XVII. *Problem. (Plate V. Fig. 2.)*

(C) *To measure any arc of a great circle.*

General Rule. Find the pole of the given circle (N. 159.), from which draw straight lines through the ends of the arc to be measured, cutting the primitive in two points, the distance between these points applied to a scale of chords, will give the measure of the arc.

Or, I. The pole of the primitive is the centre P, therefore any arc of the primitive is measured by taking the extent of the arc (as *be*) and applying it to a scale of chords.

II. To measure any part of a right circle as P*a*, P*p*, P*v*, &c. From C the pole, draw *caf*, CP*e*, *cpg*, *cvw*; then *ef* applied to a scale of chords is the measure of P*a*, *eg* of P*p*, and *gw* of P*v*.

Or, P*a*, P*p*, P*v*, applied to the line of semi-tangents, will give their respective measures. And, it is evident that *vp* is the difference between the measures of P*v*, and P*p*, and that *ap* is the sum of the measures of P*a* and P*p*.

III. To measure any part of an oblique circle as *cr*, *ro*, *oa*, &c. Find *p* the pole thus: draw *caf*, make *fg* an arc of 90 degrees, and join *gc*, then *p* is the pole. From *p* through *r*, *o*, *a*, *m*, draw *p*R, *po*, *p*A, *p*M. Then RO, applied to a scale of chords, is the measure of an arc on the sphere represented by *ro*; OA is the measure of an arc on the sphere represented by *oa*; and AM is the measure of an arc on the sphere represented by *am*, &c. (L. 159.)

PROPOSITION XVIII. *Problem. (Plate V. Fig. 2.)*

(D) *To cut any number of degrees from the arc of a great circle; or, from a given point, in a projected great circle, to cut off an arc equal to a given arc.*

This proposition is the reverse of Prop. XVII.

I. If the projected great circle be the primitive whose pole is P, and *e* the given point. Take the number of degrees in the given arc from a scale of chords and apply the extent from *e* to *b*, then *eb* is the arc required.

II. If the projected great circle be a right circle (as *x*A) whose pole is C, and *a* the given point. From the pole C, through *a*, draw C*af*, make *fg* equal to the given arc, by a scale of chords, and join C*g*, then will *pa* contain the required number of degrees.

Or, any number of degrees may be cut from *x*A by the scale of semi-tangents, in the same manner as the arcs were measured in the XVIIth Problem.

III. If the projected circle be an oblique circle (as C*ae*) whose pole is *p*, and *o* the given point. From *p* through *o* draw *po*O, make OR equal to the given number of degrees, and join *p*R, then *or* will contain the number of degrees required.

PROPOSITION XIX. *Problem. (Plate V. Fig. 3.)*

(E) *Any great circle* CA*e in the plane of projection being given: to describe another great circle* BAD, *which shall cut the given circle* CA*e*, *and also the primitive in any assigned angles.*

About the pole P of the primitive describe the parallel circle *no*, at a distance equal to the angle which BAD is required to make with the primitive (Z.162). About *p*, the pole of CA*e*, at a distance equal to the measure of the angle which BAD is required to make with CA*e*, describe the parallel circle *ab* (Z. 162.) cutting *no* in *d*.

About *d* as a pole describe the great circle BAD (A. 162.), cutting the primitive in B, and CA*e* in A: then BAD is the great circle required.

(F) This construction is evident from Q. 135.

PROPOSITION XX. *(Problem. Plate V. Fig. 4.)*

(G.) *To measure any spherical angle.*

General Rule. Find the poles of the two great circles which form the angle (N. 159.) Straight lines drawn from the angular point through these poles, will cut the primitive in two points; the distance between which, applied to a scale of chords, will give the measure of the required angle.

The truth of this appears from Q. 135.

Or, I. If the angular point be at P, and *r*P*v*, the angle to be measured, then P*r*, P*v* are quadrants; and as a spherical angle is always measured on the arc of a great circle at a quadrant's distance from the angular point (R. 135.), *rv* applied to a scale of chords is the measure of the angle *r*P*v*.

II. If the angle be formed by the primitive, and an oblique circle, as *o*CE.

Find *m* the pole of CA*e*, and from C draw C*mw*; and through P, the pole of the primitive, draw CP*e*, then *we* applied to a scale of chords gives the measure of *o*CE. To measure *o*CP, the angle formed by the right circle CP*e*, and the oblique circle CA*e*; through *m*, the pole of CA*e*, and *r*, the pole of CP*e*, draw C*w*, and C*r*; then *wr* applied to a scale of chords is the measure of *o*CP.

Or, The angle *o*CE may be measured by the line of semi-tangents; thus, let *o*E be applied to the scale of semi-tangents from 90 towards the left hand, the number of degrees contained between the points of the compasses, will be the measure of *o*CE; and P*o*, applied to the scale of semi-tangents, from the beginning of the scale towards 90, will give the measure of *o*CP.

III. If the angle be formed by two oblique circles, as CA*e* and BAD.

Find *m* the pole of CA*e*, and *n* the pole of BAD (N. 159.) From the angular point A, through *m* and *n*, draw A*ms* and A*nv*, cutting the primitive in *s*, and *v*; then *sv*, applied to a scale of chords, will give the measure of the angle BAC, or of its equal *e*AD.

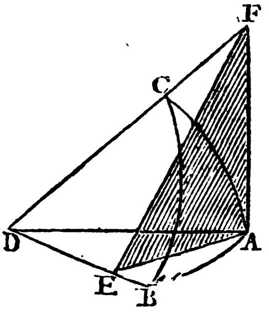
CHAP. III.

INVESTIGATION OF GENERAL RULES FOR CALCULATING THE SIDES AND ANGLES OF RIGHT-ANGLED SPHERICAL TRIANGLES.

PROPOSITION XXI.

(H) In any right-angled spherical triangle.

Radius, is to the sine of any side; as the tangent of the adjacent angle, is to the tangent of the opposite side.



DEMONSTRATION. Let ABC be a spherical triangle, right-angled at A.

rad : sine AB :: tang ∠ B : tang AC.

For, let D be the centre of the sphere, and draw the radii DC, DA, and DB; also from the right-angle A, in the plane DBA, draw AE perpendicular to DB, and it will be the sine of the arc AB.

At the point E, in the plane DBC, draw EF perpendicular to DB, then will the angle FEA be the inclination of the planes DBA and DBC, and consequently (D. 133.) equal to the spherical angle CBA.

Draw AF from the point A, a tangent to the arc AC, and produce the radius DC to F; then since the arc AC is perpendicular to the plane DBA (for by hypothesis it cuts the arc AB at right-angles) AF will be perpendicular to AE.

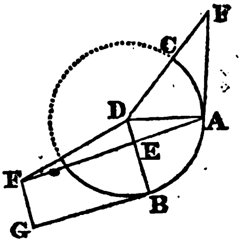
Because AC is less than a quadrant, and DA, DB, and AE, are in the same plane DBA, and that EF is at right-angles to DB (by construction); DC cannot likewise be at right-angles to DB. Therefore DC and EF being in the same plane DBC, and not parallel to each other, must meet.

The plane triangle FAE is right-angled at A; the perpendicular AF is a tangent to the spherical perpendicular AC; the base AE is the sine of the spherical base AB; and the angle FEA is equal to the spherical angle ABC.

Hence, rad : AE :: tang FEA : AF.

That is, rad : sine AB :: tang ABC : tang AC. Q. E. D.

(I) Scholium.



The foregoing proposition is of considerable importance in spherical trigonometry, and therefore ought to be clearly understood.—It will perhaps be of advantage to some students, to illustrate the nature of it in a mechanical way; in order to render the preceding demonstration more perspicuous.

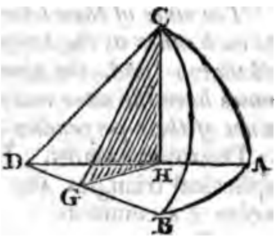
Upon a piece of pasteboard, with any radius describe a circle, make AC of any length less than a quadrant; draw AF perpendicular to the radius AD, and draw DCF through C; let AB be of any length less than a semi-circle, and draw DB. From A draw AEF cutting BD at right-angles, make DF equal to DCF, and draw BG a tangent to BD.

Then cut out the figure FABGFDCF, next cut the back of the line AD half through, do the same with the line DB, and raise AF till it is perpendicular to the plane ADB; then raise up the plane DEBGF, so that the points F may coincide, and you will have a plane triangle FAE right-angled at A, which will shew the nature of the problem as clearly as possible.

PROPOSITION XXII.

(K) In any right-angled spherical triangle.

Radius is to the sine of the hypothenuse, as the sine of any angle is to the sine of its opposite side.



DEMONSTRATION. Let ABC be a spherical triangle, right-angled at A.

rad : sine BC : : sine ∠B : sine AC.

For, let D be the centre of the sphere, draw the radii DC, DB, and DA.

In the plane DBC draw CG perpendicular to DB, and it will be the sine of the hypothenuse BC.

From the point G in the plane DBA, draw GH perpendicular to DB, and join CH; then (H. 165.) GHC will be a right-angle, and the angle CGH will be equal to the spherical angle ABC.

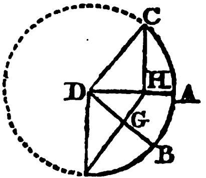
The plane triangle GHC is right-angled at H, the hypothenuse GC is the sine of the spherical hypothenuse BC; the perpendicular CH is the sine of the spherical perpendicular AC, and the angle CGH is equal to the spherical angle ABC.

Hence,

Radius : GC : : sine CGH : CH; that is,
Radius : sine of the hypoth. BC : : sine of the angle CBA : Sine of the perpendicular AC. Q. E. D.

(L) SCHOLIUM.

This proposition may perhaps be rendered more familiar to the understanding by a mechanical illustration.

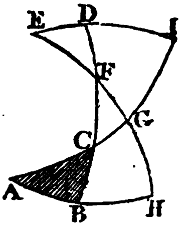


Upon a piece of pasteboard, with any radius describe a circle, make BC, the hypothenuse of any length less than a quadrant; and draw the radii BD, CD; from C draw CG at right-angles to BD; set off AB of such a length that when CG is produced it may cut AD somewhere, as in H; at H erect the perpendicular HC, and let DC be drawn.

Cut out the figure CDCABC, then cut the lines AD and BD half through, as in the XXIst proposition, raise the plane ADC perpendicular to the plane ADB, and the plane BDC, till the points C coincide; you will then have a right-angled plane triangle CHG, from which the whole proposition will appear exceedingly simple and easy.

PROPOSITION XXIII.

(M) In any two right-angled spherical triangles, having th same acute angle at the base.



The sines of their bases have the same ratio to each other as the tangents of their perpendiculars; and, the sines of their hypothenuses have the same ratio to each other as the sines of their perpendiculars.

Demonstration. Let the right-angled spherical triangles ABC and AHG have the acute ∠ A common.

Radius : sine AB :: *tang.* A : tang. BC. (H. 165.)
Radius : sine AH :: *tang.* A : tang. HG. (H. 165.)
Therefore, sine AB : sine AH :: tang. BC : tang. HG.
Radius : sine AC :: *sine* A : sine BC. (K. 166.)
Radius : sine AG :: *sine* A : sine HG. (K. 166.)
Hence, sine AC : sine AG :: sine BC : sine HG. Q. E. D.

SCHOLIUM.

(N) The different cases or varieties that may happen in the solution of right-angled spherical triangles, wherein two things, together with the right-angle, are always given to find a third, are in all sixteen, and from this proposition alone (by producing the sides of the triangle to quadrants (L. 151.) the whole may be solved. The five following corollaries, or cases, include the whole practice of right-angled spherical triangles, and are the foundation of Baron Napier's rules.

Case I. Sine AH : sine AB :: tang GH : tang BC (M. 167.)
Viz. rad : sine AB :: tang A : tang BC (L. 151.)
But rad : tang :: cot : rad (Z. 103.)
∵ Cot A : rad :: sine AB : tang BC
Also, rad : sine AC :: sine C : sine AB (K. 166.)

Case II. Rad : sine BC :: tang C : tang AB (H. 165.)
But, rad : tang :: cot : rad (Z. 103.)
∵ Cot C : rad :: sine BC : tang AB
Sine AG : sine AC :: sine HG : sine BC (M. 167.)
Viz. rad : sine AC :: sine A : sine BC (L. 151.)

Case III. Sine EI : sine ED :: tang IG : tang DF (M. 167.,
Viz. rad : cos C :: tang AC : tang BC (L. 151.)
But, rad : tang :: cot : rad. (Z. 103.)
∵ Cot AC : rad :: cos C : tang BC
Sine DC : sine ID :: sine CF : sine FG (M. 167.)
Viz. rad : sine C :: cos BC : cos A (L. 151.)
Or, rad : sine A :: cos AB : cos C

Case IV. Sine FH : sine FG :: tang BH : tang CG (M. 167.)
Viz. rad : cos A :: cot AB : cot AC (L. 151.)

But, rad : cot :: tang : rad (Z. 103.)
∴ Tang AB : rad :: cos A : cot AC
Sine CF : sine CD :: sine FG : sine ID (M. 167.)
Viz. cos BC : rad :: cos A : sine C (L. 151.)

CASE V. Sine CG : sine CI :: tang FG : tang DI (M. 167.)
Viz. cos AC : rad :: cot A : tang C (L. 151.)
But, rad : tang :: cot : rad (Z. 103.)
∴ Cos AC : cot A :: cot C : rad
Sine BH : sine CG :: sine FB : sine CF (M. 167.)
Viz. cos AB : cos AC :: rad : cos BC (L. 151.)

(O) If the extremes and means of each case be multiplied together, we shall obtain the same equations as are produced by Baron Napier's rules.

I. Rad × sine AB = tang BC × cot A
Rad × sine AB = sine AC × sine C
II. Rad × sine BC = tang AB × cot C
Rad × sine BC = sine AC × sine A
III. Rad × cos C = cot AC × tang BC
Rad × cos C = sine A × cos AB
IV. Rad × cos A = cot AC × tang AB
Rad × cos A = cos BC × sine C
V. Rad × cos AC = cot A × cot C
Rad × cos AC = cos AB × cos BC

(P) If we use the notation of Legendre *, and call the hypothenuse b, the base c, and the perpendicular a, and their opposite angles B, C, and A, the equations may be thus expressed.

I. Rad × sine c = tang a × cot A = sine b × sine C
II. Rad × sine a = tang c × cot C = sine b × sine A
III. Rad × cos C = cot b × tang a = cos c × sine A
IV. Rad × cos A = cot b × tang c = cos a × sine C
V.† Rad × cos b = cot A × cot C = cos c × cos a

Any two of these quantities, together with the right-angle, being given, the rest may be found.

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* Élémens de Géométrie, page 380.

† A late writer on trigonometry, after telling us in his preface that the well-known rules of Napier, called the FIVE CIRCULAR PARTS, are "too artificial and restricted to be generally employed in the present advanced state of the science;" gives the rules above, and actually solves all his cases of right-angled spherics by them, though they are exactly the same as those of Napier, but less commodiously expressed. And whatever improvements may have been made in trigonometry, either by the English or foreign mathematicians, it is a certain truth that "Mathematical science cannot boast of a neater compendium of results, or a more valuable aid to the memory of the student," than Napier's rules. *Vide* Monthly Review, Vol. LIII. page 280.

BARON NAPIER'S UNIVERSAL RULES FOR SOLVING RIGHT-ANGLED SPHERICAL TRIANGLES.

PROPOSITION XXIV.

(Q) I. *Radius × sine of the middle part = rectangle of the tangents of the extremes when conjunct.*

And,

II. *Radius × sine of the middle part = rectangle of the cosines of the extremes when disjunct.*

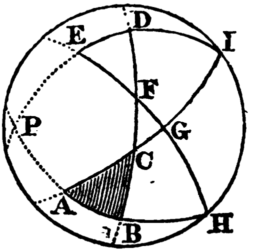
Observing to use the complements of the hypothenuse and angles.

EXPLANATION OF THE RULES.

In every right-angled spherical triangle there are FIVE CIRCULAR PARTS, exclusive of the right-angle, which is not taken into consideration: and these five parts are the hypothenuse, the two legs or sides, and their opposite angles; they are called circular parts, because the measure of each of them is the arc of a circle.

Now in every case proposed for solution, there are three of these five parts concerned, *viz.* two given and a third required.

It is therefore obvious that when the three parts follow each other in a successive order, the middle one is the *middle part*, and the two others the *extremes conjunct*, that is, joined to the middle part.



Suppose in the triangle ABC, that the angles A and C, and the hypothenuse AC are the parts concerned, it is evident AC is the middle part, and that A and C are joined, or adjacent to AC, and therefore are *conjunct*.

Again, if the three parts do not follow each other in a successive order, that which is not connected with either of the other two is *invariably* called the *middle part*; and the other two, which are not connected with it, are called the *extremes disjunct*; that is, not joined to the middle part.

Suppose the hypothenuse AC, the base AB, and the perpendicular BC, the parts concerned; AC will be the middle part, for it is not connected either with AB or BC, the angle A intervening in the former case, and the angle C in the latter; the sides AB and BC are *extremes disjunct*, that is, not joined to AC, because of the intervention of the angles A and C, but the sides AB and

BC are considered as joined though the right-angle B is between them, for, as we have before observed, it is excluded, or not taken into consideration.

It must be remembered that, in speaking of the hypothenuse or either of the angles as a middle part, or extremes conjunct or disjunct, their complements are to be used; but when we speak of the sides or legs, *their* complements are not to be used, but the *real* sides or legs.

Hence the middle part must *universally* be either the base AB, the perpendicular BC, the complement of the angle C, the complement of the angle A, or the complement of the hypothenuse AC, that is, it must be some one of the five circular parts.

(R) CASE I. FIRST, *let* AB, BC, *and the angle* A, *be the parts under consideration, in the triangle* ABC.

Here the three parts are joined together, because the right angle B is not regarded.—Therefore AB is the *middle part*, BC and the complement of A, are the *extremes conjunct.*

Hence, rad × sine AB = tang BC × cot A.

SECONDLY. *Let* AB, AC, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The base AB is still the *middle part*, for it is not connected either with AC or the angle C, therefore these parts are the *extremes disjunct;* the former being separated from AB by the angle A, and the latter by the side BC. Here we apply the second rule, recollecting that the cosine of the complement of an angle, or the hypothenuse, is the sine itself.

Hence, rad × sine AB = sine AC × sine C.

(S) CASE II. FIRST, *Let* AB, BC, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

This is exactly like the first part of the preceding case; the three parts are connected, BC is the *middle part*, AB and the complement of C are the *extremes conjunct.*

Hence, rad × sine BC = tang AB × cot C.

SECONDLY. *Let* BC, *the hypothenuse* AC, *and the angle* A, *be the parts concerned in the triangle* ABC.

This is similar to the second part of the preceding case; the perpendicular BC is still the *middle part*, for the angle C separates it from AC, and the side AB from the angle A: therefore AC and the angle A are *extremes disjunct*, that is, not joined to BC the middle part. Here we must apply the second rule, taking care to remember that the cosine of the complement of the hypothenuse, or an angle, is the sine itself.

Hence, rad × sine BC = sine AC × sine A.

(T) CASE III. FIRST, *Let* BC, AC, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The three parts follow each other, without the intervention of any other quantity; therefore the complement of C is the *middle part,* BC and the complement of AC are the *extremes conjunct,* that is, joined to the angle C.

$$\text{Hence, rad} \times \cos C = \cot AC \times \text{tang } BC.$$

SECONDLY. *Let* AB, *the angle* A, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The complement of the angle C is here the *middle part,* being separated from the angle A, by AC; and from the side AB, by the perpendicular BC; therefore, AB and the complement of the angle A, are the *extremes disjunct,* or not joined to C.

$$\text{Hence, rad} \times \cos C = \text{sine } A \times \cos AB.$$

(U) CASE IV. FIRST, *Let* AB, AC, *and the angle* A, *be the parts under consideration, in the triangle* ABC.

This is exactly similar to the first part of case 3d.

The complement of the angle A is the *middle part,* AB and the complement of AC, are the *extremes conjunct,* that is, they are joined to the angle A.

$$\text{Hence, rad} \times \cos A = \cot AC \times \text{tang } AB.$$

SECONDLY. *Let* BC, *the angle* A, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

This is exactly of the same nature with the second part of the 3d case. The complement of the angle A is the *middle part;* BC and the complement of C, are the *extremes disjunct,* the former being separated from the middle part A, by the base AB, and the latter by the hypothenuse AC.

$$\text{Hence, rad} \times \cos A = \cos BC \times \text{sine } C.$$

(W) CASE V. FIRST, *Let the hypothenuse* AC, *the angle* A, *and the angle* C, *be the parts under consideration, in the triangle* ABC.

The three parts follow each other; therefore the complement of AC is the *middle part,* the complement of A and the complement of C are the *extremes conjunct,* that is, they are joined to the middle part AC.

$$\text{Hence, rad} \times \cos AC = \cot A \times \cot C.$$

SECONDLY. *Let the hypothenuse* AC, *the base* AB, *and the perpendicular* BC, *be the parts under consideration, in the triangle* ABC.

The complement of AC is here the *middle part,* being separated from AB by the angle A, and from BC by the angle C; therefore AB and BC are the *extremes disjunct.*

$$\text{Hence, rad} \times \cos AC = \cos AB \times \cos BC.$$

SCHOLIUM.

The preceding cases include all the varieties that can possibly happen in the practice of right-angled spherical triangles.

Any of the equations may be turned into a proportion by putting the required term last, that with which it is connected first, and the other two in the middle in any order. These equations are exactly the same as those already given (O. 169.) and therefore Napier's rules are universally true.

CHAP. IV.

INVESTIGATION OF GENERAL RULES FOR SOLVING THE DIFFERENT CASES OF OBLIQUE SPHERICAL TRIANGLES, BY DRAWING A PERPENDICULAR FROM THE VERTICAL ANGLE UPON THE BASE.

PROPOSITION XXV.

Shewing the manner of applying BARON NAPIER'S *rules to oblique spherical triangles, from which several useful corollaries are deduced.*

(X) When the three *given* parts do not follow each other in a regular order, viz. when an unknown part intervenes, a perpendicular should always be drawn from the end of a *given* side, and opposite to an *adjacent given angle.*

But when the three *given* quantities follow each other without the intervention of an unknown quantity, the perpendicular should be drawn in such a manner, as to fall not only from the end of a *given* side and opposite to *an adjacent given angle,* but likewise from the end of a *required* sine, or opposite to a *required* angle, according as a side or an angle is the subject of enquiry.*

(Y) Having drawn a perpendicular, agreeably to the foregoing directions; then, if the vertical angle of the oblique triangle be given or sought, find the vertical angle of that right-angled triangle wherein two things are given; but, if the base of the oblique triangle be given or sought, find the base of that right-angled triangle wherein two quantities are given.

Compare the *perpendicular,* the part given, and the part sought, in that triangle wherein only *one* quantity is given; find the middle part, and make an equation agreeably to NAPIER's RULES, marking the term sought with an asterisk (*).

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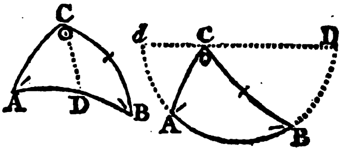
* The reason of these rules for drawing a perpendicular is founded on practice and observation. For in every right-angled triangle there must be two *given* quantities, exclusive of the right angle, and it is obvious that the perpendicular must be so drawn as to form *one* right-angled triangle wherein two quantities are given

Compare the *perpendicular* and the similar parts in the triangle where *two* quantities are given, and make an equation, which place exactly under the former, and strike out such terms as are common to both the equations.

Turn the remaining quantities into a proportion; thus, if all the terms which are to be struck out of the equations be on the same side of both, put the required term last, that with which it is connected first, and the other two in the middle in any order.

But if the terms which are not struck out of the equations stand on both sides; one unmarked term, in the lower equation, is to the unmarked term exactly above it; as the other unmarked term in the lower equation, is to the marked or required term which stands above it.

(Z) CASE I. *Suppose two angles at the base of an oblique-angled triangle, and a side opposite to one of these angles were given, to find the vertical angle.*



First, find the vertical angle BCD[†] in the triangle BDC, where two entire quantities are given.

Here BC is the *middle part*, and the other parts are the *extremes conjunct*. Hence,

$$\text{rad}\times\cos \text{BC}=\cot\angle\text{B}\times\cot\overset{*}{\text{BCD}}.$$

Therefore $\cot\angle\text{B} : \text{rad} :: \cos\text{BC} : \cot\text{BCD}$.

In the triangle ADC; $\textit{rad}\,\ddagger\times\cos\angle\text{A}=\textit{cos}\ \text{DC}\times\sin\overset{*}{\text{ACD}}$.

In the triangle BDC; $\textit{rad}\ \times\cos\angle\text{B}=\textit{cos}\ \text{DC}\times\sin\text{BCD}$.

Therefore $\cos\angle\text{B} : \cos\angle\text{A} :: \text{sine BCD} : \text{sine ACD}$.

Had the ∠ C been given, and the ∠ A required, we must have proceeded exactly in the same manner, only the ∠ A must have been marked instead of ACD, and the proportion would have been thus; $\text{sine BCD} : \text{sine ACD} :: \cos\angle\text{B} : \cos\angle\text{A}$.

The sum, or difference, of ACD and BCD, gives ACB according as the perpendicular falls within, or without the triangle.

(A) COROLLARY I. *The cosines of the angles at the base, are in proportion to each other as the sines of the angles at the*

[†] When both the angles at the base are given; if the perpendicular fall without the triangle, it is the most convenient to draw it so that *two known* parts may come between the perpendicular and the *given* angle to which it is opposite, hence CD is the proper perpendicular in this case.

[‡] The terms which are to be struck out of the equations are printed in *italics*.

vertex, made by a perpendicular drawn from the vertical angle upon the base.

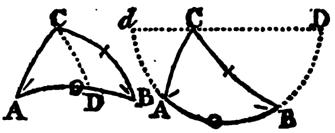
(B) COROLLARY II. *The sum of the cosines of the angles at the base,*
Is to their difference;
As the sum of the sines of the angles at the vertex,
Is to their difference. For,

$$\text{Cos B} : \text{cos A} :: \text{sine BCD} : \text{sine ACD};$$

And by composition and division,

$$\text{Cos A} + \text{cos B} : \text{cos A} \sim \text{cos B} :: \text{sine ACD} + \text{sine BCD} : \text{sine ACD} \sim \text{sine BCD}.$$

(C) CASE II. *Suppose the two angles at the base of an oblique spherical triangle, and a side opposite to one of these angles were given to find the base.*



First find the base DB in the triangle BDC, where two entire quantities are given. Here the complement of CBD is the *middle part,* and the other parts are the *extremes conjunct.* Hence,

$$\text{Rad} \times \cos \angle \text{B} = \text{tang } \overset{*}{\text{DB}} \times \text{cot BC};$$

$$\text{Therefore cot BC} : \text{rad} :: \cos \angle \text{B} : \text{tang CB},$$

$$\text{In the triangle, ADC, } rad \times \text{sine } \overset{*}{\text{AD}} = \text{cot A} \times tang \text{ DC},$$

$$\text{In the triangle, BDC, } rad \times \text{sine DB} = \text{cot B} \times tang \text{ DC},$$

$$\text{Therefore } \cot \angle \text{B} : \cot \angle \text{A} :: \text{sine DB} : \text{sine AD}.$$

Had the base AB been given, and the angle A required, the same method of solution must have been observed, only the angle A must have been marked instead of the segment AD, and the last proportion would have been,

$$\text{Sine DB} : \text{sine AD} :: \cot \angle \text{B} : \cot \angle \text{A}.$$

The sum, or difference, of AD and DB, gives the base AB, according as the perpendicular falls within, or without the triangle.

(D) COROLLARY I. *The co-tangents of the angles at the base are in proportion to each other, as the sines of the segments of the base.*

(E) COROLLARY II. *The sum of the co-tangents of the base angles,*
Is to their difference;
As the sum of the sines of the segments of the base,
Is to their difference. For,

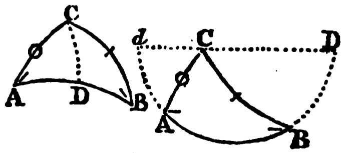
$$\cot \angle \text{B} : \cot \angle \text{A} :: \text{sine DB} : \text{sine AD},$$

And by composition and division,

Cot ∠ B + cot ∠ A : cot ∠ B ∼ cot ∠ A : : sine DB + sine AD : sine DB ∼ sine AD.

(F) CASE III. *Suppose the two angles at the base, and a side opposite to one of them were given, to find the side opposite to the other.*

Neither the base nor vertical angle being here concerned, it would be superfluous to find any thing in the triangle BDC.



*

In the triangle ADC, *rad* × *sine* DC = sine AC × sine ∠ A,

In the triangle BDC, *rad* × *sine* DC = sine BC × sine ∠ B;

Therefore sine ∠ A : sine ∠ B : : sine BC : sine AC.

Had the side AC been given, and the angle A required, the angle A must have been marked, and the proportion would have been, sine AC : sine BC : : sine B : sine A.

(G) COROLLARY I. *The sines of the sides, are in proportion to each other as the sines of their opposite angles, et contra.*

(H) COROLLARY II. *The sum of the sines of the sides,*
Is to their difference;
As the sum of the sines of the angles at the base,
Is to their difference. For,

Sine AC : sine BC : : sine ∠ B : sine ∠ A;

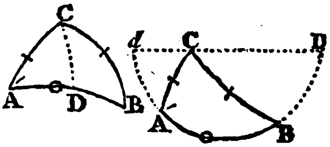
And by composition and division,

Sine AC + sine BC : sine AC ∼ sine BC : : sine ∠ B + sine ∠ A : sine ∠ B ∼ sine ∠ A.

(I) CASE IV. *Suppose two sides and an angle opposite to one of them were given, to find the base.*

First find the segment AD † in the triangle ADC where two entire quantities are given.

Here the complement of A is the *middle part*, and the other two parts are the *extremes conjunct.*



*

Hence, rad × cos ∠ A = cot AC × tang AD;

Therefore, cot AC : rad : : cos ∠ A : tang AD.

*

In the triangle BDC, *rad* × cos BC = *cos* DC × cos DB,

In the triangle ADC, *rad* × cos AC = *cos* DC × cos AD;

Therefore cos AC : cos BC : : cos AD : cos DB.

---

† When *only one* angle at the base is given; if the perpendicular fall without the triangle, let it be drawn so as to fall opposite to that given angle, whether it be acute or obtuse, hence CD is the proper perpendicular in this case.

Had the side AB been given, and BC required, the operation would have been the same, only BC must have been marked instead of DB, then we should have had this proportion:

$$\mathrm{Cos}\ AD : \cos DB :: \cos AC : \cos BC.$$

The difference between the base AB and the segment AD gives the segment DB; if AB exceeds AD the perpendicular falls within the triangle, if not, it falls without.

(K) COROLLARY I. *The cosines of the sides are in proportion to each other, as the cosines of the segments, of the base.*

(L) COROLLARY II. *The sum of the cosines of the sides,*
Is to their difference;
As the sum of the cosines of the segments of the base,
Is to their difference. For,

$$\mathrm{Cos}\ AC : \cos BC :: \cos AD : \cos DB;$$

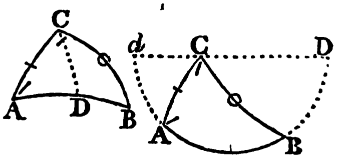
And by composition and division,

$$\mathrm{Cos}\ AC + \cos BC : \cos AC \sim \cos BC :: \cos AD + \cos DB : \cos AD \sim \cos DB.$$

(M) CASE V. *Suppose a side, and its two adjacent angles were given, to find a side opposite to one of these angles.*

Find the vertical angle ACD, in the triangle ADC.

Here the complement of AC is the *middle part*, and the complements of A and C are the *extremes conjunct.*



$$\text{Hence, } rad \times \cos AC = \cot \angle A \times \cot \overset{*}{ACD};$$

$$\text{Therefore, } \cot \angle A : rad :: \cos AC : \cot ACD.$$

Then the difference between ACB and ACD gives BCD. If ACD be less than ACB the perpendicular falls within the triangle, if greater, it falls without.

$$\text{In the triangle BDC, } rad \times \cos BCD = tang\ DC \times \cot \overset{*}{BC},$$

$$\text{In the triangle ADC, } rad \times \cos ACD = tang\ DC \times \cot AC.$$

$$\text{Therefore } \cos ACD : \cos BCD :: \cot AC : \cot BC.$$

Had the side BC been given, and the angle ACB required, the same mode of operation must have been observed, only marking BCD instead of BC, and the proportion would have been thus,

$$\mathrm{Cot}\ AC : \cot BC :: \cos ACD : \cos BCD.$$

(N) COROLLARY I. *The cosines of the vertical angles are in proportion to each other, as the co-tangents of the sides. Or the tangents of the sides are reciprocally as the cosines of the vertical angles.* For,

$$\mathrm{Cot}\ AC : \cot BC :: \cos ACD : \cos BCD;$$

But the tangents are inversely as the co-tangents,
Hence, tang BC : tang AC :: cos ACD : cos BCD.

(O) Corollary II. *The sum of the co-tangents of the sides, is to their difference; as the sum of the cosines of the vertical angles, is to their difference.*

And, *The sum of the tangents of the sides,*
Is to their difference;
As the sum of the cosines of the vertical angles,
Is to their difference. For,

Cot AC : cot BC :: cos ACD : cos BCD,
and tang BC : tang AC :: cos ACD : cos BCD.

By composition and division,

Cot AC $+$ cot BC : cot AC $\sim$ cot BC :: cos ACD $+$ cos BCD : cos ACD $\sim$ cos BCD, and

Tang BC $+$ tang AC : tang BC $\sim$ tang AC :: cos ACD $+$ cos BCD : cos ACD $\sim$ cos BCD.

(P) Scholium.

From the preceding cases and their corollaries, we deduce the following general rules for solving ten cases of oblique spherical triangles, viz.

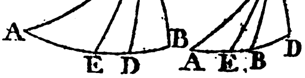
I. *The sines of the sides are directly proportional to the sines of their opposite angles, et contra.*

II. *The cosines of the vertical angles (made by a perpendicular) are as the co-tangents of their adjacent sides; and the sines thereof, as the cosines of the base angles.*

III. *The cosines of the segments of the base (made by a perpendicular) are as the cosines of their adjacent sides; and the sines thereof, as the co-tangents of their adjacent base angles.*

PROPOSITION XXVI.

(Q) In any spherical triangle.
I. If the perpendicular fall within the triangle,
The co-tangent of half the sum of the two sides,
Is to tangent of half their difference;
As the co-tangent of half the base,
Is to the tangent of the distance of a perpendicular from the middle of the base.



II. Or, If the perpendicular fall without the triangle,
Tangent of half the difference of the sides, is to co-tangent of half their sum; as tangent of half the base, is to co-tangent of the distance of a perpendicular from the middle of the base.

Demonstration. Cos AC $+$ cos BC : cos AC $\sim$ cos BC :: cos AD $+$ cos DB : cos AD $\sim$ cos DB (L. 177.) But,

Cos AC $+$ cos BC : cos AC $\sim$ cos BC :: co-tang $\frac{1}{2}$(AC $+$ BC) : tang $\frac{1}{2}$ (AC $\sim$ BC) (R. 112.) And,

$\text{Cos}\ AD + \cos BD : \cos AD \sim \cos BD :: \text{co-tang}\ \frac{1}{2}(AD+DB) : \text{tang}\ \frac{1}{2}(AD \sim DB)$ (R. 112.)

$\therefore$ $\text{Co-tang}\ \frac{1}{2}(AC+BC) : \text{tang}\ \frac{1}{2}(AC \sim BC) :: \text{co-tang}\ \frac{1}{2}(AD+DB) : \text{tang}\ \frac{1}{2}(AD \sim DB.)$

But $\frac{1}{2}(AD+DB)=AE$, and $\frac{1}{2}(AD \sim DB)=DE$, hence $\text{co-tang}\ \frac{1}{2}(AC+BC) : \text{tang}\ \frac{1}{2}(AC \sim BC) :: \text{co-tang}\ AE : \text{tang}\ DE$.

But when the perpendicular falls without the triangle, $\frac{1}{2}(AD \sim DB)=AE$ and $\frac{1}{2}(AD+DB)=DE$.

$\therefore$ $\text{Co-tang}\ \frac{1}{2}(AC+BC) : \text{tang}\ \frac{1}{2}(AC \sim BC) :: \text{co-tang}\ DE : \text{tang}\ AE$; and by inversion, $\text{tang}\ \frac{1}{2}(AC \sim BC) : \text{co-tang}\ \frac{1}{2}(AC+BC) :: \text{tang}\ AE : \text{co-tang}\ DE$. Q. E. D.

(R) COROLLARY I. The *distance* of a perpendicular from the middle of the base, or as some writers call it the *altern*, or *alternate* base, is always equal to half the difference of the segments of the base, when the perpendicular falls within the triangle; or equal to half the sum of the segments, when a perpendicular falls without the triangle. Either of the above rules will bring the same conclusion, whether the perpendicular falls within or without the triangle; only in the first case, the fourth number will be less than half the base, and in the second case, it will be greater.

That the rules are both the same may be shewn thus,

When the perpendicular falls within the triangle.

$\text{Co-tang}\ \frac{1}{2}(AC+BC) : \text{tang}\ \frac{1}{2}(AC \sim BC) :: \text{co-tang}\ AE : \text{tang}\ DE$; and inversely, $\text{tang}\ \frac{1}{2}(AC \sim BC) : \text{co-tang}\ \frac{1}{2}(AC+BC) :: \text{tang}\ DE : \text{co-tang}\ AE$.

But the tangents are reciprocally as their co-tangents.

$\therefore$ $\text{Tang}\ \frac{1}{2}(AC \sim BC) : \text{co-tang}\ \frac{1}{2}(AC+BC) :: \text{tang}\ AE : \text{co-tang}\ DE$; the same conclusion as when the perpendicular falls without the triangle.

(S) COROLLARY II. *The tangent of half the base,*
Is to the tangent of half the sum of the sides;
As the tangent of half the difference of the sides,
Is to the tangent of the distance of a perpendicular from the middle of the base. Or,

The tangent of half the sum of the sides, is to the co-tangent of half their difference; as the tangent of half the base, is to the co-tangent of the distance of a perpendicular from the middle of the base. And,

According as this distance is less or greater than half the base, the perpendicular falls within, or without the triangle. For we have already shewn,

$\text{Co-tangent}\ \frac{1}{2}(AC+BC) : \text{tang}\ \frac{1}{2}(AC \sim BC) :: \text{co-tang}\ AE : \text{tang}\ DE$; and $\text{tang}\ \frac{1}{2}(AC \sim BC) : \text{co-tang}\ \frac{1}{2}(AC+BC) :: \text{tang}\ AE : \text{co-tang}\ DE$.

But, since the tangents are reciprocally as the co-tangents, Tang AE : tang $\frac{1}{2}$ (AC $+$ BC) :: tang $\frac{1}{2}$ (AC $\sim$ BC) : tang DE; and tang $\frac{1}{2}$ (AC $+$ BC) : co-tang $\frac{1}{2}$ (AC $\sim$ BC) :: tang AE : co-tang DE.

(T) COROLLARY III. If the triangle be isosceles, or equilateral, the perpendicular will fall on the middle of the base, except the sides be quadrants, and then it may fall in any part of the base.

(U) COROLLARY IV. In a right-angled triangle, *the rectangle of the tangents of half the sum and half the difference, of the hypothenuse and one leg, is equal to the square of the tangent of half the other leg.* For in this case B and D will coincide, and DE will be equal to $\frac{1}{2}$ AB, or equal AE.

PROPOSITION XXVII. (*See the Fig. to* PROP. XXVI.)

(W) In any spherical triangle,

I. If the perpendicular fall within the triangle.

The co-tangent of half the sum of the angles at the base,
Is to the tangent of half their difference;
As the tangent of half the vertical angle,
Is to the tangent of the excess of the greater of the two vertical angles, made by a perpendicular, above half the aforesaid vertical angle.

II. Or, If the perpendicular fall without the triangle.

The tangent of half the difference between the base angles,
Is to co-tangent of half their sum;
As the co-tangent of half the vertical angle,
Is to the co-tangent of the excess of the greater of the two vertical angles, formed by a perpendicular upon the base, above half the aforesaid vertical angle.

DEMONSTRATION. Cos A $+$ cos B : cos A $\sim$ cos B :: sine ACD $+$ sine BCD : sine ACD $\sim$ sine BCD. (B. 175.)

Cos A $+$ *cos* B : *cos* A $\sim$ *cos* B :: cot $\frac{1}{2}$ (A $+$ B) : tang $\frac{1}{2}$ (A $\sim$ B) (R. 112.) *Sine* ACD $+$ *sine* BCD : *sine* ACD $\sim$ *sine* BCD :: tang $\frac{1}{2}$ (ACD $+$ BCD) : tang $\frac{1}{2}$ (ACD $\sim$ BCD) (P. 111.) Therefore,

Cot $\frac{1}{2}$ (A $+$ B) : tang $\frac{1}{2}$ (A $\sim$ B) :: tang $\frac{1}{2}$ (ACD $+$ BCD) : tang $\frac{1}{2}$ (ACD $\sim$ BCD.)

But $\frac{1}{2}$ (ACD $+$ BCD) $=$ $\frac{1}{2}$ ACB, and $\frac{1}{2}$ (ACD $\sim$ BCD) $=$ ECD, hence cot $\frac{1}{2}$ (A $+$ B) : tang $\frac{1}{2}$ (A $\sim$ B) :: tang $\frac{1}{2}$ ACB : tang ECD.

But when the perpendicular falls without the triangle, $\frac{1}{2}$ (ACD $-$ BCD) $=$ $\frac{1}{2}$ ACB, and ECD $=$ $\frac{1}{2}$ (ACD $+$ BCD.)

Cot $\frac{1}{2}$ (A $\sim$ B) : tang $\frac{1}{2}$ (A $+$ B) :: tang $\frac{1}{2}$ (ACD $-$ BCD) : tang $\frac{1}{2}$ (ACD $+$ BCD); by inversion, tang $\frac{1}{2}$ (A $+$ B) : cot $\frac{1}{2}$ (A $\sim$ B) :: tang ECD : tang $\frac{1}{2}$ ACB.

But the tangents and co-tangents are reciprocally proportional.

Tang $\frac{1}{2}$ (A$\sim$B) : cot $\frac{1}{2}$ (A+B) :: cot $\frac{1}{2}$ ACB : cot. ECD. Q.E.D.

(X) COROLLARY I. The excess of the greater of the two vertical angles (formed by a perpendicular) above half the vertical angle, is equal to half the difference of those vertical angles, when the perpendicular falls within the triangle; or half their sum, when it falls without. Either of the above rules will bring the same conclusion, whether the perpendicular falls within or without the triangle; only in the former case, the fourth number will be less than half the vertical angle, and in the latter it will be greater. That the rules are the same may be shewn thus,

When the perpendicular falls without the triangle.

Tang $\frac{1}{2}$ (A$\sim$B) : co-tang $\frac{1}{2}$ (A + B) :: cot $\frac{1}{2}$ ACB : cot ECD; by inversion, cot $\frac{1}{2}$ (A + B) : tang $\frac{1}{2}$ (A$\sim$B) :: cot ECD : cot $\frac{1}{2}$ ACB.

But the tangents are reciprocally as the co-tangents.

$\because$ Cot $\frac{1}{2}$ (A+B) : tang $\frac{1}{2}$ (A$\sim$B) :: tang $\frac{1}{2}$ ACB : tang ECD, the same expression as when the perpendicular falls within the triangle.

(Y) COROLLARY II. *The co-tangent of half the sum of the angles at the base, is to the tangent of half their difference; as the tangent of half the vertical angle, is to tangent of half the difference between the two vertical angles, formed by a perpendicular, or to tangent of half their sum, according as the perpendicular falls within or without the triangle.*

CHAP. V.

INVESTIGATION OF GENERAL RULES FOR CALCULATING THE SIDES AND ANGLES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES WITHOUT MAKING USE OF A PERPENDICULAR.

PROPOSITION XXVIII.*

(Z) *If the cosine of any side of a spherical triangle be multiplied by the radius, and the rectangle of the cosines of the other two sides be deducted from the product; the remainder divided by the rectangle of the sines of these two sides, will be equal to the cosine of the included angle divided by the radius,*

viz. in any spherical triangle ABC,
$$\frac{(\cos AB \times \text{rad})-(\cos AC \times \cos BC)}{\text{sine } AC \times \text{sine } BC} = \frac{\cos \angle C}{\text{rad}}.$$

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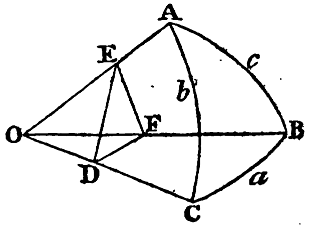
[*] Legendre's Geometry, 6th Edition, page 386, et seq.



Demonstration. Let ABC be the triangle proposed, and let O be the centre of the sphere; join AO, BO, and CO. Take any point D in OC, and in the planes AOC, BOC, draw DE and DF each at right angles to OC, and join EF.

Then because the ∠ EDF is the measure of the inclination of the planes AOC, BOC, it is also the measure of the spherical ∠ ACB. (D. 133.)

In the plane triangle EDF, $\frac{\cos \mathrm{EDF}}{\mathrm{rad}} = \frac{\mathrm{ED}^2 + \mathrm{DF}^2 - \mathrm{EF}^2}{2\mathrm{DE} \times \mathrm{DF}}$ (N. 92.)

also in the plane triangle EOF, $\frac{\cos \mathrm{EOF}}{\mathrm{rad}} = \frac{\mathrm{EO}^2 + \mathrm{OF}^2 - \mathrm{EF}^2}{2\mathrm{EO} \times \mathrm{OF}}$.

From the second of these equations $\mathrm{EF}^2 = \mathrm{EO}^2 + \mathrm{OF}^2 - \frac{2\mathrm{EO} \times \mathrm{OF} \times \cos \mathrm{EOF}}{\mathrm{rad}}$, which substituted in the first equation gives

$$\cos \mathrm{EDF} = \frac{\mathrm{rad}\ (\mathrm{ED}^2 + \mathrm{DF}^2 - \mathrm{EO}^2 - \mathrm{OF}^2) + (2\mathrm{EO} \times \mathrm{OF} \times \cos\ \mathrm{EOF})}{2\mathrm{DE} \times \mathrm{DF}}.$$

But $\mathrm{OE}^2 - \mathrm{ED}^2 = \mathrm{OD}^2$ (Euclid 47 *of* I.) and $\mathrm{OF}^2 - \mathrm{DF}^2 = \mathrm{OD}^2$, consequently $\cos\ \mathrm{EDF} = \frac{(\mathrm{EO} \times \mathrm{OF} \times \cos\ \mathrm{EOF}) - (\mathrm{OD}^2 \times \mathrm{rad})}{\mathrm{DE} \times \mathrm{DF}}$.

Now, $\angle\ \mathrm{EDF} = \angle\ \mathrm{C}$; $\angle\ \mathrm{EOF} = \mathrm{AB}$; $\frac{\mathrm{OE}}{\mathrm{DE}} = \frac{\mathrm{rad}}{\mathrm{sine}\ \angle\ \mathrm{DOE}}$ (O. 44.) $= \frac{\mathrm{rad}}{\mathrm{sine\ AC}}$; $\frac{\mathrm{OF}}{\mathrm{DF}} = \frac{\mathrm{rad}}{\mathrm{sine}\ \angle\ \mathrm{DOF}}$ (O. 44.) $= \frac{\mathrm{rad}}{\mathrm{sine\ BC}}$; $\frac{\mathrm{OD}}{\mathrm{DE}} = \frac{\cos\ \angle\ \mathrm{DOE}}{\mathrm{sine}\ \angle\ \mathrm{DOE}} = \frac{\cos \mathrm{AC}}{\mathrm{sine\ AC}}$; $\frac{\mathrm{OD}}{\mathrm{OF}} = \frac{\cos\ \angle\ \mathrm{DOF}}{\mathrm{sine}\ \angle\ \mathrm{DOF}} = \frac{\cos \mathrm{BC}}{\mathrm{sine\ BC}}$; and if these values be substituted in the equation, $\cos \angle\ \mathrm{EDF} = \frac{(\mathrm{rad}^2 \times \cos\ \mathrm{AB}) - (\mathrm{rad} \times \cos\ \mathrm{AC} \times \cos\ \mathrm{BC})}{\mathrm{sine\ AC} \times \mathrm{sine\ BC}}$; viz. $\frac{\cos\ \angle\ \mathrm{C}}{\mathrm{rad}} = \frac{(\cos\ \mathrm{AB} \times \mathrm{rad}) - (\cos\ \mathrm{AC} \times \cos\ \mathrm{BC})}{\mathrm{sine\ AC} \times \mathrm{sine\ BC}}$. Q. E. D.

(A) Since the preceding conclusion does not depend on any peculiar relation which the ∠ C has to the other angles, a similar equation will be equally true for the angles A and B. Hence

$$\mathrm{Cos}\ \angle\ \mathrm{A} = \frac{(\mathrm{rad}^2 \,.\, \cos a) - (\mathrm{rad} \,.\, \cos b \,.\, \cos c)}{\mathrm{sine}\ b \,.\, \mathrm{sine}\ c},$$

$$\mathrm{Cos}\ \angle\ \mathrm{B} = \frac{(\mathrm{rad}^2 \,.\, \cos b) - (\mathrm{rad} \,.\, \cos a \,.\, \cos c)}{\mathrm{sine}\ a \,.\, \mathrm{sine}\ c}$$

$$\text{Cos} \angle \mathrm{C} = \frac{(\text{rad}^2 \,.\, \cos c) - (\text{rad} \,.\, \cos b \,.\, \cos a)}{\text{sine } b \,.\, \text{sine } a}.$$

These are the formulæ from which Lagrange and Legendre* begin their investigations, and of the four quantities involved, any three being given the fourth may be found. They are applicable to every species of spherical triangles, whether right-angled, quadrantal, or oblique-angled, but the formulæ for right-angled and quadrantal triangles have already been given. (O. 169, and P. 169.)

(B) If A, B, C represent the three angles of a spherical triangle, the opposite sides may be represented by 180° − A, 180° − B, 180° − C; and if a, b, c represent the three sides, their opposite angles may be represented by $180° - a$, $180° - b$, $180° - c$ (U. 197.) Hence

$$\text{Cos } (180° - a) = \frac{\text{rad}^2 \,.\, \cos (180° - \mathrm{A}) - \text{rad} \,.\, \cos (180° - \mathrm{B}) \,.\, \cos (180° - \mathrm{C})}{\text{sine } (180° - \mathrm{B}) \,.\, \text{sine } (180° - \mathrm{C})}$$

But $\cos (180° - a) = -\cos a$; $\cos (180° - \mathrm{A}) = -\cos \mathrm{A}$, &c. (N. 101.) consequently

$$\cos a = \frac{(\text{rad}^2 \,.\, \cos \mathrm{A}) + (\text{rad} \,.\, \cos \mathrm{B} \,.\, \cos \mathrm{C})}{\text{sine } \mathrm{B} \,.\, \text{sine } \mathrm{C}}.$$

In the same manner the cosines of the other sides may be determined.

(C) It has been shewn (G. 176.) that the sines of the sides of any spherical triangle have the same ratio to each other as the sines of their opposite angles; hence by using the notation of Legendre, we shall have

$$\text{I. Sine } \mathrm{A} = \frac{\text{sine } a \,.\, \text{sine } \mathrm{B}}{\text{sine } b} = \frac{\text{sine } a \,.\, \text{sine } \mathrm{C}}{\text{sine } c}.$$

$$\text{Sine } \mathrm{B} = \frac{\text{sine } b \,.\, \text{sine } \mathrm{A}}{\text{sine } a} = \frac{\text{sine } b \,.\, \text{sine } \mathrm{C}}{\text{sine } c}.$$

$$\text{Sine } \mathrm{C} = \frac{\text{sine } c \,.\, \text{sine } \mathrm{A}}{\text{sine } a} = \frac{\text{sine } c \,.\, \text{sine } \mathrm{B}}{\text{sine } b}.$$

$$\text{II. Sine } a = \frac{\text{sine } b \,.\, \text{sine } \mathrm{A}}{\text{sine } \mathrm{B}} = \frac{\text{sine } c \,.\, \text{sine } \mathrm{A}}{\text{sine } \mathrm{C}}.$$

$$\text{Sine } b = \frac{\text{sine } a \,.\, \text{sine } \mathrm{B}}{\text{sine } \mathrm{A}} = \frac{\text{sine } c \,.\, \text{sine } \mathrm{B}}{\text{sine } \mathrm{C}}.$$

$$\text{Sine } c = \frac{\text{sine } a \,.\, \text{sine } \mathrm{C}}{\text{sine } \mathrm{A}} = \frac{\text{sine } b \,.\, \text{sine } \mathrm{C}}{\text{sine } \mathrm{B}}.$$

(D) The general expressions for the cosines, which have been obtained by this proposition, may be arranged thus:

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* The former in the *Journal de L'Ecole Polytechnique,* and the latter in his *Elémens de Géométrie.*

$$\text{III. Cos A}=\frac{(\text{rad}^2 \,.\, \text{cos } a)-(\text{rad} \,.\, \text{cos } b \,.\, \text{cos } c)}{\text{sine } b \,.\, \text{sine } c}.$$

$$\text{Cos B}=\frac{(\text{rad}^2 \,.\, \text{cos } b)-(\text{rad} \,.\, \text{cos } a \,.\, \text{cos } c)}{\text{sine } a \,.\, \text{sine } c}.$$

$$\text{Cos C}=\frac{(\text{rad}^2 \,.\, \text{cos } c)-(\text{rad} \,.\, \text{cos } b \,.\, \text{cos } a)}{\text{sine } b \,.\, \text{sine } a}.$$

And by reducing these last equations.

$$\text{IV. Cos } a=\frac{(\text{cos A} \,.\, \text{sine } b \,.\, \text{sine } c)+(\text{rad} \,.\, \text{cos } b \,.\, \text{cos } c)}{\text{rad}^2}.$$

$$\text{Cos } b=\frac{(\text{Cos B} \,.\, \text{sine } a \,.\, \text{sine } c)+(\text{rad} \,.\, \text{cos } a \,.\, \text{cos } c)}{\text{rad}^2}.$$

$$\text{Cos } c=\frac{(\text{Cos C} \,.\, \text{sine } b \,.\, \text{sine } a)+(\text{rad} \,.\, \text{cos } b \,.\, \text{cos } a)}{\text{rad}^2}.$$

$$\text{V. Cos } a=\frac{(\text{rad}^2 \,.\, \text{cos A})+(\text{rad} \,.\, \text{cos B} \,.\, \text{cos C})}{\text{sine B} \,.\, \text{sine C}}. \text{ (B. 183.)}$$

$$\text{Cos } b=\frac{(\text{rad}^2 \,.\, \text{cos B})+(\text{rad} \,.\, \text{cos A} \,.\, \text{cos C})}{\text{sine A} \,.\, \text{sine C}}.$$

$$\text{Cos } c=\frac{(\text{rad}^2 \,.\, \text{cos C})+(\text{rad} \,.\, \text{cos A} \,.\, \text{cos B})}{\text{sine A} \,.\, \text{sine B}}.$$

And by reducing these equations, we shall have

$$\text{VI. Cos A}=\frac{(\text{cos } a \,.\, \text{sine B} \,.\, \text{sine C})-(\text{rad} \,.\, \text{cos B} \,.\, \text{cos C})}{\text{rad}^2},$$

$$\text{Cos B}=\frac{(\text{cos } b \,.\, \text{sine A} \,.\, \text{sine C})-(\text{rad} \,.\, \text{cos A} \,.\, \text{cos C})}{\text{rad}^2}.$$

$$\text{Cos C}=\frac{(\text{cos } c \,.\, \text{sine A} \,.\, \text{sine B})-(\text{rad} \,.\, \text{cos A} \,.\, \text{cos B})}{\text{rad}^2}.$$

(E) The six preceding articles afford solutions to all the different cases of oblique-angled spherical triangles.

The Ist. Finds the angles, when two sides and an angle opposite to one of them are given.
The IId. Finds a side, when two angles and a side opposite to one of them are given.
The IIId. Finds the angles, from the three sides being given.
The IVth. Finds the third side, when two sides and their contained angle are given.
The Vth. Finds the sides, from the three angles being given.
The VIth. Finds the third angle, when two angles and the side adjacent to both of them are given.

(F) But none of the foregoing formulæ are conveniently adapted to logarithmical calculation.

Let the value of the cosine of C (D. 183.) be substituted in the formula $\text{rad}^2 - \text{rad} \,.\, \cos \text{C} = 2 \text{ sine}^2 \tfrac{1}{2} \text{C}$ (2d equation I. 117.) we shall have $1 - \dfrac{\cos \text{C}}{\text{rad}} = \dfrac{2 \text{ sine}^2 \frac{1}{2}\text{C}}{\text{rad}^2} = 1 - \dfrac{(\text{rad}^2 \cos c) - (\text{rad} . \cos a . \cos b)}{\text{sine } a \,.\, \text{sine } b \,.\, \text{rad}}$ $= \dfrac{(\text{sine } a \,.\, \text{sine } b) + (\cos a \,.\, \cos b) - (\text{rad} \,.\, \cos c)}{\text{sine } a \,.\, \text{sine } b}$; but $(\text{sine } a \,.\, \text{sine } b) + (\cos a \,.\, \cos b) = \text{rad} \,.\, \cos (a-b)$ (D. 115.), hence $\dfrac{2 \text{ sine}^2 \frac{1}{2}\text{C}}{\text{rad}^2} = \dfrac{\text{rad} \,.\, \cos (a-b) - (\text{rad} \,.\, \cos c)}{\text{sine } a \,.\, \text{sine } b}$.

From the 4th equation (F. 116.) $\text{rad} \,.\, \cos \text{Q} - \text{rad} \,.\, \cos \text{P} = 2 \text{ sine } \tfrac{1}{2} (\text{P}+\text{Q}) \,.\, \text{sine } \tfrac{1}{2} (\text{P}-\text{Q})$, which, by putting $\text{Q} = (a-b)$ and $\text{P} = c$, becomes $\text{rad} \,.\, \cos (a-b) - \text{rad} \,.\, \cos c = 2 \text{ sine } \tfrac{1}{2} (c+a-b) \,.\, \text{sine } \tfrac{1}{2} (c-a+b)$, hence we obtain $\dfrac{2 \text{ sine}^2 \frac{1}{2}\text{C}}{\text{rad}^2} = \dfrac{2 \,.\, \text{sine } \frac{1}{2} (c+a-b) \,.\, \text{sine } (c-a+b)}{\text{sine } a \,.\, \text{sine } b}$, that is $\text{sine } \tfrac{1}{2} \text{C} = \text{rad} \sqrt{\dfrac{\text{sine } \frac{1}{2} (c+a-b) \,.\, \text{sine } \frac{1}{2} (c-a+b).}{\text{sine } a \,.\, \text{sine } b}}$ But $\tfrac{1}{2}(c+a-b) = \tfrac{1}{2}(a+b+c)-b$, and $\tfrac{1}{2}(c+b-a) = \tfrac{1}{2}(a+b+c)-a$, hence $\text{sine } \tfrac{1}{2} \text{C} = \text{rad} \sqrt{\dfrac{\text{sine} \frac{1}{2}(a+b+c)-b \,.\, \text{sine} \frac{1}{2}(a+b+c)-a,}{\text{sine } a \,.\, \text{sine } b}}$ and it is evident that the same formula will be obtained, with only a change of the letters, for the angles A and B.

When $\tfrac{1}{2} \angle \text{C}$ is near 90° this will not be a convenient rule for producing an accurate result, because the difference of the logarithmical sines for 1″ is then very small (see the note page 54); if $\tfrac{1}{2} \angle \text{C}$ be less than 45°, it will be proper to use this rule.

(G) Again, if the value of the cosine of C (D. 183.) be substituted in the formula $\text{rad}^2 + \text{rad} \,.\, \cos \text{C} = 2 \cos^2 \tfrac{1}{2} \text{C}$ (1st Equation I. 117. we shall have $1 + \dfrac{\cos \text{C}}{\text{rad}} = \dfrac{2 \cos^2 \frac{1}{2} \text{C}}{\text{rad}^2} = 1 + \dfrac{(\text{rad} \,.\, \cos c) - (\cos b \,.\, \cos a)}{\text{sine } a \,.\, \text{sine } b} = \dfrac{(\text{sine } a \,.\, \text{sine } b) + (\text{rad} \,.\, \cos c) - (\cos b \,.\, \cos a)}{\text{sine } a \,.\, \text{sine } b}$; but $(\cos a \,.\, \cos b) - (\text{sine } a \,.\, \text{sine } b) = \text{rad} \,.\, \cos (a+b)$, (D. 115.) Hence $(\text{sine } a \,.\, \text{sine } b) - (\cos a \,.\, \cos b) = -\text{rad} \,.\, \cos (a+b)$, therefore $\dfrac{2 \cos^2 \frac{1}{2} \text{C}}{\text{rad}^2} = \dfrac{(\text{rad} \,.\, \cos c) - \text{rad} \,.\, \cos (a+b)}{\text{sine } a \,.\, \text{sine } b.}$.

From the 4th equation (F. 116.) $(\text{rad} \,.\, \cos \text{Q}) - \text{rad} \,.\, \cos \text{P} = 2 \text{ sine } \tfrac{1}{2} (\text{P}+\text{Q}) \,.\, \text{sine } \tfrac{1}{2} (\text{P}-\text{Q})$, which, by putting $\text{Q} = c$ and $\text{P} =$

$(a+b)$, becomes $(\mathrm{rad}.\cos c)-\mathrm{rad}.\cos(a+b)=2\,\mathrm{sine}\,\frac{1}{2}(a+b+c)$ $.\,\mathrm{sine}\,\frac{1}{2}(a+b-c)$, hence we obtain

$$\frac{2\cos^2\frac{1}{2}\mathrm{C}}{\mathrm{rad}^2}=\frac{2\,\mathrm{sine}\,\frac{1}{2}(a+b+c)\,.\,\mathrm{sine}\,\frac{1}{2}(a+b-c)}{\mathrm{sine}\,a\,.\,\mathrm{sine}\,b};$$ that is

$$\cos\tfrac{1}{2}\mathrm{C}=\mathrm{rad}\sqrt{\frac{\mathrm{sine}\,\frac{1}{2}(a+b+c)\,.\,\mathrm{sine}\,\frac{1}{2}(a+b-c)}{\mathrm{sine}\,a\,.\,\mathrm{sine}\,b}}.$$ But $\frac{1}{2}(a+b-c)=\frac{1}{2}(a+b+c)-c$ hence

$$\cos\tfrac{1}{2}\mathrm{C}=\mathrm{rad}\sqrt{\frac{\mathrm{sine}\,\frac{1}{2}(a+b+c)\,.\,\mathrm{sine}\,\frac{1}{2}(a+b+c)-c}{\mathrm{sine}\,a\,.\,\mathrm{sine}\,b}},$$ and it is obvious that the same formula will be obtained, with only a change of the letters, for the angles A and B.

When $\frac{1}{2}\angle\mathrm{C}$ is very small, this rule should not be used where a very accurate result is wanted, because the logarithmical cosines of very small arcs, in a table carried to seven places of figures, differ but little from each other (see the note page 54); if $\frac{1}{2}\angle\mathrm{C}$ be between 45°, and 90°, this rule may be used with advantage.

(H) Also, because $\frac{\mathrm{rad}\,.\,\mathrm{sine}\,\frac{1}{2}\mathrm{C}}{\cos\frac{1}{2}\mathrm{C}}=\mathrm{tang}\,\frac{1}{2}\mathrm{C}$ (N. 104), and $\frac{\mathrm{rad}\,.\,\cos\frac{1}{2}\mathrm{C}}{\mathrm{sine}\,\frac{1}{2}\mathrm{C}}=\cot\frac{1}{2}\mathrm{C}$ (O. 104.) we shall obtain by division,

$$\mathrm{Tang}\tfrac{1}{2}\mathrm{C}=\mathrm{rad}\sqrt{\frac{\mathrm{sine}\frac{1}{2}(a+b+c)-b\,.\,\mathrm{sine}\frac{1}{2}(a+b+c)-a}{\mathrm{sine}\frac{1}{2}(a+b+c)\,.\,\mathrm{sine}\frac{1}{2}(a+b+c)-c}},$$ and

$$\mathrm{Cot}\tfrac{1}{2}\mathrm{C}=\mathrm{rad}\sqrt{\frac{\mathrm{sine}\frac{1}{2}(a+b+c)\,.\,\mathrm{sine}\frac{1}{2}(a+b+c)-c}{\mathrm{sine}\frac{1}{2}(a+b+c)-b\,.\,\mathrm{sine}\frac{1}{2}(a+b+c)-a}}.$$

(I) Any of the three preceding formulæ (F. 184, G. 185, or H. 186.) will determine an angle when the three sides are given, and by continuing Legendre's* mode of investigation, formulæ for determining a side in terms of the three angles may be obtained; thus, let the value of $\cos a$ (B. 183.) be substituted in the formula $\mathrm{rad}^2-\mathrm{rad}\,.\,\cos a=2\,\mathrm{sine}^2\frac{1}{2}a$ (2d Equation I. 117.) we have

$$1-\frac{\cos a}{\mathrm{rad}}=\frac{2\,\mathrm{sine}^2\frac{1}{2}a}{\mathrm{rad}^2}=1-\frac{(\mathrm{rad}^2.\cos\mathrm{A})+(\mathrm{rad}.\cos\mathrm{B}.\cos\mathrm{C})}{\mathrm{sine}\,\mathrm{B}\,.\,\mathrm{sine}\,\mathrm{C}\,.\,\mathrm{rad}}=$$

$$\frac{(\mathrm{sine}\,\mathrm{B}\,.\,\mathrm{sine}\,\mathrm{C})-(\mathrm{rad}\,.\cos\mathrm{A})-(\cos\mathrm{B}\,.\cos\mathrm{C})}{\mathrm{sine}\,\mathrm{B}\,.\,\mathrm{sine}\,\mathrm{C}};$$ but

$\mathrm{rad}\,.\cos(\mathrm{B}+\mathrm{C})=(\cos\mathrm{B}\,.\cos\mathrm{C})-(\mathrm{sine}\,\mathrm{B}\,.\,\mathrm{sine}\,\mathrm{C})$, (D. 115.) or, $(\mathrm{sine}\,\mathrm{B}\,.\,\mathrm{sine}\,\mathrm{C})-(\cos\mathrm{B}\,.\cos\mathrm{C})=-\mathrm{rad}\,.\cos(\mathrm{B}+\mathrm{C})$, therefore

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* Eléments de Géométrie, 6th edition, page 391, et seq.

$\frac{2\,\text{sine}^2\frac{1}{2}a}{\text{rad}^2}=\frac{-\text{rad}\,.\cos(\mathrm{B}+\mathrm{C})-(\text{rad}\,.\cos\mathrm{A})}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}$. From the 3d equation (F. 116.) $\text{rad}.\cos\mathrm{P}+\text{rad}.\cos\mathrm{Q}=2\cos\frac{1}{2}(\mathrm{P}+\mathrm{Q}).\cos\frac{1}{2}(\mathrm{P}-\mathrm{Q})$, which, by putting $\mathrm{P}=\mathrm{B}+\mathrm{C}$ and $\mathrm{Q}=\mathrm{A}$, becomes $\text{rad}.\cos(\mathrm{B}+\mathrm{C})+\text{rad}\,.\cos\mathrm{A}=2\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}+\mathrm{A})\,.\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})$, hence we obtain

$\frac{2\,\text{sine}^2\frac{1}{2}a}{\text{rad}^2}=\frac{-2\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}+\mathrm{A})\,.\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}$, that is

$\text{Sine}\,\frac{1}{2}a=\text{rad}\sqrt{\frac{-\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}+\mathrm{A})\,.\cos(\frac{1}{2}\mathrm{B}+\mathrm{C}-\mathrm{A})}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}}$. But

$\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{A})=\frac{1}{2}(\mathrm{B}+\mathrm{C}+\mathrm{A})-\mathrm{A}$, hence

$\text{sine}\,\frac{1}{2}a=\text{rad}\sqrt{\frac{-\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\,.\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{A}}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}}$.[*]

It is evident that the same formula will be obtained for the sides *b* and *c*, by a change of the letters only.

When ½ *a* is nearer 90°, this rule should not be used where extraordinary accuracy is required; but if ½ *a* be less than 45°, it may then be used with advantage. See the conclusion of (F. 184.)

(K) Again, if the value of the cosine of *a* (B. 183.) be substituted in the formula $\text{rad}^2+\text{rad}\,.\cos a=2\cos^2\frac{1}{2}a$ (1st Equation I. 117.), we have

$1+\frac{\cos a}{\text{rad}}=\frac{2\cos^2\frac{1}{2}a}{\text{rad}^2}=1+\frac{(\text{rad}\,.\cos\mathrm{A})+(\cos\mathrm{B}\,.\cos\mathrm{C})}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}=$

$\frac{(\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C})+(\cos\mathrm{B}\,.\cos\mathrm{C})+(\text{rad}\,.\cos\mathrm{A})}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}$; but

$\text{rad}\,.\cos(\mathrm{B}-\mathrm{C})=(\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C})-(\cos\mathrm{B}\,.\cos\mathrm{C})$, (D. 115.)

therefore $\frac{2\cos^2-\frac{1}{2}a}{\text{rad}^2}=\frac{\text{rad}\,.\cos(\mathrm{B}-\mathrm{C})+\text{rad}\,.\cos\mathrm{A}}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}$. From the 3d Equation (F. 116.) $\text{rad}\,.\cos\mathrm{P}+\text{rad}\,.\cos\mathrm{Q}=2\,.\cos\frac{1}{2}(\mathrm{P}+\mathrm{Q})\,.\cos\frac{1}{2}(\mathrm{P}-\mathrm{Q})$, which, by putting $\mathrm{Q}=(\mathrm{B}-\mathrm{C})$ and $\mathrm{P}=\mathrm{A}$, becomes $\text{rad}.\cos\mathrm{A}+\text{rad}\,.\cos(\mathrm{B}-\mathrm{C})=2\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C}).\cos\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B})$, hence we obtain $\frac{2\cos^2\frac{1}{2}a}{\text{rad}}=\frac{2\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C})\,.\cos\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B})}{\text{sine}\,\mathrm{B}\,.\,\text{sine}\,\mathrm{C}}$

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[*] Though the quantity under the radical sign appears under a negative form, it is always positive; for, since the three angles of every spherical triangle are together greater than two right angles (T. 137.) $\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})$ must be greater than 90°, and the cosine of an arc greater than 90°, is negative, (K. 100,) therefore the expression $-\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})$ becomes positive. The other part under the radical sign is always positive; for, if the supplements of the three angles A, B, C be taken, they will give the three sides of a new triangle $=180°-\mathrm{A}$, $180°-\mathrm{B}$, $180°-\mathrm{C}$, (U. 137.) any two of which taken together are greater than the third (Y. 138); hence $180°-\mathrm{A}$ *is less than* $180°-\mathrm{B}+180°-\mathrm{C}$, that is, $-\mathrm{A}+\mathrm{B}+\mathrm{C}$ *is less than* 180°, consequently $\frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})$ or its equal $\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{A}$ *is less than* 90°, and the cosine of an arc less than 90° is always affirmative. (K. 100.) Legendre, page 392.

that is, $\cos\frac{1}{2}a = \text{rad}\sqrt{\frac{\cos -(\mathrm{A}+\mathrm{B}-\mathrm{C})\,.\,\cos\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B})}{\text{sine }\mathrm{B}\,.\,\text{sine }\mathrm{C}}}$, but $\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C}) = \frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{C}$, and $\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B}) = \frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{B}$, hence $\cos\frac{1}{2}a =$ $\text{rad}\sqrt{\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{C}\,.\,\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{B}}{\text{sine }\mathrm{B}\,.\,\text{sine }\mathrm{C}}}$. And it is plain that the same formula, with only a change of the letters, will be found for the sides b and c.

When $\frac{1}{2}a$ is very small it will not be proper to use this rule, but if $\frac{1}{2}a$ be between 45° and 90°, it may be used with advantage. (See the conclusion of G. 185.)

(L) Also, because $\text{rad}\,.\,\frac{\text{sine }\frac{1}{2}a}{\cos\frac{1}{2}a} = \text{tang }\frac{1}{2}a$ (N. 104.) and $\frac{\text{rad}\,.\,\cos\frac{1}{2}a}{\text{sine }\frac{1}{2}a} = \cot\frac{1}{2}a$ (O. 104.) we shall obtain by division

$$\text{Tang }\frac{1}{2}a = \text{rad}\sqrt{\frac{-\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\,.\,\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{A}}{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{C}\,.\,\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{B}}},$$

$$\text{and } \cot\frac{1}{2}a = \text{rad}\sqrt{\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{C}\,.\,\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{B}}{-\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\,.\,\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})-\mathrm{A}}}.$$

And by the same process a similar formula will be obtained for the other sides. Also for the reasons already given (Note I. 187.) the quantities which appear with a negative sign are positive.

Any one of the three preceding formulæ (I. 186. K. 187. L. 188.) will determine a side when the three angles are given.

(M) From the third set of equations (D. 183.) we have

$$\text{Cos }\mathrm{A}\,.\,\text{sine }b\,.\,\text{sine }c = \text{rad}^2\,.\,\cos a - \text{rad}\,.\,\cos b\,.\,\cos c,$$

$$\text{and }\cos\mathrm{C}\,.\,\text{sine }b\,.\,\text{sine }a = \text{rad}^2\,.\,\cos c - \text{rad}\,.\,\cos b\,.\,\cos a.$$

By exterminating $\cos c$ and reducing the equations, we get $\text{rad}\,.\,\cos\mathrm{A}\,.\,\text{sine }c = (\text{rad}\,.\,\cos a\,.\,\text{sine }b) - (\cos\mathrm{C}.\,\text{sine }a\,.\,\cos b.)$

This last equation, by a simple permutation, gives $\text{rad}\,.\,\cos\mathrm{B}\,.\,\text{sine }c = (\text{rad}\,.\,\cos b\,.\,\text{sine }a) - (\cos\mathrm{C}\,.\,\text{sine }b\,.\,\cos a)$.*
By adding the last two equations together, and reducing them we obtain $\text{sine }c\,.\,(\cos\mathrm{A}+\cos\mathrm{B}) = (\text{rad}-\cos\mathrm{C})\,.\,\text{sine }(a+b)$.

But since $\frac{\text{sine }c}{\text{sine }\mathrm{C}} = \frac{\text{sine }a}{\text{sine }\mathrm{A}} = \frac{\text{sine }b}{\text{sine }\mathrm{B}}$ (C. 183.) we have

$$\text{sine }c\,.\,(\text{sine }\mathrm{A}+\text{sine }\mathrm{B}) = \text{sine }\mathrm{C}\,.\,(\text{sine }a+\text{sine }b).$$

And $\text{sine }c\,.\,(\text{sine }\mathrm{A}-\text{sine }\mathrm{B}) = \text{sine }\mathrm{C}\,.\,(\text{sine }a-\text{sine }b)$.
Dividing successively these two equations by the preceding equation, we obtain

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* Legendre's Geometry, page 394.

$$\frac{\text{sine A}+\text{sine B}}{\cos\text{A}+\cos\text{B}}=\frac{\text{sine C}}{\text{rad}-\cos\text{C}}\cdot\frac{\text{sine }a+\text{sine }b}{\text{sine }(a+b)}$$

$$\text{and}\frac{\text{sine A}\quad\text{sine B}}{\cos\text{A}+\cos\text{B}}=\frac{\text{sine C}}{\text{rad}-\cos\text{C}}\cdot\frac{\text{sine }a-\text{sine }b}{\text{sine }(a+b)}.$$

Reducing these equations by the formulæ G. 116, and I. 117,

$$\text{we get, tang }\tfrac{1}{2}(\text{A}+\text{B})=\cot\tfrac{1}{2}\text{C}\cdot\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)},$$

$$\text{and tang }\tfrac{1}{2}(\text{A}-\text{B})=\cot\tfrac{1}{2}\text{C}\cdot\frac{\text{sine }\frac{1}{2}(a-b)}{\text{sine }\frac{1}{2}(a+b)}.$$

These equations give the analogies of Baron Napier for finding the angles A and B, when the two sides a and b, and the angle C comprehended between them are given, viz.

$\cos\frac{1}{2}(a+b) : \cos\frac{1}{2}(a-b) :: \cot\frac{1}{2}\text{C} : \text{tang}\frac{1}{2}(\text{A}+\text{B})$.

$\text{sine}\frac{1}{2}(a+b) : \text{sine}\frac{1}{2}(a-b) :: \cot\frac{1}{2}\text{C} : \text{tang}\frac{1}{2}(\text{A}\sim\text{B})$.

(N) Again, if A, B, C, represent the three angles of a spherical triangle, the opposite sides may be represented by 180° − A, 180° − B, 180° − C; and if a, b, c represent the three sides, their opposite angles may be represented by $180°-a$, $180°-b$, $180°-c$ (U. 137.) By substituting 180° − A, 180° − B for a and b; $180°-c$ for C; and $180°-a$, $180°-b$ for A and B respectively, in the preceding analogies, we shall obtain the following

$\cos\frac{1}{2}(\text{A}+\text{B}) : \cos\frac{1}{2}(\text{A}-\text{B}) :: \text{tang}\frac{1}{2}c : \text{tang}\frac{1}{2}(a+b.)$

$\text{sine}\frac{1}{2}(\text{A}+\text{B}) : \text{sine}\frac{1}{2}(\text{A}-\text{B}) :: \text{tang}\frac{1}{2}c : \text{tang}\frac{1}{2}(a-b.)$

These are Baron Napier's analogies for finding the sides a and b when two angles A and B, and the side c adjacent to both of them are given.

(O) From the preceding formulæ, which are more than sufficient for solving all the cases of oblique-angled spherical triangles various others may easily be deduced.

$(\text{rad}.\cos\text{A}.\text{sine}c)+(\cos\text{C}.\text{sine}a.\cos b)=\text{rad}.\cos a.\text{sine}b$ (M. 188.)

By substituting in this equation $\text{sine C}=\dfrac{\text{sine }a\,.\,\text{sine C}}{\text{sine A}}$ (C. 183.)

$\text{Cot A}=\dfrac{\text{rad}\,.\,\cos\text{A}}{\text{sine A}}$ and $\cot a=\dfrac{\text{rad}\,.\,\cos a}{\text{sine }a}$, (O. 104.) we shall obtain $\cot\text{A}=\dfrac{(\cot a\,.\,\text{sine }b)-(\cos\text{C}\,.\,\cos b)}{\text{sine C}}$, and by substituting B for A, and b for a,

$$\cot\text{B}=\frac{(\cot b\,.\,\text{sine }a)-(\cos\text{C}\,.\,\cos a)}{\text{sine C}}.$$

These rules may be applied, when two sides and contained between them are given, to find the other for it is evident that similar formulæ may be obtained for the angles A and C, or B and C, by merely, changing the letters.

(P) By substituting $180°-a$ for A, $180°-B$ for b, $180°-A$ for a, and $180°-c$ for C, and reducing the equation,

$$\cot a = \frac{(\cot A\ .\ \mathrm{sine}\ B)+(\cos c\ .\ \cos B)}{\mathrm{sine}\ c},$$ and by substituting B for A, &c. as in the preceding article, we get

$$\cot b = \frac{(\cot B\ .\ \mathrm{sine}\ A)+(\cos c\ .\ \cos A)}{\mathrm{sine}\ c}.$$

These rules may be applied, when two angles and the side adjacent to both of them, are given, to find the third angle.

(Q) The foregoing formulæ (O. and P. 189, 190.) are not conveniently adapted to logarithmical calculation, but they may be rendered so by the assistance of other quantities; thus,

$$(\cot A\ .\ \mathrm{sine}\ C)+(\cos C\ .\ \cos b)=\cot a\ .\ \mathrm{sine}\ b \quad (\mathrm{O}.\ 189.)$$

Let $\mathrm{tang}\ \varphi = \frac{\cos b\ .\ \mathrm{tang}\ A}{\mathrm{rad}}$, then because $\mathrm{tang}\ A = \frac{\mathrm{rad}^2}{\cot A}$ and $\mathrm{tang}\ \varphi = \frac{\mathrm{rad}\ .\ \mathrm{sine}\ \varphi}{\cos \varphi}$ (N. 104.) we obtain $\cot A = \frac{\cos b\ .\ \cos \varphi}{\mathrm{sine}\ \varphi}$

hence, by substitution and reduction .

$$\frac{\cos b}{\mathrm{sine}\ \varphi}\ .\ [(\cos \varphi\ .\ \mathrm{sine}\ C)+(\mathrm{sine}\ \varphi\ .\ \cos C)]=\cot a\ .\ \mathrm{sine}\ b.$$

But $(\cos \varphi\ .\ \mathrm{sine}\ C)+(\mathrm{sine}\ \varphi\ .\ \cos C)=\mathrm{sine}(\varphi+C)\ .\ \mathrm{rad}$ (D. 115.)

hence, $\mathrm{sine}(\varphi+C)=\frac{\cot a\ .\ \mathrm{sine}\ b\ .\ \mathrm{sine}\ \varphi}{\cos b\ .\ \mathrm{rad}}$, and because $\cot a = \frac{\mathrm{rad}^2}{\mathrm{tang}\ a}$ (O. 104.) and $\frac{\mathrm{sine}\ b\ .\ \mathrm{rad}}{\cos b}=\mathrm{tang}\ b$ (N. 104.) we derive

$$\mathrm{sine}\ (\varphi+C)=\frac{\mathrm{tang}\ b\ .\ \mathrm{sine}\ \varphi}{\mathrm{tang}\ a}.$$

Now the value of ϕ is already known, therefore the value of C is easily obtained; these formulæ may be applied when two sides a and b, and an angle A, opposite to one of them, are given to find the included angle C.

(R) From the fourth equation (D. 183.) we have

$$(\cos A\ .\ \mathrm{sine}\ b\ .\ \mathrm{sine}\ C)+(\mathrm{rad}\ .\ \cos b.\ \cos c)=\mathrm{rad}^2\ .\ \cos a.$$

Let $\cos A\ .\ \mathrm{sine}\ b = \frac{\mathrm{rad}\ .\ \cos b\ .\ \mathrm{sine}\ \varphi}{\cos \varphi} = \mathrm{tang}\ \varphi\ .\ \cos b$ (N. 104.)

hence $\mathrm{tang}\ \varphi = \frac{\cos A\ .\ \mathrm{sine}\ b}{\cos b}$, but $\frac{\mathrm{sine}\ b\ .\ \mathrm{rad}}{\cos b}=\mathrm{tang}\ b$ (N. 104.)

therefore $\mathrm{tang}\ \phi = \frac{\cos A\ .\ \mathrm{tang}\ b}{\mathrm{rad}}$. By substitution

$\frac{\cos b}{\cos \varphi}.\,[(\cos c \,.\, \cos \varphi)+(\text{sine } c \,.\, \text{sine } \varphi)]=\text{rad} \,.\, \cos a.$ But $(\cos c \,.\, \cos \varphi)+(\text{sine } c \,.\, \text{sine } \varphi)=\cos (c=\varphi) \,.\, \text{rad}$ (D. 115.) hence $\cos (c-\varphi)=\frac{\cos a \,.\, \cos \varphi}{\cos b}$. Now the value of φ is already known, therefore the value of c is easily determined. These formulæ may be applied when two sides a and b, and an angle A opposite to one of them, are given to find the third side.

(S) Again, $\text{rad}^2 \,.\, \cos c=(\text{sine } a \,.\, \text{sine } b \,.\, \cos \mathrm{C})+\text{rad} \,.\, \cos a \,.\, \cos b$ from the 4th equation. (D. 183.)

Let $\text{tang } \varphi=\frac{\cos \mathrm{C} \,.\, \text{tang } b}{\text{rad}}$; then $\cos b \,.\, \text{tang } \varphi = \cos \mathrm{C}.\frac{\text{tang } b \,.\, \cos b}{\text{rad}}=\cos \mathrm{C} \,.\, \sin b$ (L. 104.); hence, by substitution, $\text{rad}^2 \,.\, \cos c=(\text{sine } a \,.\, \cos b \,.\, \text{tang}\,\varphi)+\text{rad} \,.\, \cos a \,.\, \cos b$, or $\frac{\text{rad}^2 \cos c}{\cos b}=(\text{sine } a \,.\, \text{tang } \varphi) + (\text{rad} \,.\, \cos a)$, but $\text{tang } \varphi = \frac{\text{rad} \,.\, \text{sine } \varphi}{\cos \varphi}$, therefore $\frac{\text{rad} \,.\, \cos c}{\cos b}=$ $\frac{(\text{sine } a \,.\, \text{sine } \varphi)+(\cos a \,.\, \cos \varphi)}{\cos \varphi}=\frac{\cos (a-\varphi) \,.\, \text{rad}}{\cos \varphi}$ (D.115.) Consequently $\cos c=\frac{\cos b}{\cos \varphi} \,.\, \cos (a-\varphi)$.

These formulæ may be applied when two sides a and b and the angle C, contained between them, are given to find the third side c.

(T) From the 6th set of equations (D. 183.) we have

$\text{rad}^2 \,.\, \cos \mathrm{C}= (\cos c \,.\, \text{sine } \mathrm{A} \,.\, \text{sine } \mathrm{B})-(\text{rad} \,.\, \cos \mathrm{A} \,.\, \cos \mathrm{B}.)$

Let $\cot \varphi=\frac{\cos c \,.\, \text{tang } \mathrm{B}}{\text{rad}}$; then $\cos \mathrm{B} \,.\, \cot \varphi = \cos c\,\frac{\text{tang. } \mathrm{B} \,.\, \cos \mathrm{B}}{\text{rad}}=\cos c \,.\, \text{sine } \mathrm{B}$ (L. 104.); hence by substitution, $\text{Rad}^2 \,.\, \cos \mathrm{C}=(\cos \mathrm{B} \,.\, \cot \varphi \,.\, \text{sine } \mathrm{A})-(\text{rad} \,.\, \cos \mathrm{A} \,.\, \cos \mathrm{B})$, but $\cot \varphi = \frac{\cos \varphi \,.\, \text{rad}}{\text{sine } \varphi}$ (O. 104.) and by substitution and reduction, we get $\frac{\text{rad} \,.\, \cos \mathrm{C}}{\cos \mathrm{B}}=\frac{(\cos \varphi \,.\, \text{sine } \mathrm{A})-(\cos \mathrm{A} \,.\, \text{sine } \varphi)}{\text{sine } \varphi}=$ $\frac{\text{sine } (\mathrm{A}-\varphi) \,.\, \text{rad}}{\text{sine } \varphi}$ (D. 115.) therefore $\cos \mathrm{C} = \cos \mathrm{B} \,.\frac{\text{sine}(\mathrm{A}-\varphi)}{\text{sine } \varphi}$

These formulæ may be applied when two angles A and B, and the side c, adjacent to both of them, are given to find the third angle C.

(U) $(\text{Cot}\, a \,.\, \text{sine}\, c) - (\cos c \,.\, \cos \text{B}) = \cot \text{A} \,.\, \text{sine}\, \text{B}$. (P. 190.) Let $\text{tang}\,\varphi = \frac{\cos \text{B} \,.\, \text{tang}\, a}{\text{rad}}$, then $\text{rad} \,.\, \text{tang}\,\varphi = \cos \text{B} \,.\, \text{tang}\, a$, but $\text{rad}.\text{tang}\,\varphi = \frac{\text{rad}^2 \,.\, \text{sine}\,\varphi}{\cos\varphi}$ (N. 104.) hence $\frac{\text{rad}^2}{\text{tang}\,a} \,.\, \text{sine}\,\varphi = \cos \text{B} \,.\, \cos\varphi$, that is $\cot a \,.\, \text{sine}\,\varphi = \cos \text{B} \,.\, \cos\varphi$ (O. 104.), or $\cot a = \cos \text{B} \,.\, \frac{\cos\varphi}{\text{sine}\,\varphi}$, hence, by substitution $\frac{\cos \text{B}}{\text{sine}\,\varphi}[(\text{sine}\, c \,.\, \cos\varphi) - (\cos c.\text{sine}\,\varphi)] = \cot \text{A}.\text{sine}\, \text{B}$. But $(\text{sine}\, c \,.\, \cos\varphi) - (\cos c.\, \text{sine}\,\varphi) = \text{sine}\,(c-\varphi) \,.\, \text{rad}$ (D. 115.), by substitution $\cos \text{B} \,.\, \text{sine}\,(c-\varphi).\,\text{rad} = \cot \text{A} \,.\, \text{sine}\, \text{B} \,.\, \text{sine}\,\varphi$. and $\text{sine}\,(c-\varphi).\,\text{rad}^2 = \cot \text{A}.\text{sine}\,\varphi \,.\, \frac{\text{sine}\, \text{B} \,.\, \text{rad}}{\cos \text{B}}$, that is $\text{sine}\,(c-\varphi) \,.\, \frac{\text{rad}^2}{\cot \text{A}} = \text{sine}\,\varphi.\text{tang}\,\text{B}$ (N. 104.) and $\text{sine}(c-\varphi) = \frac{\text{sine}\,\varphi \,.\, \text{tang}\, \text{B}}{\text{tang}\, \text{A}}$ (O. 104.)

These formulæ may be applied when two angles A and B, and a side a, opposite to one of them, are given to find the side c adjacent to both the given angles.

(W) From the 6th set of equations (D. 183.) we have $(\cos a \,.\, \text{sine}\, \text{B} \,.\, \text{sine}\, \text{c}) - (\text{rad} \,.\, \cos \text{B} \,.\, \cos \text{c}) = \text{rad}^2 \,.\, \cos \text{A}$.

Let $\cot\varphi = \frac{\cos a \,.\, \text{tang}\, \text{B}}{\text{rad}} = \cos a \,.\, \frac{\text{sine}\, \text{B}}{\cos \text{B}}$ (N. 104.), but $\cot\varphi = \frac{\text{rad} \,.\, \cos\varphi}{\text{sine}\,\varphi}$, (O. 104.) hence $\cos a \,.\, \text{sine}\, \text{B} = \frac{\text{rad} \,.\, \cos \text{B} \,.\, \cos\varphi}{\text{sine}\,\varphi}$, and by substitution we have $\frac{\cos \text{B}}{\text{sine}\,\varphi} \,.\, [(\text{sine}\, \text{c} \,.\, \cos\varphi) - (\cos \text{c} \,.\, \text{sine}\,\varphi)] = \text{rad} \,.\, \cos \text{A}$.

But $(\text{sine}\, \text{c} \,.\, \cos\varphi) - (\cos \text{c} \,.\, \text{sine}\,\varphi) = \text{sine}\,(\text{c} - \varphi) \,.\, \text{rad}$ (D. 115.) therefore $\cos \text{B} \,.\, \text{sine}\,(\text{c}-\varphi) = \cos \text{A} \,.\, \text{sine}\,\varphi$, hence

$$\text{sine}\,(\text{c}-\varphi) = \frac{\cos \text{A} \,.\, \text{sine}\,\varphi}{\cos \text{B}}.$$

These formulæ may be applied when two angles A and B, and a side a, opposite to one of them, are given to find the third angle.

(X) Any of the preceding formulæ may be turned into proportions, and by introducing the versed sines, &c. they may be extended almost without limit, but the versed sines are seldom used in trigonometry.

$$\frac{2\ \text{sine}^2\ \tfrac{1}{2}\,\mathrm{C}}{\text{rad}^2}.\ \text{sine}\ a\ .\ \text{sine}\ b = \text{rad}\ .\ \cos\,(a-b) - (\text{rad}\ .\ \cos c);$$

(F. 184.) hence $2\ \text{sine}^2\tfrac{1}{2}\mathrm{C}\,.\,\text{sine}\,a\,.\,\text{sine}\,b = \text{rad}^3\,.\,[\cos(a-b) - \cos c]$.

$$\text{And}\ \cos c = \cos\ (a-b) - \sin\ a\ .\ \text{sine}\ b\ .\ \text{sine}^2\ \tfrac{1}{2}\,\mathrm{C}\ .\ \frac{2}{\text{rad}^3}.$$

This formula, by the assistance of a table of logarithms and a table of natural sines, furnishes us with a very convenient rule for finding the third side of a spherical triangle, when two sides and the included angles are given.

(Y) $2\cos^2\tfrac{1}{2}\,a\,.\,\text{sine}\ \mathrm{B}\,.\,\text{sine}\ \mathrm{C} = \text{rad}\,.\,\cos(\mathrm{B}-\mathrm{C}) + (\text{rad}\,.\,\cos\mathrm{A})$; (K. 187.)

Hence $2\cos^2\tfrac{1}{2}\,a\,.\,\text{sine}\ \mathrm{B}\,.\,\text{sine}\ \mathrm{C} = \text{rad}^3\,.\,[\cos\,(\mathrm{B}-\mathrm{C}) + \cos\mathrm{A}]$:

$$\text{And}\ \cos\mathrm{A} = (\text{sine}\ \mathrm{B}\ .\ \text{sine}\ \mathrm{C}\ .\ \cos\tfrac{1}{2}\,a\,.\frac{2}{\text{rad}^3}) - \cos\,(\mathrm{B}-\mathrm{C}).$$

This formula, by the assistance of a table of logarithms and a table of natural sines, gives a useful rule for finding the third angle of a spherical triangle, when two angles and the sides adjacent to both of them are given.

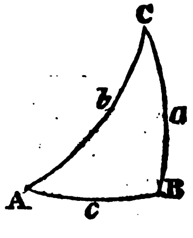
(Z) SCHOLIUM.

Spherical triangles whose sides are very small arcs, may be considered as straight lined, and, therefore, the sides may become nearly equal to their sines or tangents; hence, all the foregoing proportions and formulæ, wherein *cosines* or *cotangents* of the sides are not concerned, are equally applicable to plane trigonometry, using the word *side* instead of *sine of a side* or *tangent of a side*.

SOLUTIONS OF THE DIFFERENT CASES OF RIGHT-ANGLED SPHERICAL TRIANGLES.

Rules for solving the different cases both of right-angled and oblique-angled spherical triangles, have already been given; the subject is here resumed for the express purpose of concentrating some of the principal formulæ.

The three angles of the triangle are represented by A, B, C, and their opposite sides by a, b, c, as in the figure annexed; r = radius = sine of 90°.



CASE I. Given the hypothenuse and an angle, to find the side adjacent to the given angle.

Solution. $$\text{Tang } c=\frac{\cos A\,.\,r}{\cot b}=\frac{\cos A\,.\,\text{tang } b}{r}=\frac{\text{tang } b\,.\,r}{\sec A}.$$

$$\text{Cot } c=\frac{\cot b\,.\,r}{\cos A}=\frac{\sec A\,.\,r}{\text{tang } b}=\frac{\cot b\,.\,\sec A}{r}.$$

$$\text{And, Tang } a=\frac{\cos C\,.\,r}{\cot b}=\frac{\cos C\,.\,\text{tang } b}{r}=\frac{\text{tang } b\,.\,r}{\sec C}.$$

$$\text{Cot } a=\frac{\cot b\,.\,r}{\cos C}=\frac{\sec C\,.\,r}{\text{tang } b}=\frac{\cot b\,.\,\sec C}{r}.$$

Case II. Given the hypothenuse and an angle, to find the side opposite to that given angle.

Solution. $$\text{Sine } a=\frac{\text{sine } A\,.\,\text{sine } b}{r}=\frac{\text{sine } A\,.\,r}{\text{cosec } b}=\frac{\text{sine } b\,.\,r}{\text{cosec } A}.$$

$$\text{Cosec } a=\frac{\text{cosec } b\,.\,r}{\text{sine } A}=\frac{\text{cosec } b\,.\,\text{cosec } A}{r}=\frac{\text{cosec } A\,.\,r}{\text{sine } b}.$$

$$\text{And, Sine } c=\frac{\text{sine } C\,.\,\text{sine } b}{r}=\frac{\text{sine } C\,.\,r}{\text{cosec } b}=\frac{\sin b\,.\,r}{\text{cosec } C}.$$

$$\text{Cosec } c=\frac{\text{cosec } b\,.\,r}{\text{sine } C}=\frac{\text{cosec } b\,.\,\text{cosec } C}{r}=\frac{\text{cosec } C\,.\,r}{\text{sine } b}.$$

Case III. Given the hypothenuse and one angle, to find the other angle.

Solution. $$\text{Tang } C=\frac{\sec b\,.\,r}{\text{tang } A}=\frac{\sec b\,.\,\cot A}{r}=\frac{\cot A\,.\,r}{\cos b}.$$

$$\text{Cot } C=\frac{\cos b\,.\,r}{\cot A}=\frac{\cos b\,.\,\text{tang } A}{r}=\frac{\text{tang } A\,.\,r}{\sec b}.$$

$$\text{And, Tang } A=\frac{\sec b\,.\,r}{\text{tang } C}=\frac{\sec b\,.\,\cot C}{r}=\frac{\cot C\,.\,r}{\cos b}.$$

$$\text{Cot } A=\frac{\cos b\,.\,r}{\cot C}=\frac{\cos b\,.\,\text{tang } C}{r}=\frac{\text{tang } C\,.\,r}{\sec b}.$$

Case IV. Given the hypothenuse and a leg, to find the angle adjacent to that given leg.

Solution. $$\text{Cos } A=\frac{\text{tang } c\,.\,\cot b}{r}=\frac{\text{tang } c\,.\,r}{\text{tang } b}=\frac{\cot.b\,.\,r}{\cot c}.$$

$$\text{Sec } A=\frac{\cot c\,.\,r}{\cot b}=\frac{\cot c\,.\,\text{tang } b}{r}=\frac{\text{tang } b\,.\,r}{\text{tang } c}.$$

$$\text{And, Cos } C=\frac{\text{tang } a\,.\,\cot b}{r}=\frac{\text{tang } a\,.\,r}{\text{tang } b}=\frac{\cot b\,.\,r}{\cot a}.$$

$$\text{Sec C} = \frac{\text{cot } a \,.\, r}{\text{cot } b} = \frac{\text{cot } a \,.\, \text{tang } b}{r} = \frac{\text{tang } b \,.\, r}{\text{tang } a}.$$

$$\text{Also, Tang } \tfrac{1}{2}\,\text{A} = \sqrt{\frac{\text{sine } (b-c)}{\text{sine } (c+b)}} \text{ and tang C} = r\sqrt{\frac{\text{sine } (b-a)}{\text{sine } (b+a)}}$$

CASE V. Given the hypothenuse and a leg, to find the angle opposite to that given leg.

$$\text{SOLUTION. Sine C} = \frac{\text{sine } c \,.\, r}{\text{sine } b} = \frac{\text{sine } c \,.\, \text{cosec } b}{r} = \frac{\text{cosec } b \,.\, r}{\text{cosec } c}.$$

$$\text{Cosec C} = \frac{\text{cosec } c \,.\, r}{\text{cosec } b} = \frac{\text{sine } b \,.\, r}{\text{sine } c} = \frac{\text{sine } b \,.\, \text{cosec } c}{r}.$$

$$\text{And, Sine A} = \frac{\text{sine } a \,.\, r}{\text{sine } b} = \frac{\text{sine } a \,.\, \text{cosec } b}{r} = \frac{\text{cosec } b \,.\, r}{\text{cosec } a}.$$

$$\text{Cosec A} = \frac{\text{cosec } a \,.\, r}{\text{cosec } b} = \frac{\text{sine } b \,.\, r}{\text{sine } a} = \frac{\text{sine } b \,.\, \text{cosec } a}{r}.$$

$$\text{Also, tang } (45^\circ + \tfrac{1}{2}\,\text{C}) = \pm\sqrt{\text{tang } \tfrac{1}{2}\,(b+c) \,.\, \text{cot } \tfrac{1}{2}\,(b-c)}.$$

$$\text{And, tang } (45^\circ + \tfrac{1}{2}\,\text{A}) = \pm\sqrt{\text{tang } \tfrac{1}{2}\,(b+a) \,.\, \text{cot } \tfrac{1}{2}\,(b-a)}.$$

CASE VI. Given the hypothenuse and one side or leg, to find the other leg.

$$\text{SOLUTION. Cos } a = \frac{\text{cos } b \,.\, r}{\text{cos } c} = \frac{\text{cos } b \,.\, \text{sec } c}{r} = \frac{\text{sec } c \,.\, r}{\text{sec } b}.$$

$$\text{Sec } a = \frac{\text{cos } c \,.\, r}{\text{cos } b} = \frac{\text{cos } c \,.\, \text{sec } b}{r} = \frac{\text{sec } b \,.\, r}{\text{sec } c}.$$

$$\text{And, Cos } c = \frac{\text{cos } b \,.\, r}{\text{cos } a} = \frac{\text{cos } b \,.\, \text{sec } a}{r} = \frac{\text{sec } a \,.\, r}{\text{sec } b}.$$

$$\text{Sec } c = \frac{\text{cos } a \,.\, r}{\text{cos } b} = \frac{\text{cos } a \,.\, \text{sec } b}{r} = \frac{\text{sec } b \,.\, r}{\text{sec } a}.$$

$$\text{Also, tang } \tfrac{1}{2}\, a = \sqrt{\text{tang } \tfrac{1}{2}\,(b+c) \,.\, \text{tang } \tfrac{1}{2}\,(b-c)}.$$

$$\text{And, tang } \tfrac{1}{2}\, c = \sqrt{\text{tang } \tfrac{1}{2}\,(b+a) \,.\, \text{tang } \tfrac{1}{2}\,(b-a)}.$$

CASE VII. Given a side of a right-angled spherical triangle, and the angle adjacent, or joining to it, to find the side opposite to the given angle.

$$\text{SOLUTION. Tang } a = \frac{\text{sine } c \,.\, r}{\text{cot A}} = \frac{\text{tang A} \,.\, \text{sin } c}{r} = \frac{\text{tang A} \,.\, r}{\text{cosec } c}.$$

$$\text{Cot } a = \frac{\text{cot A} \,.\, r}{\text{sine } c} = \frac{\text{cosec } c \,.\, r}{\text{tang A}} = \frac{\text{cosec } c \,.\, \text{cot A}}{r}.$$

$$\text{And, Tang } c = \frac{\mathrm{sin}\ a\,.\,r}{\mathrm{cot}\ \mathrm{C}} = \frac{\mathrm{tang}\ \mathrm{C}\,.\,\mathrm{sin}\ a}{r} = \frac{\mathrm{tang}\ \mathrm{C}\,.\,r}{\mathrm{cosec}\ a}.$$

$$\text{Cot } c = \frac{\mathrm{cot}\ \mathrm{C}\,.\,r}{\mathrm{sine}\ a} = \frac{\mathrm{cosec}\ a\,.\,r}{\mathrm{tang}\ \mathrm{C}} = \frac{\mathrm{cosec}\ a\,.\,\mathrm{cot}\ \mathrm{C}}{r}.$$

Case VIII. Given a side of a right-angled spherical triangle, and its adjacent angle, to find the angle opposite to the given side.

Solution.

$$\text{Cos C} = \frac{\mathrm{cos}\ c\,.\,\mathrm{sin}\ \mathrm{A}}{r} = \frac{\mathrm{cos}\ c\,.\,r}{\mathrm{cosec}\ \mathrm{A}} = \frac{\mathrm{sin}\ \mathrm{A}\,.\,r}{\mathrm{sec}\ c}.$$

$$\text{Sec C} = \frac{\mathrm{cosec}\ \mathrm{A}\,.\,r}{\mathrm{cos}\ c} = \frac{\mathrm{cosec}\ \mathrm{A}\,.\,\mathrm{sec}\ c}{r} = \frac{\mathrm{sec}\ c\,.\,r}{\mathrm{sin}\ \mathrm{A}}.$$

$$\text{And, Cos A} = \frac{\mathrm{cos}\ a\,.\,\mathrm{sine}\ \mathrm{C}}{r} = \frac{\mathrm{cos}\ a\,.\,r}{\mathrm{cosec}\ \mathrm{C}} = \frac{\mathrm{sine}\ \mathrm{C}\,.\,r}{\mathrm{sec}\ a}.$$

$$\text{Sec A} = \frac{\mathrm{cosec}\ \mathrm{C}\,.\,r}{\mathrm{cos}\ a} = \frac{\mathrm{cosec}\ \mathrm{C}\,.\,\mathrm{sec}\ a}{r} = \frac{\mathrm{sec}\ a\,.\,r}{\mathrm{sine}\ \mathrm{C}}.$$

Case IX. Given a side and its adjacent angle, to find the hypothenuse.

Solution.

$$\text{Cot } b = \frac{\mathrm{cos}\ \mathrm{A}\,.\,r}{\mathrm{tang}\ c} = \frac{\mathrm{cot}\ c\,.\,\mathrm{cos}\ \mathrm{A}}{r} = \frac{\mathrm{cot}\ c\,.\,r}{\mathrm{sec}\ \mathrm{A}}.$$

$$\text{Tang } b = \frac{\mathrm{tang}\ c\,.\,r}{\mathrm{cos}\ \mathrm{A}} = \frac{\mathrm{sec}\ \mathrm{A}\,.\,r}{\mathrm{cot}\ c} = \frac{\mathrm{sec}\ \mathrm{A}\,.\,\mathrm{tang}\ c}{r}.$$

$$\text{And, Cot } b = \frac{\mathrm{cos}\ \mathrm{C}\,.\,r}{\mathrm{tang}\ a} = \frac{\mathrm{cot}\ a\,.\,\mathrm{cos}\ \mathrm{C}}{r} = \frac{\mathrm{cot}\ a\,.\,r}{\mathrm{sec}\ \mathrm{C}}.$$

$$\text{Tang } b = \frac{\mathrm{tang}\ a\,.\,r}{\mathrm{cos}\ \mathrm{C}} = \frac{\mathrm{sec}\ \mathrm{C}\,.\,r}{\mathrm{cot}\ a} = \frac{\mathrm{sec}\ \mathrm{C}\,.\,\mathrm{tang}\ a}{r}.$$

Case X. Given a side of a right-angled spherical triangle, and its opposite angle, to find the side adjacent to that angle.

Solution.

$$\text{Sine } c = \frac{\mathrm{tang}\ a\,.\,\mathrm{cot}\ \mathrm{A}}{r} = \frac{\mathrm{tang}\ a\,.\,r}{\mathrm{tang}\ \mathrm{A}} = \frac{\mathrm{cot}\ \mathrm{A}\,.\,r}{\mathrm{cot}\ a}.$$

$$\text{Cosec } c = \frac{\mathrm{tang}\ \mathrm{A}\,.\,r}{\mathrm{tang}\ a} = \frac{\mathrm{tang}\ \mathrm{A}\,.\,\mathrm{cot}\ a}{r} = \frac{\mathrm{cot}\ a\,.\,r}{\mathrm{cot}\ \mathrm{A}}.$$

$$\text{And, Sine } a = \frac{\mathrm{tang}\ c\,.\,\mathrm{cot}\ \mathrm{C}}{r} = \frac{\mathrm{tang}\ c\,.\,r}{\mathrm{tang}\ \mathrm{C}} = \frac{\mathrm{cot}\ \mathrm{C}\,.\,r}{\mathrm{cot}\ c}.$$

$$\text{Cosec } a = \frac{\mathrm{tang}\ \mathrm{C}\,.\,r}{\mathrm{tang}\ c} = \frac{\mathrm{tang}\ \mathrm{C}\,.\,\mathrm{cot}\ c}{r} = \frac{\mathrm{cot}\ c\,.\,r}{\mathrm{cot}\ \mathrm{C}}.$$

Also, Tang $(45^\circ+\tfrac{1}{2}c)=\pm r\sqrt{\dfrac{\text{sine }(A+a)}{\text{sine }(A-a)}}$.

And, Tang $(45^\circ+\tfrac{1}{2}a)=\pm r\sqrt{\dfrac{\text{sine }(C+c)}{\text{sine }(C-c)}}$.

CASE XI. Given a side of a right-angled spherical triangle, and its opposite angle, to find the adjacent angle.

SOLUTION. $\text{Sine } C=\dfrac{\cos A\,.\,r}{\cos a}=\dfrac{\cos A\,.\,\sec a}{r}=\dfrac{\sec a\,.\,r}{\sec A}$.

$$\text{Cosec } C=\frac{\sec A\,.\,r}{\sec a}=\frac{\sec A\,.\,\cos a}{r}=\frac{\cos a\,.\,r}{\cos A}.$$

$$\text{And, Sine } A=\frac{\cos C\,.\,r}{\cos c}=\frac{\cos C\,.\,\sec c}{r}=\frac{\sec c\,.\,r}{\sec C}.$$

$$\text{Cosec } A=\frac{\sec C\,.\,r}{\sec c}=\frac{\sec C\,.\,\cos c}{r}=\frac{\cos c\,.\,r}{\cos C}.$$

Also, Tang $(45^\circ+\tfrac{1}{2}C)=\pm\sqrt{\cot\tfrac{1}{2}(A+a)\,.\,\cot\tfrac{1}{2}(A-a)}$.

And, Tang $(45^\circ+\tfrac{1}{2}A)=\pm\sqrt{\cot\tfrac{1}{2}(C+c)\,.\,\cot\tfrac{1}{2}(C-c)}$.

CASE XII. Given a side of a right-angled spherical triangle, and its opposite angle, to find the hypothenuse.

SOLUTION. $\text{Sine } b=\dfrac{\text{sine } a\,.\,r}{\text{sine } A}=\dfrac{\text{sine } a\,.\,\text{cosec } A}{r}=\dfrac{\text{cosec } A\,.\,r}{\text{cosec } a}$.

$$\text{Cosec } b=\frac{\text{sine } A\,.\,r}{\text{sine } a}=\frac{\text{sine } A\,.\,\text{cosec } a}{r}=\frac{\text{cosec } a\,.\,r}{\text{cosec } A}.$$

$$\text{And, Sine } b=\frac{\text{sine } c\,.\,r}{\text{sine } C}=\frac{\text{sine } c\,.\,\text{cosec } C}{r}=\frac{\text{cosec } C\,.\,r}{\text{cosec } c}.$$

$$\text{Cosec } b=\frac{\text{sine } C\,.\,r}{\text{sine } c}=\frac{\text{sine } C\,.\,\text{cosec } c}{r}=\frac{\text{cosec } c\,.\,r}{\text{cosec } C}.$$

Also, tang $(45^\circ+\tfrac{1}{2}b)=\pm\sqrt{\text{tang}\,\tfrac{1}{2}(C+c)\,.\,\cot\tfrac{1}{2}(C-c)}$.

And, tang $(45^\circ+\tfrac{1}{2}b)=\pm\sqrt{\text{tang}\,\tfrac{1}{2}(A+a)\,.\,\cot\tfrac{1}{2}(A-a)}$.

CASE XIII. Given the two sides, or legs, of a right-angled spherical triangle, to find an angle.

SOLUTION. $\text{Cot } A=\dfrac{\text{sine } c\,.\,r}{\text{tang } a}=\dfrac{\text{sine } c\,.\,\cot a}{r}=\dfrac{\cot a\,.\,r}{\text{cosec } c}$

$$\text{Tang A}=\frac{\text{tang }a\,.\,r}{\text{sine }c}=\frac{\text{tang }a\,.\,\text{cosec }c}{r}=\frac{\text{cosec }c\,.\,r}{\text{cot }a}.$$

$$\text{And, Cot C}=\frac{\text{sine }a\,.\,r}{\text{tang }c}=\frac{\text{sine }a\,.\,\text{cot }c}{r}=\frac{\text{cot }c\,.\,r}{\text{cosec }a}.$$

$$\text{Tang C}=\frac{\text{tang }c\,.\,r}{\text{sine }a}=\frac{\text{tang }c\,.\,\text{cosec }a}{r}=\frac{\text{cosec }a\,.\,r}{\text{cot C}}.$$

Case XIV. Given the two sides, or legs, of a right-angled spherical triangle, to find the hypothenuse.

Solution. $\text{Cos }b=\frac{\text{cos }a\,.\,\text{cos }c}{r}=\frac{\text{cos }a\,.\,r}{\text{sec }c}=\frac{\text{cos }c\,.\,r}{\text{sec }a}.$

$$\text{Sec }b=\frac{\text{sec }a\,.\,r}{\text{cos }c}=\frac{\text{sec }a\,.\,\text{sec }c}{r}=\frac{\text{sec }c\,.\,r}{\text{cos }a}.$$

$$\text{And, Cos }b=\frac{\text{cos }c\,.\,\text{cos }a}{r}=\frac{\text{cos }c\,.\,r}{\text{sec }a}=\frac{\text{cos }a\,.\,r}{\text{sec }c}.$$

$$\text{Sec }b=\frac{\text{sec }c\,.\,r}{\text{cos }a}=\frac{\text{sec }c\,.\,\text{sec }a}{r}=\frac{\text{sec }a\,.\,r}{\text{cos }c}.$$

Case XV. Given two angles of a right-angled spherical triangle, to find a side, or leg.

Solution. $\text{Cos }a=\frac{\text{cos A}\,.\,r}{\text{sine C}}=\frac{\text{cos A}\,.\,\text{cosec C}}{r}=\frac{\text{cosec C}\,.\,r}{\text{sec A}}.$

$$\text{Sec }a=\frac{\text{sine C}\,.\,r}{\text{cos A}}=\frac{\text{sine C}\,.\,\text{sec A}}{r}=\frac{\text{sec A}\,.\,r}{\text{cosec C}}.$$

$$\text{And, Cos }c=\frac{\text{cos C}\,.\,r}{\text{sine A}}=\frac{\text{cos C}\,.\,\text{cosec A}}{r}=\frac{\text{cosec A}\,.\,r}{\text{sec C}}.$$

$$\text{Sec }c=\frac{\text{sine A}\,.\,r}{\text{cos C}}=\frac{\text{sine A}\,.\,\text{sec C}}{r}=\frac{\text{sec C}\,.\,r}{\text{cosec A}}.$$

$$\text{Also, tang}\tfrac{1}{2}c=\sqrt{\text{tang}\left(\frac{\text{A}+\text{C}}{2}-45^\circ\right).\,\text{cot}\left(\frac{\text{A}-\text{C}}{2}+45^\circ\right)}.$$

$$\text{And, tang}\tfrac{1}{2}a=\sqrt{\text{tang}\left(\frac{\text{C}+\text{A}}{2}-45^\circ\right).\,\text{cot}\left(\frac{\text{C}-\text{A}}{2}+45^\circ\right)}.$$

Case XVI. Given two angles, of a right-angled spherical triangle, to find the hypothenuse.

Solution. $\text{Cos }b=\frac{\text{cot C}\,.\,\text{cot A}}{r}=\frac{\text{cot A}\,.\,r}{\text{tang C}}=\frac{\text{cot C}\,.\,r}{\text{tang A}}.$

$$\text{Sec }b=\frac{\text{tang C}\,.\,r}{\text{cot A}}=\frac{\text{tang C}\,.\,\text{tang A}}{r}=\frac{\text{tang A}\,.\,r}{\text{cot C}}.$$

$$\text{And, Cos } b = \frac{\cot \text{A} \,.\, \cot \text{C}}{r} = \frac{\cot \text{C} \,.\, r}{\text{tang A}} = \frac{\cot \text{A} \,.\, r}{\text{tang C}}.$$

$$\text{Sec } b = \frac{\text{tang A} \,.\, r}{\cot \text{C}} = \frac{\text{tang A} \,.\, \text{tang C}}{r} = \frac{\text{tang C} \,.\, r}{\cot \text{A}}.$$

$$\text{Also, tang } \tfrac{1}{2} b = \sqrt{\frac{-\cos(\text{A}+\text{C})}{\cos(\text{A} \sim \text{C})}}.$$

GENERAL OBSERVATIONS ON THE SPECIES AND AMBIGUITY OF THE CASES.

(A) The species of the sides and angles may be determined from the equations produced by Baron Napier's Rules, or from the preceding formulæ, by attending to the signs of the quantities which compose the equations or formulæ.

The sides which contain the right angle are each of the same species as their opposite angles, viz. a is of the same species with A, and b is of the same species with B. (R. 145.)

It may be proper to observe that where a quantity is to be determined by the sines only, and a side or angle opposite to the quantity sought does not enter into the equation, the case will be ambiguous, thus in the XIIth case, where sine $b = \frac{\text{sine } a \,.\, r}{\text{sine A}}$, the hypothenuse b is ambiguous.

Again, in the IId case, where sine $a = \frac{\text{sine A} \,.\, \text{sine } b}{r}$, the sine of a is evidently determinate, because it is of the same species with A which is a given quantity.

(B) When an unknown quantity is to be determined by its cosine, tangent, or cotangent, the sign of this value will always determine its species; for, if its proper sign be +, the arc will be less than 90°; if the proper sign be −, the arc will be greater than 90°. (K. 100.)

(C) Again, in Case VIth, where rad × cos b = cos c × cos a, it is obvious that the three sides are each less than 90°, or that two of them are greater than 90°, and the third less; as no other combination can render the sign of cos c × cos a like that of cos b as the equation requires.*

QUADRANTAL TRIANGLES.

(D) Any spherical triangle of which A, B, C, are the angles, and a, b, c, the opposite sides, may be changed into a spherical triangle of which the angles are supplements of the sides a, b, c, and the sides supplements of the angles A, B, C, (U 137.) viz. if we call A', B', C' the angles of the supplemental triangle, and a', b', c' the sides opposite to these angles, we shall have

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* Legendre's Geometry, 6th Edition, page 381.

$$A'=180^\circ-a;\ B'=180^\circ-b;\ C'=180^\circ-c$$
$$a'=180^\circ-A;\ b'=180^\circ-B;\ c'=180^\circ-C$$

Hence it is plain, that if a spherical triangle, has a side b equal to a quadrant, the corresponding angle B' of the supplemental triangle will be a right angle, and since there are always three *given* parts in a triangle, the supplemental triangle will be a right-angled triangle, having *two* parts given to find the rest; consequently, by finding the required parts in the supplemental right-angled triangle, the different parts of the quadrantal triangle will be known.

(E) Formulæ might have been inserted for solving the different cases of quadrantal triangles, but this would be making an increase of formulæ to very little purpose, since all quadrantal triangles are easily turned into right-angled triangles.

SOLUTIONS OF THE DIFFERENT CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

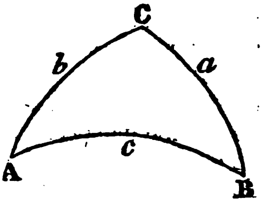
CASE I. Given two sides of an oblique-angled spherical triangle, and an angle opposite to one of them, to find the angle opposite to the other.

SOLUTION.

$$\text{sine } A=\frac{\text{sine } B\ .\ \text{sine } a}{\text{sine } b}=\frac{\text{sine } C\ .\ \text{sine } A}{\text{sine } c}$$

$$\text{sine } B=\frac{\text{sine } A\ .\ \text{sine } b}{\text{sine } a}=\frac{\text{sine } C\ .\ \text{sine } b}{\text{sine } c}$$

$$\text{sine } C=\frac{\text{sine } A\ .\ \text{sine } c}{\text{sine } A}=\frac{\text{sine } B\ .\ \text{sine } c}{\text{sine } b}$$



CASE II. Given two sides of an oblique-angled spherical triangle, and an angle opposite to one of them, to find the angle contained between these sides.

SOLUTION. Find the angle opposite to the other given side by Case I.

$$\text{Then, Cot } \tfrac{1}{2}\, A=\frac{\text{sine } \frac{1}{2}\,(c+b)}{\text{sine } \frac{1}{2}\,(c\sim b)}.\ \text{tang } \tfrac{1}{2}\,(C\sim B)$$

$$\text{Cot } \tfrac{1}{2}\, B=\frac{\text{sine } \frac{1}{2}\,(c+a)}{\text{sine } \frac{1}{2}\,(c\sim a)}.\ \text{tang } \tfrac{1}{2}\,(C\sim A)$$

$$\text{Cot } \tfrac{1}{2}\,\mathrm{C} = \frac{\text{sine } \frac{1}{2}\,(a+b)}{\text{sine } \frac{1}{2}\,(a \sim b)} .\ \text{tang } \tfrac{1}{2}\,(\mathrm{A} \sim \mathrm{B})$$

$$\text{Also, I. Tang } \phi = \frac{\cos c . \text{tang } \mathrm{B}}{\text{rad}};\ \text{tang,sine}(\phi + \mathrm{A}) = \frac{\text{tang}\, c .\ \text{sine}\, \phi}{\text{tang } b}$$

$$\text{II. Tang } \varphi = \frac{\cos c . \text{tang } \mathrm{C}}{\text{rad}};\ \text{then,sine}(\phi + \mathrm{B}) = \frac{\text{tang}\, a . \text{sine}\, \phi}{\text{tang } c}$$

$$\text{III. Tang } \phi = \frac{\cos b . \text{tang } \mathrm{A}}{\text{rad}};\ \text{then,sine}(\phi + \mathrm{C}) = \frac{\text{tang}\, b . \text{sine}\, \phi}{\text{tang } a}$$

CASE III. Given two sides of an oblique-angled spherical triangle, and an angle opposite to one of them, to find the other side.

SOLUTION. Find the angle opposite to the other given side by Case I.

$$\text{Then, Tang } \tfrac{1}{2}\, a = \frac{\text{sine } \frac{1}{2}\,(\mathrm{C}+\mathrm{B})}{\text{sine } \frac{1}{2}\,(\mathrm{C} \sim \mathrm{B})} .\ \text{tang } \tfrac{1}{2}\,(c \sim b)$$

$$\text{Tang } \tfrac{1}{2}\, b = \frac{\text{sine } \frac{1}{2}\,(\mathrm{C}+\mathrm{A})}{\text{sine } \frac{1}{2}\,(\mathrm{C} \sim \mathrm{A})} .\ \text{tang } \tfrac{1}{2}\,(c \sim a)$$

$$\text{Tang } \tfrac{1}{2}\, c = \frac{\text{sine } \frac{1}{2}\,(\mathrm{A}+\mathrm{B})}{\text{sine } \frac{1}{2}\,(\mathrm{A} \sim \mathrm{B})} .\ \text{tang } \tfrac{1}{2}\,(a \sim b)$$

$$\text{Also, I. Tang } \phi = \frac{\cos \mathrm{B} . \text{tang } c}{\text{rad}};\ \text{then, } \cos(a \sim \phi) = \frac{\cos b . \cos \varphi}{\cos c}.$$

$$\text{II. Tang } \varphi = \frac{\cos \mathrm{C} . \text{tang } a}{\text{rad}};\ \text{then, } \cos(b \sim \varphi) = \frac{\cos c . \cos \varphi}{\cos a}.$$

$$\text{III. Tang } \varphi = \frac{\cos \mathrm{A} . \text{tang } b}{\text{rad}};\ \text{then, } \cos(c \sim \varphi) = \frac{\cos a . \cos \varphi}{\cos b}.$$

CASE IV. Given two angles of an oblique-angled spherical triangle, and a side opposite to one of them, to find the other opposite side.

$$\text{SOLUTION. Sine}\, a = \frac{\text{sine } \mathrm{A} . \text{sine } b}{\text{sine } \mathrm{B}} = \frac{\text{sine } \mathrm{A} . \text{sine } c}{\text{sine } \mathrm{C}}$$

$$\text{Sine } b = \frac{\text{sine } \mathrm{B} . \text{sine } a}{\text{sine } \mathrm{A}} = \frac{\text{sine } \mathrm{B} . \text{sine } c}{\text{sine } \mathrm{C}}$$

$$\text{Sine } c = \frac{\text{sine } \mathrm{C} . \text{sine } a}{\text{sine } \mathrm{A}} = \frac{\text{sine } \mathrm{C} . \text{sine } b}{\text{sine } \mathrm{B}}$$

CASE V. Given two angles of an oblique-angled spherical triangle, and a side opposite to one of them, to find the side adjacent to these angles.

Solution. Find the side opposite to the other given angle by Case IV.

$$\text{Then, Tang } \tfrac{1}{2}\, a = \frac{\text{sine } \frac{1}{2}\,(\mathrm{C}+\mathrm{B})}{\text{sine } \frac{1}{2}\,(\mathrm{C}\sim\mathrm{B})} . \text{ tang } \tfrac{1}{2}\,(c\sim b)$$

$$\text{Tang } \tfrac{1}{2}\, b = \frac{\text{sine } \frac{1}{2}\,(\mathrm{A}+\mathrm{C})}{\text{sine } \frac{1}{2}\,(\mathrm{A}\sim\mathrm{C})} . \text{ tang } \tfrac{1}{2}\,(a\sim c)$$

$$\text{Tang } \tfrac{1}{2}\, c = \frac{\text{sine } \frac{1}{2}\,(\mathrm{A}+\mathrm{B})}{\text{sine } \frac{1}{2}\,(\mathrm{A}\sim\mathrm{B})} . \text{ tang } \tfrac{1}{2}\,(a\sim b)$$

$$\text{Also, I. Tang } \varphi = \frac{\cos \mathrm{C} \,.\, \text{tang}\, b}{\text{rad}};\ \text{then, sine}(a\sim\varphi) = \frac{\text{sine}\,\varphi \,.\, \text{tang}\,\mathrm{C}}{\text{tang } \mathrm{B}}.$$

$$\text{II. Tang } \varphi = \frac{\cos \mathrm{A} \,.\, \text{tang}\, c}{\text{rad}};\ \text{then, sine}(b\sim\varphi) = \frac{\text{sine}\,\varphi \,.\, \text{tang}\,\mathrm{A}}{\text{tang } \mathrm{C}}.$$

$$\text{III. Tang } \varphi = \frac{\cos \mathrm{B} \,.\, \text{tang}\, a}{\text{rad}};\ \text{then, sine}(c\sim\varphi) = \frac{\text{sine}\,\varphi \,.\, \text{tang}\,\mathrm{B}}{\text{tang } \mathrm{A}}.$$

Case VI. Given two angles of an oblique-angled spherical triangle, and a side opposite to one of them, to find the third angle.

Solution. Find the side opposite to the other given angle by Case IV.

$$\text{Then, Cot } \tfrac{1}{2}\, \mathrm{A} = \frac{\text{sine } \frac{1}{2}\,(c+b)}{\text{sine } \frac{1}{2}\,(c\sim b)} . \text{ tang } \tfrac{1}{2}\,(\mathrm{C}\sim\mathrm{B})$$

$$\text{Cot } \tfrac{1}{2}\, \mathrm{B} = \frac{\text{sine } \frac{1}{2}\,(a+c)}{\text{sine } \frac{1}{2}\,(a\sim c)} . \text{ tang } \tfrac{1}{2}\,(\mathrm{A}\sim\mathrm{C})$$

$$\text{Cot } \tfrac{1}{2}\, \mathrm{C} = \frac{\text{sine } \frac{1}{2}\,(a+b)}{\text{sine } \frac{1}{2}\,(a\sim b)} . \text{ tang } \tfrac{1}{2}\,(\mathrm{A}\sim\mathrm{B})$$

$$\text{Also, I. Cot}\,\varphi = \frac{\cos b \,.\, \text{tang } \mathrm{G}}{\text{rad}};\ \text{then, sine}(\mathrm{A}\sim\varphi=)\frac{\cos \mathrm{B} \,.\, \text{sine}\,\varphi}{\cos \mathrm{C}}.$$

$$\text{II. Cot}\,\varphi = \frac{\cos c \,.\, \text{tang } \mathrm{A}}{\text{rad}};\ \text{then, sine}(\mathrm{B}\sim\varphi) = \frac{\cos \mathrm{C} \,.\, \text{sine}\,\varphi}{\cos \mathrm{A}}.$$

$$\text{III. Cot}\,\varphi = \frac{\cos a \,.\, \text{tang } \mathrm{B}}{\text{rad}};\ \text{then, sine}(\mathrm{C}\sim\varphi) = \frac{\cos \mathrm{A} \,.\, \text{sine}\,\varphi}{\cos \mathrm{B}}.$$

Case VII. Given two sides of an oblique-angled spherical triangle, and the angle contained between them, to find the other angles.

Solution.

$$\text{I.}\begin{cases} \text{Tang}\tfrac{1}{2}\,(\mathrm{A}+\mathrm{B}) = \dfrac{\cos \frac{1}{2}\,(a\sim b)}{\cos \frac{1}{2}\,(a+b)} . \cot \tfrac{1}{2}\, \mathrm{C} \\ \text{Tang}\tfrac{1}{2}\,(\mathrm{A}\sim\mathrm{B}) = \dfrac{\text{sine } \frac{1}{2}\,(a\sim b)}{\text{sine } \frac{1}{2}\,(a+b)} . \cot \tfrac{1}{2}\, \mathrm{C} \end{cases}$$

$$\text{II.}\left\{\begin{array}{l}\text{Tang}\ \frac{1}{2}\,(\text{A}+\text{C})=\dfrac{\cos\frac{1}{2}\,(a\sim c)}{\cos\frac{1}{2}\,(a+c)}\,.\,\cot\frac{1}{2}\,\text{B}\\[2ex] \text{Tang}\ \frac{1}{2}\,(\text{A}\sim\text{C})=\dfrac{\text{sine}\,\frac{1}{2}\,(a\sim c)}{\text{sine}\,\frac{1}{2}\,(a+c)}\,.\,\cot\frac{1}{2}\,\text{B}\end{array}\right.$$

$$\text{III.}\left\{\begin{array}{l}\text{Tang}\ \frac{1}{2}\,(\text{C}+\text{B})=\dfrac{\cos\frac{1}{2}\,(c\sim b)}{\cos\frac{1}{2}\,(c+b)}\,.\,\cot\frac{1}{2}\,\text{A}\\[2ex] \text{Tang}\ \frac{1}{2}\,(\text{C}\sim\text{B})=\dfrac{\text{sine}\,\frac{1}{2}\,(c\sim b)}{\text{sine}\,\frac{1}{2}\,(c+b)}\,.\,\cot\frac{1}{2}\,\text{A}\end{array}\right.$$

$$\text{Also, IV.}\left\{\begin{array}{l}\text{Cot}\ \text{A}=\dfrac{(\cot a\,.\,\text{sine}\,b)-(\cos\text{C}\,.\,\cos b)}{\text{sine C}}\\[2ex] \text{Cot}\ \text{B}=\dfrac{(\cot b\,.\,\text{sine}\,a)-(\cos\text{C}\,.\,\cos a)}{\text{sine C}}\end{array}\right.$$

$$\text{V.}\left\{\begin{array}{l}\text{Cot}\ \text{C}=\dfrac{(\cot c\,.\,\text{sine}\,a)-(\cos\text{B}\,.\,\cos a)}{\text{sine B}}\\[2ex] \text{Cot}\ \text{A}=\dfrac{(\cot a\,.\,\text{sine}\,c)-(\cos\text{B}\,.\,\cos c)}{\text{sine B}}\end{array}\right.$$

$$\text{VI.}\left\{\begin{array}{l}\text{Cot}\ \text{B}=\dfrac{(\cot b\,.\,\text{sine}\,c)-(\cos\text{A}\,.\,\cos b)}{\text{sine A}}\\[2ex] \text{Cot}\ \text{C}=\dfrac{(\cot c\,.\,\text{sine}\,b)-(\cos\text{A}\,.\,\cos b)}{\text{sine A}}\end{array}\right.$$

$$\text{VII.}\left\{\begin{array}{l}\text{Tang}\,\varphi=\dfrac{\cos\text{A}\,.\,\text{tang}\,b}{\text{rad}};\ \text{then,}\cot\text{B}=\dfrac{\cot\text{A}}{\text{sine}\,\varphi}.\text{sine}(\text{C}\sim\varphi)\\[2ex] \text{Tang}\,\varphi=\dfrac{\cos\text{A}\,.\,\text{tang}\,c}{\text{rad}};\ \text{then,}\cot\text{C}=\dfrac{\cot\text{A}}{\text{sine}\,\varphi}.\,\text{sine}(b\sim\varphi)\end{array}\right.$$

Here b, c, and the included $\angle$ A are given, and formulæ will be obtained by a mere change of letters, if a, b, and the $\angle$ C; or if a, c, and the $\angle$ B are given.

CASE VIII. Given two sides of an oblique-angled spherical triangle, and the angle contained between them, to find the third side.

SOLUTION. Find the other two angles by Case VII, and then find the third side by Case V.

$$\text{Or, I. Tang}\,\varphi=\frac{\cos\text{A}\,.\,\text{tang}\,c}{\text{rad}};\ \text{then, }\cos a=\frac{\cos c}{\cos\varphi}.\cos\,(b\sim\varphi)$$

$$\text{II. Tang}\,\varphi=\frac{\cos\text{B}\,.\,\text{tang}\,a}{\text{rad}};\ \text{then, }\cos b=\frac{\cos a}{\cos\varphi}.\cos\,(c\sim\varphi)$$

$$\text{III. Tang}\,\varphi=\frac{\cos\text{C}\,.\,\text{tang}\,b}{\text{rad}};\ \text{then, }\cos c=\frac{\cos b}{\cos\varphi}.\cos(a\sim\varphi)$$

Or, the sides a, b, or c respectively, may be found by the *fourth* set of equations, page 184.

CASE IX. Given two angles of an oblique-angled spherical triangle, and the side adjacent to both of them, to find a side opposite to one of the given angles.

SOLUTION.

$$\text{I.}\left\{\begin{aligned}&\text{Tang}\ \tfrac{1}{2}(a+b)=\frac{\cos\tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})}{\cos\tfrac{1}{2}(\mathrm{A}+\mathrm{B})}.\ \text{tang}\ \tfrac{1}{2}c\\&\text{Tang}\ \tfrac{1}{2}(a\sim b)=\frac{\text{sine}\ \tfrac{1}{2}(\mathrm{A}\sim\mathrm{B})}{\text{sine}\ \tfrac{1}{2}(\mathrm{A}+\mathrm{B})}.\ \text{tang}\ \tfrac{1}{2}c\end{aligned}\right.$$

$$\text{II.}\left\{\begin{aligned}&\text{Tang}\ \tfrac{1}{2}(a+c)=\frac{\cos\tfrac{1}{2}(\mathrm{A}\sim\mathrm{C})}{\cos\tfrac{1}{2}(\mathrm{A}+\mathrm{C})}.\ \text{tang}\ \tfrac{1}{2}b\\&\text{Tang}\ \tfrac{1}{2}(a\sim c)=\frac{\text{sine}\tfrac{1}{2}(\mathrm{A}\sim\mathrm{C})}{\text{sine}\ \tfrac{1}{2}(\mathrm{A}+\mathrm{C})}.\ \text{tang}\ \tfrac{1}{2}b.\end{aligned}\right.$$

$$\text{III.}\left\{\begin{aligned}&\text{Tang}\ \tfrac{1}{2}(c+b)=\frac{\cos\tfrac{1}{2}(\mathrm{C}\sim\mathrm{B})}{\cos\tfrac{1}{2}(\mathrm{C}+\mathrm{B})}.\ \text{tang}\ \tfrac{1}{2}a\\&\text{Tang}\quad \tfrac{1}{2}\sim b)=\frac{\text{sine}\tfrac{1}{2}(\mathrm{C}\sim\mathrm{B})}{\text{sine}\tfrac{1}{2}(\mathrm{C}+\mathrm{B})}.\ \text{tang}\ \tfrac{1}{2}a\end{aligned}\right.$$

Also,

$$\text{IV.}\left\{\begin{aligned}&\text{Cot}\ a=\frac{(\cot\mathrm{A}\ .\ \text{sine}\ \mathrm{B})+(\cos\mathrm{B}\ .\cos c)}{\text{sine}\ c}\\&\text{Cot}\ b=\frac{(\cot\mathrm{B}\ .\ \text{sine}\ \mathrm{A})+(\cos\mathrm{A}\ .\cos c)}{\text{sine}\ c}\end{aligned}\right.$$

$$\text{V.}\left\{\begin{aligned}&\text{Cot}\ c=\frac{(\cot\mathrm{C}\ .\ \text{sine}\ \mathrm{A})+(\cos\mathrm{A}\ .\cos b)}{\text{sine}\ b}\\&\text{Cot}\ a=\frac{(\cot\mathrm{A}\ .\ \text{sine}\ \mathrm{C})+(\cos\mathrm{C}\ .\cos b)}{\text{sine}\ b}\end{aligned}\right.$$

$$\text{VI.}\left\{\begin{aligned}&\text{Cot}\ b=\frac{(\cot\mathrm{B}\ .\ \text{sine}\ \mathrm{C})+(\cos\mathrm{C}\ .\cos a)}{\text{sine}\ a}\\&\text{Cot}\ c=\frac{(\cot\mathrm{C}\ .\ \text{sine}\ \mathrm{B})+(\cos\mathrm{B}\ .\cos a)}{\text{sine}\ a}\end{aligned}\right.$$

$$\text{VII.}\left\{\begin{aligned}&\text{Cot}\ \varphi=\frac{\text{tang}\ \mathrm{A}\ .\cos b}{\text{rad}};\ \text{then,}\ \cot a=\frac{\cot b}{\cos\varphi}.\ \cos(\mathrm{C}\sim\varphi)\\&\text{Cot}\ \varphi=\frac{\text{tang}\ \mathrm{C}.\cos b}{\text{rad}};\ \text{then,}\ \cot c=\frac{\cot b}{\cos\varphi}.\ \cos(\mathrm{A}\sim\varphi)\end{aligned}\right.$$

Here ∠ A, ∠ C and the included side *b* are given; and formulæ will be obtained by a change of letters, if ∠ A and ∠ B and the included side *c* are given; or if ∠ B and ∠ C and the included side *a* are given.

CASE X. Given two angles of an oblique-angled spherical triangle, and the side adjacent to both of them, to find the other angle.

Solution. Find the other two sides by Case IX, and then find the other angle by Case VI.

Or, I. $\text{Cot } \varphi = \frac{\cos a \,.\, \text{tang C}}{\text{rad}}$; then, $\cos \text{A} = \cos \text{C} \,.\, \frac{\text{sine } (\text{B} \sim \varphi)}{\text{sine } \varphi}$

II. $\text{Cot } \varphi = \frac{\cos b \,.\, \text{tang A}}{\text{rad}}$; then, $\cos \text{B} = \cos \text{A} \,.\, \frac{\text{sine } (\text{C} \sim \varphi)}{\text{sine } \varphi}$

III. $\text{Cot } \varphi = \frac{\cos c \,.\, \text{tang B}}{\text{rad}}$; then, $\cos \text{C} = \cos \text{B} \,.\, \frac{\text{sine } (\text{A} \sim \varphi)}{\text{sine } \varphi}$

Or, the angles A, B, or C, respectively, may be found by the *sixth* set of equations, page 184.

Case XI. Given the three sides of an oblique-angled spherical triangle, to find the angles.

Solution. Let $a+b+c=s$, then

I. $\text{Sine } \frac{1}{2} \text{A} = \text{rad} \sqrt{\frac{\text{sine } (\frac{1}{2} s - b) \,.\, \text{sine } (\frac{1}{2} s - c)}{\text{sine } b \,.\, \text{sine } c}}$

II. $\text{Cos } \frac{1}{2} \text{A} = \text{rad} \sqrt{\frac{\text{sine } \frac{1}{2} s \,.\, \text{sine } (\frac{1}{2} s - a)}{\text{sine } b \,.\, \text{sine } c}}$

III. $\text{Tang } \frac{1}{2} \text{A} = \text{rad} \sqrt{\frac{\text{sine } (\frac{1}{2} s - b) \,.\, \text{sine } (\frac{1}{2} s - c)}{\text{sine } \frac{1}{2} s \,.\, \text{sine } (\frac{1}{2} s - a)}}$

IV. $\text{Cot } \frac{1}{2} \text{A} = \text{rad} \sqrt{\frac{\text{sine } \frac{1}{2} s \,.\, \text{sine } (\frac{1}{2} s - a)}{\text{sine } (\frac{1}{2} s - b) \,.\, \text{sine } (\frac{1}{2} s - c)}}$

I. $\text{Sine } \frac{1}{2} \text{B} = \text{rad} \sqrt{\frac{\text{sine } (\frac{1}{2} s - a) \,.\, \text{sine } (\frac{1}{2} s - c)}{\text{sine } a \,.\, \text{sine } c}}$

II. $\text{Cos } \frac{1}{2} \text{B} = \text{rad} \sqrt{\frac{\text{sine } \frac{1}{2} s \,.\, \text{sine } (\frac{1}{2} s - b)}{\text{sine } a \,.\, \text{sine } c}}$

III. $\text{Tang } \frac{1}{2} \text{B} = \text{rad} \sqrt{\frac{\text{sine } (\frac{1}{2} s - a) \,.\, \text{sine } (\frac{1}{2} s - c)}{\text{sine } \frac{1}{2} s \,.\, \text{sine } (\frac{1}{2} s - b)}}$

IV. $\text{Cot } \frac{1}{2} \text{B} = \text{rad} \sqrt{\frac{\text{sine } \frac{1}{2} s \,.\, \text{sine } (\frac{1}{2} s - b)}{\text{sine } (\frac{1}{2} s - a) \,.\, \text{sine } (\frac{1}{2} s - c)}}$

I. $\text{Sine } \frac{1}{2} \text{C} = \text{rad} \sqrt{\frac{\text{sine } (\frac{1}{2} s - b) \,.\, \text{sine } (\frac{1}{2} s - a)}{\text{sine } a \,.\, \text{sine } b}}$

II. $\text{Cos } \frac{1}{2} \text{C} = \text{rad} \sqrt{\frac{\text{sine } \frac{1}{2} s \,.\, \text{sine } (\frac{1}{2} s - c)}{\text{sine } a \,.\, \text{sine } b}}$

$$\text{III. Tang}\ \tfrac{1}{2}\,c=\text{rad}\sqrt{\frac{\text{sine}\,(\frac{1}{2}s-b)\,.\,\text{sine}\,(\frac{1}{2}s-a)}{\text{sine}\,\frac{1}{2}s\,.\,\text{sine}\,(\frac{1}{2}s-c)}}$$

$$\text{IV. Cot}\ \tfrac{1}{2}\,c=\text{rad}\sqrt{\frac{\text{sine}\,\frac{1}{2}s\,.\,\text{sine}\,(\frac{1}{2}s-c)}{\text{sine}\,(\frac{1}{2}s-b)\,.\,\text{sine}\,(\frac{1}{2}s-c)}}$$

Or, the angles A, B, or C, respectively, may be found by the formulæ A. 182.

CASE XII. Given the three angles of an oblique-angled spherical triangle, to find the sides.

SOLUTION. Let $\text{A}+\text{B}+\text{C}=\text{S}$, then

$$\text{I. Sine}\ \tfrac{1}{2}\,a=\text{rad}\sqrt{\frac{\cos\frac{1}{2}\text{S}\,.\,\cos(\frac{1}{2}\text{S}-\text{A})}{\text{sine B}\,.\,\text{sine C}}}$$

$$\text{II. Cos}\ \tfrac{1}{2}\,a=\text{rad}\sqrt{\frac{\cos(\frac{1}{2}\text{S}-\text{C})\,.\,\cos(\frac{1}{2}\text{S}-\text{B})}{\text{sine B}\,.\,\text{sine C}}}$$

$$\text{III. Tang}\ \tfrac{1}{2}\,a=\text{rad}\sqrt{\frac{\cos\frac{1}{2}\text{S}\,.\,\cos(\frac{1}{2}\text{S}-\text{A})}{\cos(\frac{1}{2}\text{S}-\text{C})\,.\,\cos(\frac{1}{2}\text{S}-\text{B})}}$$

$$\text{IV. Cot}\ \tfrac{1}{2}\,a=\text{rad}\sqrt{\frac{\cos(\frac{1}{2}\text{S}-\text{C})\,.\,\cos(\frac{1}{2}\text{S}-\text{B})}{\cos\frac{1}{2}\text{S}\,.\,\cos(\frac{1}{2}\text{S}-\text{A})}}$$

$$\text{I. Sine}\ \tfrac{1}{2}\,b=\text{rad}\sqrt{\frac{\cos\frac{1}{2}\text{S}\,.\,\cos(\frac{1}{2}\text{S}-\text{B})}{\text{sine A}\,.\,\text{sine C}}}$$

$$\text{II. Cot}\ \tfrac{1}{2}\,b=\text{rad}\sqrt{\frac{\cos(\frac{1}{2}\text{S}-\text{A})\,.\,\cos(\frac{1}{2}\text{S}-\text{C})}{\text{sine A}\,.\,\text{sine C}}}$$

$$\text{III. Tang}\ \tfrac{1}{2}\,b=\text{rad}\sqrt{\frac{\cos\frac{1}{2}\text{S}\,.\,\cos(\frac{1}{2}\text{S}-\text{B})}{\cos(\frac{1}{2}\text{S}-\text{A})\,.\,\cos(\frac{1}{2}\text{S}-\text{C})}}$$

$$\text{IV. Cot}\ \tfrac{1}{2}\,b=\text{rad}\sqrt{\frac{\cos(\frac{1}{2}\text{S}-\text{A})\,.\,\cos(\frac{1}{2}\text{S}-\text{C})}{\cos\frac{1}{2}\text{S}\,.\,\cos(\frac{1}{2}\text{S}-\text{B})}}$$

$$\text{I. Sine}\ \tfrac{1}{2}\,c=\text{rad}\sqrt{\frac{\cos(\frac{1}{2}\text{S}\,.\,\cos(\frac{1}{2}\text{S}-\text{C})}{\text{sine A}\,.\,\text{sine B}}}$$

$$\text{II. Cos}\ \tfrac{1}{2}\,c=\text{rad}\sqrt{\frac{\cos(\frac{1}{2}\text{S}-\text{A})\,.\,\cos(\frac{1}{2}\text{S}-\text{B})}{\text{sine A}\,.\,\text{sine B}}}$$

$$\text{III. Tang } \tfrac{1}{2}c = \text{rad}\sqrt{\frac{\cos \frac{1}{2} S \cdot \cos(\frac{1}{2} S - C)}{\cos(\frac{1}{2} S - A) \cdot \cos(\frac{1}{2} S - C)}}$$

$$\text{IV. Cot } \tfrac{1}{2}c = \text{rad}\sqrt{\frac{\cos(\frac{1}{2} S - A) \cdot \cos(\frac{1}{2} S - B)}{\cos \frac{1}{2} S \cdot \cos(\frac{1}{2} S - C)}}$$

Or, the sides a, b, or c, respectively, may be found by the formulæ in the *fifth* set of equations, page 184.

OF THE AMBIGUITY OF THE DIFFERENT CASES.

(F) When two sides, and an angle opposite to one of them, are given, to find the rest (see Case 1st, IId, and IIId), the values of these required parts are sometimes ambiguous.

(G) Practical Rules for determining whether the quantities sought are acute, obtuse, or ambiguous, are given in the solutions of the different cases.

The two following tables are the same as those given by Legendre at pages 400 and 401 of the 6th edition of his Geometry, and are deduced from Prop. XVII, page 149, and Prop. XVIII, page 150 of this treatise.

TABLE I. Let A, a and b be the given parts. Then,

$$1.\ A \sqsubset 90^\circ,\ b \sqsubset 90^\circ \begin{cases} a \sqsupset b \text{ one solution.} \\ a \sqsubset b \text{ two solutions.} \end{cases}$$

$$2.\ A \sqsubset 90^\circ,\ b \sqsupset 90^\circ \begin{cases} a + b \sqsupset 180^\circ \text{ one solution.} \\ a + b \sqsubset 180^\circ \text{ two solutions.} \end{cases}$$

$$3.\ A \sqsupset 90^\circ,\ b \sqsubset 90^\circ \begin{cases} a + b \sqsupset 180^\circ \text{ two solutions.} \\ a + b \sqsubset 180^\circ \text{ one solution.} \end{cases}$$

$$4.\ A \sqsupset 90^\circ,\ b \sqsupset 90^\circ \begin{cases} a \sqsupset b \text{ two solutions.} \\ a \sqsubset b \text{ one solution.} \end{cases}$$

If $A = 90^\circ$, $a = b$, or $a + b = 180$, the cases will not be ambiguous, but if $b = 90^\circ$, there will be two solutions.

When two angles, and a side opposite to one of them, are given, to find the rest (see Case IVth, Vth, and VIth), the values of the required parts are subject to ambiguity; this triangle being supplemental to that wherein two sides and an angle opposite to one of them are given.

TABLE II. Let A, B, and a be the given parts. Then,

$$1.\ a \sqsupset 90^\circ,\ B \sqsupset 90^\circ \begin{cases} A \sqsubset B \text{ one solution.} \\ A \sqsupset B \text{ two solutions.} \end{cases}$$

$$2.\ a \sqsupset 90^\circ,\ B \sqsubset 90^\circ \begin{cases} A + B \sqsubset 180^\circ \text{ one solution.} \\ A + B \sqsupset 180^\circ \text{ two solutions.} \end{cases}$$

$$3.\ a \sqsubset 90^\circ,\ B \sqsupset 90^\circ \begin{cases} A + B \sqsubset 180^\circ \text{ two solutions.} \\ A + B \sqsupset 180^\circ \text{ one solution.} \end{cases}$$

$$4.\ a \sqsubset 90^\circ,\ B \sqsubset 90^\circ \begin{cases} A \sqsubset B \text{ two solutions.} \\ A \sqsupset B \text{ one solution.} \end{cases}$$

If $a=90°$, $A=B$, or $A+B=180°$, there will be but one solution, but if $B=90°$, there will be two solutions.

(H) We may likewise remark, that since any side or angle of a spherical triangle is less than 180°, the half of any angle, or half the difference between any two sides, or half the difference between any two angles, must be acute.

Hence in the equation, where $\cot\frac{1}{2}c \cdot \cos\frac{1}{2}(a-b)=\tan\frac{1}{2}(A+B), \cos\frac{1}{2}(a+b)$ (M. 188) it is plain that $\cot\frac{1}{2}c$ and $\cos\frac{1}{2}(a-b)$ are both positive (K. 100.), and therefore $\tan\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ must be both positive; consequently, *half the sum of any two sides of a spherical triangle is of the same species as half the sum of their opposite angles.* This rule is applied in the practical solutions of the different cases, and will frequently remove the ambiguity which would otherwise arise, where a quantity sought is to be determined by means of a sine.

CHAP. VI.

I. PRACTICAL RULES FOR THE SOLUTIONS OF ALL THE DIFFERENT CASES OF RIGHT-ANGLED SPHERICAL TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

Every spherical triangle consists of six parts, three sides, and three angles; any three of which, being given, the rest may be found.

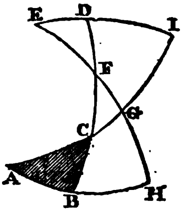
In a right-angled spherical triangle, two given parts, besides the right angle, are sufficient to determine the rest.

The questions arising from a variation of the given and required parts are 16, but if distinguished by the *data*, the number of cases is 6.

THE GIVEN QUANTITIES ARE, EITHER

1. The hypothenuse and an angle.
2. The hypothenuse and one side.
3. A side and its adjacent angle.
4. A side and its opposite angle.
5. The two sides.
6. The two angles.

(I) RULE I.



Draw a rough figure as in the margin; and let AG, AH; FB, FH; CI, CD; EI, EG, be considered as quadrants, or 90° each; then you have eight right-angled spherical triangles, every two of which will have equal angles at their bases: And the triangles CGF and EDF will have their respective sides and angles either equal to those of ABC, the triangle under consideration, or they will be the complements thereof. (L. 151.)

Then,

In triangles having equal angles at their bases. The sines of their bases have the same ratio to each other, as the tangents of their perpendiculars.

And,

The sines of the hypothenuses have the same ratio to each other, as the sines of the perpendiculars. (M. 167.)

(K) Illustration. 1*st. In the triangles* ABC *and* AHG.

1. Sine AG : sine AC : : sine HG : sine BC.
2. Sine AH : sine AB : : tang HG : tang BC.

and these are proportional by inversion.

2*d. In the triangles* FGC *and* FHB.

3. Sine BH : sine CG : : sine FB : sine FC.
4. Sine FH : sine FG : : tang BH : tang CG.

and inversely, &c.

3*d. In the triangles* CGF *and* CID.

5. Sine DC : sine FC : : sine ID : sine FG.
6. Sine CG : sine CI : : tang FG : tang ID.

and inversely, &c.

4*th. In the triangles* EDF *and* EIG.

7. Sine EI : sine ED : : tang IG : tang DF, and inversely, &c.

The student must remember that ABC is the proper triangle in the preceding proportions, and that AG, AH, &c. are each 90°, consequently sine AG, sine AH, &c. are each equal to radius.

BH is the complement of the base AB.
CG is the complement of the hypoth. AC.
FC is the complement of the perp. BC.
FG is the complement of the angle A.
ED is the complement of the angle C.

In any of these cases, for sine, or tangent, write co-sine, or co-tangent.

HG = EF is the measure of the angle A.
ID is the measure of the angle C.
IG = AC the hypothenuse, and DF = BC the perpendicular.

Since AB and BC are perpendicular to each other, either of them may be considered as the base, and to avoid a number of different figures it will sometimes be necessary to make A and C change places, as is done in some of the succeeding cases; then, in considering these remarks, where AB occurs read BC, and instead of the angle A read C; and the contrary.

(L) RULE II. BARON NAPIER'S RULES.

1. In every right-angled spherical triangle there are five parts, exclusive of the right-angle, which is not taken into consideration; and these five parts are the hypothenuse, the two sides or legs, and their opposite angles. Now in every case proposed for solution, there are three of these five parts concerned, that is, two given (together with the right-angle) and a third required.

2. If the three quantities under consideration, viz. the two which are given and that which is required, are joined together, or follow each other in a successive order, without the intervention of a side, or angle, not concerned in the question; the middle one is called the *middle part*, and the other two the *extremes conjunct*, because they are joined to the middle part.

3. If only two of the three things under consideration are connected, or joined together, they are invariably called *extremes disjunct*, that is, not joined to the middle part, and the other term which is not joined with them, is called the *middle part.*

Then,

4. *Radius* × *sine of the middle part=rectangle, or product, of the tangents of the extremes when they are conjunct.*

5. *Radius* × *sine of the middle part=rectangle, or product, of the cosines of the extremes when they are disjunct.*

Observe to write cosine or co-tangent, instead of sine or tangent; and sine instead of cosine, &c. when you make use of either of the angles, or the hypothenuse; but not when you use the sides or legs.

6. Having found which is the *middle part*, and written the terms down as expressed in one or other of the above rules, according as the extremes are *conjunct* or *disjunct*, turn the terms into a proportion, *thus:* Put the required term *last*, that with which it is connected *first*, and the remaining two in the middle in any order.

(M) RULE III.

1. Radius × sine of either side=sine of the opposite angle × sine of the hypothenuse.

2. Radius × sine of either side = tangent of the other side × cot of its opposite ∠.

3. Radius × cosine of either of the angles = tangent of the adjacent side × cot hypothenuse.

4. Radius × cosine of either of the angles = sine of the other angle × cosine of its adjacent side.

5. Radius × cosine of the hypothenuse = cot of one angle × cot of the other ∠.

6. Radius × cosine of the hypothenuse = cosine of one side × cosine of the other side.

Then put the required term last, that with which it is connected first, and the remaining two in the middle in any order, and you will have a correct proportion.

NOTE. This rule is the same as BARON NAPIER's (O. 169.)

OF THE DIFFERENT SPECIES OR AFFECTIONS OF RIGHT-ANGLED SPHERICAL TRIANGLES.

(N) I. *When the hypothenuse and an angle are given.*

1. The side opposite to the *given* angle, is of the same species with the *given* angle.

2. The side adjacent to the *given* angle, is acute or obtuse, according as the hypothenuse is of the same, or of different species with the *given* angle.

3. The other angle is acute or obtuse, according as the hypothenuse and the *given* angle are of the same or of different species.

(O) II. *When the hypothenuse and one side are given.*

1. The angle opposite to the *given* side, is of the same species with the *given* side.

2. The angle adjacent to the *given* side, is acute or obtuse, according as the hypothenuse is of the same or of different species with the *given* side.

3. The other side is acute or obtuse, according as the hypothenuse is of the same, or of different species with the *given* side.

(P) III. *When a side and its adjacent angle are given.*

1. The other angle is of the same species as the *given* side.

2. The other side is of the same species as the *given* angle.

3. The hypothenuse is acute or obtuse, according as the *given* angle is of the same or of different species with the *given* side.

(Q) IV. *When a side and its opposite angle are given.*

The required parts are always ambiguous or doubtful, that is, they may be either greater or less than 90°, and therefore admit of two answers.

(R) V. *When the two sides, containing the right-angle, are given.*

1. Each angle is of the same species as its opposite side.

2. The hypothenuse is acute or obtuse, according as the sides are of the same or of different species.

(S) VI. *When the two angles, adjacent to the hypothenuse, are given.*

1. Each side is of the same species as its opposite angle.

2. The hypothenuse is acute or obtuse, according as the angles are of the same or of different species.

(T) Case I. *Given the hypothenuse and an angle, to find the side adjacent to the given angle.*

Example. In the spherical triangle ABC.

Given { The hypothenuse AC = 78°20′ } Required the adjacent leg AB.
{ The angle A = 37°25′ }

BY NAPIER'S RULE.

The complement of the angle A is the *middle part*, AB and the complement of AC are the *extremes conjunct*, or joined to the middle part.

$$\text{Rad} \times \cos A = \text{tang}\ \overset{*}{AB} \times \cot AC.$$

∵ (10 + log cos A) − log cot AC = log tang AB.

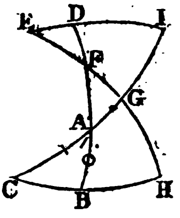
or, cot AC - - = 78°.20′ - 9·31489

: radius, sine of - 90°. - - 10·00000

:: cos ∠ A - - = 37°.25′ - 9·89995

: tang AB - - = 75°.25′ - 10·58506

Here AB *is acute.* (N. 211.)



BY RULE I.

Sine EI : sine ED :: tang IG : tang DF. But EI is radius, DI the measure of the angle A, and ED is complement, IG = AC, and DF = AB. Hence,

rad : cos ∠ A :: tang AC : tang AB.

BY CONSTRUCTION. (*Plate V. Fig.* 8.)

1. With the chord of 60 degrees describe the primitive circle; and through any point A in the primitive, draw a great circle A*ce*, making an angle BAC = 37°.25′. (P. 160.)

2. Set off the hypothenuse 78°.20′ from A to *m*, find *p* the pole of the oblique circle A*ce* (N. 159.); join *pm*, and through the point C, where it cuts A*ce*, draw BP*n*; then ABC is the triangle required.

To measure the required parts.

3. AB applied to a scale of chords gives the measure of AB = 75°.25′.

4. Take BC in your compasses, set one foot of the compasses on 90° on the scale of semi-tangents, and the other towards the left-hand of the scale, the degrees between the points $=36°.31'$ will be the measure of BC.

5. Produce *mp* to *b*, and find *a* the pole of *n*PB (N. 159.); then *ab* applied to the scale of chords will be the measure of the angle $C=81°.12'$.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The hypothenuse AC} & =61°.4'.56''\\ \text{The angle A} & =61.50.29.\end{array}\right\}\begin{array}{l}\text{Required the ad-}\\ \text{jacent leg AB.}\end{array}$$

Answer. $AB=40°.30'.20''$. acute. (N. 211.)

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The hypothenuse AC} & =113°.55'\\ \text{The angle A} & =31°.51'\end{array}\right\}\begin{array}{l}\text{Required the ad-}\\ \text{jacent leg AB.}\end{array}$$

Answer. $AB=117°.34'$. obtuse. (N. 211.)

(U) CASE II. *Given the hypothenuse and one angle, to find the side opposite to that given angle.*

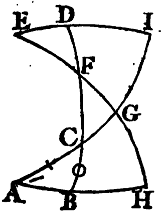
Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The hypothenuse AC} & =78°.20'\\ \text{The angle A} & =37°.25'\end{array}\right\}\begin{array}{l}\text{Required the oppo-}\\ \text{site leg BC.}\end{array}$$

BY NAPIER'S RULE.

Here BC is *the middle part*, and the complement of A and the complement of AC are the *extremes disjunct*; that is, not joined to the middle part, because of the intervention of AB and the angle C.

Rad × sine BC = sine AC × sine A.
∵ (log sine AC + log sine A) − 10 = log sine BC.
or, Radius, sine of - - 90°. - 10.00000
: sine AC - - - $=78°.20'$ - 9.99093
:: sine of ∠ A - $=37°.25'$ - 9.78362
: sine BC - - - $=36°.31'$ - 9.77455
Here BC *is acute.* (N. 211.)



BY RULE I.

Sine AG : sine AC :: sine HG : sine BC. But AG is radius, and HG is the measure of the angle A; hence,
Rad. : sine AC :: sine A : sine BC.

BY CONSTRUCTION.

See the construction of CASE I. (T. 212.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC=61°.4'.56" } Required the opposite leg BC.
{ The angle A =61°.50'.29" }

Answer. BC=50°.30'.30" acute. (N. 211.)

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC=113°.55' } Required the opposite leg BC.
{ The angle A =104°. 8' }

Answer. BC=117°.34'. obtuse. (N. 211.)

(W) CASE III. *Given the hypothenuse and one angle, to find the other angle.*

Example. In the right-angled spherical triangle ABC.

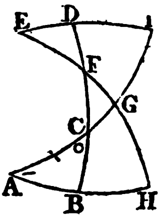
Given { The hypothenuse AC=78°.20' } Required the angle C.
{ The angle A =37. 25 }

BY NAPIER'S RULE.

Here the three parts are connected or joined together, therefore the complement of AC is the *middle part;* and the complement of A and the complement of C are *extremes conjunct.*

Rad × cos AC =cot. A × cot. C.*
∵ (10+log cos AC)−log cot A=log cot C.
or, Cot∠A - =37°.25' - - 10·11633
: radius, sine of 90° - - 10·00000
:: cosine AC =78°.20' - 9·30582
: cot∠C - =81°.12' - 9·18949

Here angle C *is acute.* (N. 211.)



BY RULE I.

Sine CG : sine CI :: tang FG : tang DI. But CG is the complement of AC, CI radius, FG the complement of the angle A, and DI the measure of the angle C. Hence,

Cos AC : rad :: cot A : tang C.

BY CONSTRUCTION.

See the construction of CASE I. (T. 212.)

PRACTICAL EXAMPLES.

I. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC=61°. 4'.56" } Required the angle C.
{ The angle A =61°.50'.29" }

Answer. The angle C=47°.54'.20" acute. (N. 211.)

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC=113°.55' } Required the angle C.
{ The angle A = 31.°51' }

Answer. The angle C=104 . obtuse. (N. 211.)

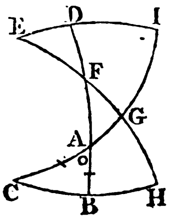
(X) CASE IV. *Given the hypothenuse and a leg, to find the angle adjacent to that given leg.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The hypothenuse AC} = 78^\circ.20' \\ \text{The side or leg AB} = 75^\circ.25'\end{array}\right\}$ To find the angle A adjacent to AB.

BY NAPIER'S RULE.

The three parts are here connected; therefore the complement of A is the *middle part*, AB and the complement of AC are the *extremes conjunct*.



Rad × cos A = cot AC × tang. AB.

∵ (log cot AC + log tang AB) — 10 = log cos A.

or, Radius, sine of - 90° - 10·00000

: cot AC - - - = 78°.20′ 9·31489

:: tang AB - - = 75°.25′ 10·58474

: cos ∠ A - - - = 37°.28′ 9·89963

Here the angle A *is acute.* (O. 211.)

BY RULE I.

Tang GI : tang DF :: sine EI : sine ED.

But GI = AC, DF = AB, EI is the radius, and ED the complement of the angle A. Hence,

Tang AC : tang AB :: rad : cos ∠ A.

BY CONSTRUCTION. (*Plate V. Fig.* 9.)

1. With the chord of 60° describe the primitive circle, and through the centre P draw CP*e*, and *a*P*b* at right angles to it.

2. Set off the hypothenuse, by a scale of chords, from C to *m*, and draw the small circle *m*A*m* parallel to the right circle *a*P*b*. (Z. 162.)

3. Take the complement of AB (viz. 14°.35′) from the scale of semi-tangents, with which as a radius and the centre P describe a small circle parallel to the primitive (Z. 162.) intersecting the circle *m*A*m* in A.

4. Through the three points CA*e* draw a great circle, then ABC is the triangle required.

To measure the required parts.

5. The angles A and C (G. 164.) are 37°.28′ and 81°.11′.

6. BC, applied to a scale of chords, will be 36°.35′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The hypothenuse AC} = 61^\circ.4'.56'' \\ \text{The side or leg, AB} = 40^\circ.30'.20''\end{array}\right\}$ To find the angle A, adjacent to AB.

Answer. The angle A = 61°.50′.29″. acute. (O. 211.)

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78°.20′ } To find the adjacent
{ The side or leg AB = 117°.34′ } angle A.

Answer. The angle A = 113°.18′. obtuse. (O. 211.)

(Y) CASE V. *Given the hypothenuse and a side or leg, to find the angle opposite to that given leg.*

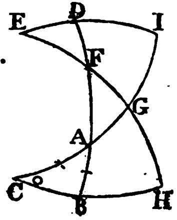
Example. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78°.20′ } To find the angle C,
{ The side AB = 75°.25′ } opposite to AB.

BY NAPIER'S RULE.

The three parts are here disjoined by the intervention of the angle A; therefore AB is the *middle part*, the complement of AC and the complement of C are the *extremes disjunct.*

Rad × sine AB = sine DC × sine C.*
∵ (10 + log sine AB) − log sine AC = log sine C.
or, Sine AC - - - = 78°.20′ - 9·99093
: radius, sine of - 90° - 10·00000
:: sine of AB - - = 75°.25′ - 9·98578
: sine ∠ C - - = 81°.11′ - 9·99485
Here the angle C *is acute.* (O. 211.)



BY RULE I.

Sine AC : sine CG :: sine AB : sine GH. But CG is radius, and GH is the measure of the angle C. Hence,

Sine AC : radius :: sine AB : sine C.

BY CONSTRUCTION.

See the construction of CASE IV. (X. 215.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 61°.4′.56″ } Required the angle
{ The side or leg AB = 40°.30′.20″ } C, opposite to AB.

Answer. The angle C = 47°.54′.20″, acute. (O. 211.)

2. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78°.20′ } Required the angle
{ The side or leg AB = 117°.34′ } C, opposite to AB.

Answer. The angle C = 115°.9′. obtuse. (O. 211.)

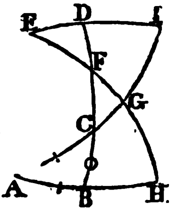
(Z) CASE VI. *Given the hypothenuse and one side or leg, to find the other leg.*

Example. In the right-angled spherical triangle ABC.

Given { The hypothenuse AC = 78°.20′ } To find the other
{ The side AB = 75°.25′ } side BC.

BY NAPIER'S RULE.

Here the complement of AC is the *middle part*, because it is not joined either to AB or BC, the angles A and C coming between them: hence, AB and BC are *extremes disjunct*, viz. not joined to the middle part.



Rad × cos AC = cos AB × cos $\overset{*}{\text{BC}}$.
∵ (10 + log cos AC) − log cos AB = log cos BC.
or, Cos AB - - 75°.25′ - = 9·40103
: radius, sine of - 90° - - - 10·00000
:: cos AC - - 78°.20′ - = 9·30582
: cos BC - - 36°.35′ - = 9·90479
Here BC *is acute.* (O. 211.)

BY RULE I.

Sine BH : sine CG :: sine FB : sine CF. But BH is the complement of AB, CG the complement of AC, FC the complement of BC, and FB radius. Hence,

Cos AB : cos AC :: rad : cos BC.

BY CONSTRUCTION.

See the construction of CASE IV. (X. 215.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The hypothenuse AC} = 61°.4'.56''\\ \text{The side, or leg, AB} = 40°.30'.20''\end{array}\right\}\begin{array}{l}\text{Required the}\\ \text{other leg BC.}\end{array}$$

Answer. BC = 50°.30′.30″. acute. (O. 211.)

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The hypothenuse AC} = 78°.20'\\ \text{The side AB} = 117°.34'\end{array}\right\}\begin{array}{l}\text{Required the}\\ \text{other side BC.}\end{array}$$

Answer. BC = 115°.55′. obtuse. (O. 211.)

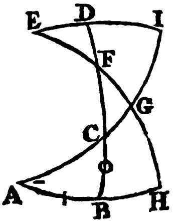
(A) CASE VII. *Given a side and the angle adjacent, or joining to it, to find the side opposite to the given angle.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The side or leg AB} = 75°.25'\\ \text{The adjacent angle A} = 37°.25'\end{array}\right\}\begin{array}{l}\text{To find the side BC,}\\ \text{opposite to A.}\end{array}$$

BY NAPIER'S RULE.

Here the three parts are joined together, because the right angle B is not regarded in the rule; therefore AB is the *middle part*, the complement of A, and BC are the *extremes conjunct*.



Rad × sine AB = cot A × tang. $\overset{*}{\text{BC}}$.
∵ (10 + log sine AB) − log cot A = log tang BC.
or, Cot A - - - = 37°.25′ - 10·11633
: rad, sine of - - 90° - - 10·00000
:: sine AB - - = 75°.25′ - 9·98578
: tang BC - - = 36°.31′ - 9·86945
Here BC *is acute.* (P. 211.)

BY RULE I.

Sine AH : sine AB :: tang GH : tang BC. But AH is radius and GH is the measure of the angle A. Hence,

Rad : sine AB :: tang ∠ A : tang BC.

BY CONSTRUCTION. (*Plate V. Fig.* 11.)

1. With the chord of 60 degrees describe the primitive circle, and through the centre P draw AP*a*, and *mnr* at right angles to it.

2. Set one foot of your compasses on 90 degrees in the line of semi-tangents, extend the other towards the beginning of the scale, till the degrees between them be equal to the angle A 37°.25′, and apply this extent from *m* to *n*.

3. Through the three points A*na* draw a great circle.

4. Set off AB=75°.25′ from A to B, by a scale of chords, and draw BP cutting the oblique circle in C, then ABC is the triangle required.

To measure the required parts.

5. BC measured by the scale of semi-tangents (as in Case I.) will be 36°.31′.

6. The angle C measured (G. 164.) will be 81°.12′.

7. The hypothenuse AC measured (C. 163.) will be 78°.20′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The leg AB} & =40^\circ.30'.20'' \\ \text{The adjacent angle A} & =61^\circ.50'.29''\end{array}\right\}\begin{array}{l}\text{To find the leg} \\ \text{BC, opposite to A.}\end{array}$$

Answer. BC=50°.30′.30″. acute. (P. 211.)

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side AB} & = 54^\circ.22' \\ \text{The adjacent angle A} & =136^\circ.40'\end{array}\right\}\begin{array}{l}\text{To find the leg BC,} \\ \text{opposite to A.}\end{array}$$

Answer. BC=142°.31′. obtuse. (P. 211.)

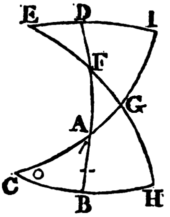
(B) Case VIII. *Given a side and its adjacent angle, to find the angle opposite to the given side.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side AB} & =75^\circ.25' \\ \text{Its adjacent angle A} & =37^\circ.25'\end{array}\right\}\begin{array}{l}\text{To find the angle C,} \\ \text{opposite to AB.}\end{array}$$

BY NAPIER'S RULE.

Here the complement of the angle C is the *middle part;* and the complement of A, and AB are the *extremes disjunct,* being neither of them joined to the middle part.



*
Rad × cos c = sine A × cos AB
∵ (log sine A + log cos AB) − 10 = log cos c.
or, Radius, sine of - - 90° - 10·00000
: sine ∠ A - - = 37°.25′ - 9·78362
:: cos of AB - = 75°.25′ - 9·40103
: cos ∠ c. - - = 81°.12′ - 9·18465

Here c *is acute.* (P. 211.)

BY RULE I.

Sine AD : sine AF :: sine ID : sine FG. But AD is radius, AF the complement of AB, ID the measure of the angle A; and FG the complement of GH, which measures the angle C. Hence,

Radius : cos AB :: sine A : cos c.

BY CONSTRUCTION.

See the construction of CASE VII. (A. 217.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The side AB = 40°.30′.20″
Its adjacent angle A = 61°.50′.29″ } Required the angle c, opposite to AB.

Answer. The angle c = 47°.54′.20″. acute. (P. 211.)

2. In the right-angled spherical triangle ABC.

Given { The side AB = 117°.34′
Its adjacent angle A = 31°.51′ } Required the angle c, opposite to AB.

Answer. The angle c = 104°.8′. obtuse. (P. 211.)

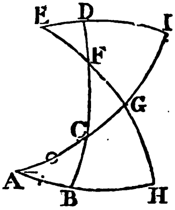
(C) CASE IX. *Given a side and its adjacent angle, to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

Given { The side AB = 75°.25′
Its adjacent angle A = 37°.25′ } Required the hypothenuse AC.

BY NAPIER'S RULE.

The three parts are here connected; therefore the complement of A is the *middle part;* AB and the complement of AC, are the *extremes conjunct.*



*
Rad × cos A = cot AC × tang. AB.
∴ (10 + log cos A) − log tang AB = log cot AC.
or, Tang AB - - = 75°.25′ - 10·58474
: rad, sine of - 90° - - 10·00000
:: cos ∠ A - - = 37°.25′ - 9·89995
: cot AC - - = 78°.20′ - 9·31521

Here AC *is acute.* (P. 211.)

BY RULE I.

Sine FH : sine FG :: tang BH : tang CG. But FH is radius, FG is the complement of the angle A, BH the complement of AB, and CG the complement of AC. Hence,

Rad : cos ∠ A : : cot AB : cot AC.

BY CONSTRUCTION.

See the construction of CASE VII. (A. 217.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{ll}\text{The side AB} & = 50^\circ.30'.30'' \\ \text{Its adjacent angle A} & = 47^\circ.54'.20''\end{array}\right\}$ Required the hypothenuse AC.

Answer. AC = 61°.4′.56″. acute. (P. 211.)

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{ll}\text{The side AB} & = 117^\circ.34'' \\ \text{Its adjacent angle A} & = 31^\circ.51'\end{array}\right\}$ Required the hypothenuse AC.

Answer. AC = 113°.55′. obtuse. (P. 211.)

(D) CASE X. *Given a side and its opposite angle, to find the side adjacent to that angle.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{ll}\text{The side BC} & = 36^\circ.31' \\ \text{Its opposite angle A} & = 37^\circ.25'\end{array}\right\}$ To find the side AB, adjacent to A.

BY NAPIER'S RULE.

The right angle B having no concern in the rule, the three parts are joined together; therefore AB is the *middle part*, the complement of A, and BC are the *extremes conjunct.*

Rad × sine $\overset{*}{\text{AB}}$ = cot A × tang. BC.

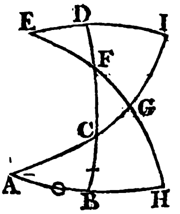
∵ (log cot A + log tang BC) − 10 = log sine AB.

or, Radius, sine of - 90° - 10·00000

: cot ∠ A - = 37°.25′ - 10·11633

: : tang BC - = 36°.31′ - 9·86947

: sine AB $\left\{\begin{array}{l} 75^\circ.25'.\ \text{or} \\ 104^\circ.35'. \end{array}\right\}$ - 9·98580



Here AB *is ambiguous.* (Q. 211.)

BY RULE I.

Tang HG : tang BC : : sine AH . sine AB. But HG is the measure of the angle A, and AH radius. Hence,

tang ∠ A : tang BC : : rad : sine AB.

BY CONSTRUCTION. (*Plate V. Fig.* 10.)

1. With the chord of 60 degrees describe the primitive circle, draw APa, and mnPr at right angles to it.

2. Set one foot of your compasses on 90° degrees in the line of semi-tangents, extend the other towards the beginning of

the scale till the degrees between the two points be equal to the angle A $37^\circ.25'$, and apply this extent from *m* to *n*.

3. Through the three points A*na* draw a great circle.

4. Take the complement of BC (viz. $53^\circ.29'$) from the scale of semi-tangents, with which as a radius and the centre P, describe a small circle *o*C parallel to the primitive (Z. 162.), meeting the great circle A*na* in C.

5. Through C and P draw BCP, then ABC is the triangle required.

This problem is ambiguous, for ABC and *a*BC have the same data.

To measure the required parts.

6. The angle C (G. 164.) will be $81^\circ.12'$.

7. The side AC (C. 163.) is $78^\circ.20'$.

8. The side AB applied to a scale of chords will be $75^\circ.25'$.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side BC} = 40^\circ.30'.20'' \\ \text{Its opposite angle A} = 47^\circ.54'.20''\end{array}\right\}$ To find the other side.

Answer. AB $= 50^\circ.30'.30''$ or $129^\circ.29'.30''$. ambiguous. (Q. 211.)

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side BC} = 51^\circ.30' \\ \text{Its opposite angle A} = 58^\circ.35'\end{array}\right\}$ To find the other side.

Answer. AB $= 50^\circ.10'$, or $129^\circ.50'$. ambiguous. (Q. 211.)

(E) CASE XI. *Given a side and its opposite angle, to find the adjacent angle.*

Example. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side BC} = 36^\circ.31' \\ \text{Its opposite angle A} = 37^\circ.25'\end{array}\right\}$ To find the angle C, adjacent to BC.

BY NAPIER'S RULE.

The complement of A is the *middle part;* and BC and the complement of C are the *extremes disjunct,* being neither of them joined to A, the middle part.

Rad × cos A = sine C × cos BC.

$\because$ (10 + log cos A) − log cos BC = log sine A.

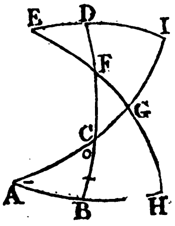
or, cos BC - $=36^\circ.31'$ - 9·90509

: rad, sine of 90° - - 10·00000

:: cos ∠ A $=37^\circ.25'$ - 9·89995

: sine ∠ C - $\left\{\begin{array}{l}81^\circ.12' \text{ or} \\ 98^\circ.48'\end{array}\right\}$ - 9·99486

Here the angle C *is ambiguous.* (Q. 211.)



BY RULE I.

Sine CF : sine CD : : sine FG : sine ID. But CF is the complement of BC, CD radius, FG the complement of the angle A, and ID the measure of the angle C. Hence,

Cos BC : rad : : cos ∠ A : sine C.

BY CONSTRUCTION.

See the construction of Case X. (D. 220.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side BC} & =40^\circ.30'.20''\\ \text{Its opposite angle A} & =47^\circ.54'.20''\end{array}\right\}\begin{array}{l}\text{Required the adjacent angle C.}\end{array}$$

Answer. Angle C $=61^\circ.50'.29''$ or $118^\circ.9'.31''$. ambiguous. (Q. 211.)

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side BC} & =42^\circ.12'\\ \text{Its opposite angle A} & =48^\circ.\end{array}\right\}\begin{array}{l}\text{To find the other angle C.}\end{array}$$

Answer. Angle C $=64^\circ.35'$. or $115^\circ.25'$. ambiguous. (Q. 211.)

(F) Case XII. *Given a side and its opposite angle, to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{ll}\text{The side BC} & =36^\circ.31'\\ \text{Its opposite angle A} & =37^\circ.25'\end{array}\right\}\begin{array}{l}\text{To find the hypothenuse AC.}\end{array}$$

BY NAPIER'S RULE.

Here BC is the *middle part*; the complement of A, and the complement of AC, are the *extremes disjunct*.

$$\text{Rad}\times\text{sine BC}=\overset{*}{\text{sine AC}}\times\text{sine A}.$$

∵ (10 + log sine BC) − log sine A = log sine AC.

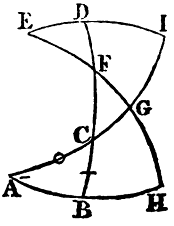
or, Sine ∠ A - $=37^\circ.25'$ - 9·78362

: rad, sine of 90° - - 10·00000

: : sine BC $=36^\circ.31'$ - 9·77456

: sine AC $\left\{\begin{array}{l}78^\circ.20' \text{ or}\\ 101^\circ.40'\end{array}\right\}$ - 9·99094

Here AC *is ambiguous.* (Q. 211.)



BY RULE I.

Sine GH : sine BC : : sine AG : sine AC. But GH is the measure of the angle A, and AG radius. Hence,

Sine ∠ A : sine BC : : rad : sine AC.

BY CONSTRUCTION.

See the construction of CASE X. (D. 220.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side BC} = 40°.30'.20''\\ \text{Its opposite angle A} = 47°.54'.20''\end{array}\right\}$ To find the hypothenuse AC.

Answer. AC = 61°.4′.56″ or 118°.55′.4″. ambiguous. (Q. 211.)

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side BC} = 11°.30'\\ \text{Its opposite angle A} = 23°.30'\end{array}\right\}$ To find the hypothenuse AC.

Answer. AC = 30° or 150°. ambiguous. (Q. 211.)

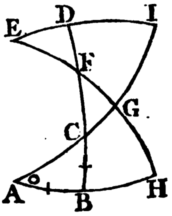
(G) CASE XIII. *Given the two sides, or legs, to find an angle.*

Example. In the right angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The side AB} = 75°.25'\\ \text{The side BC} = 36°.31'\end{array}\right\}$ To find the angle A.

BY NAPIER'S RULE.

The three parts are here connected, because the right angle B is not regarded; therefore AB is the *middle part*, the complement of A, and BC are the *extremes conjunct*.



*
Rad × sine AB = cot A × tang BC.
∵ (10 + log sine AB) − log tang AC = log cot A.
Tang BC - - = 36°.31′ - 9·86947
: rad, sine of - 90° - - 10·00000
∷ sine AB - = 75°.25′ - 9·98578
: cot ∠ A - = 37°.25′ - 10·11631
Here A *is acute.* (R. 212.)

BY RULE I.

Sine AB : sine AH ∷ tang BC : tang GH. But AH is radius, and GH is the measure of the angle A. Hence,
Sine AB : rad ∷ tang BC : tang ∠ A.

BY CONSTRUCTION. (*Plate. V. Fig.* 7.)

1. With the chord of 60 degrees describe the primitive circle, and through the centre P draw *a*B.

2. Set off BC = 36°.31′ from a scale of chords, and AB = 75°. 25′. (D. 163.) Or set one foot of the compasses on 90 degrees on the line of semi-tangents, and extend the other towards the beginning of the scale, till the distance between the

points be 75°.25′; this distance applied from B to A will give the point A.

3. From C through P draw CP*e*, and through the three points CA*e* describe a circle, then ABC is the triangle required.

To measure the required parts.

4. The angles C and A (G. 164.) will be found to be 81°.12′, and 37°.25′.

5. The hypothenuse AC (C. 163.) will be 78°.20′.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The side AB}=50^\circ.30'.30''\\ \text{The side BC}=40^\circ.30'.20''\end{array}\right\}\text{To find the angle A.}$$

Answer. A=47°.54′.20″. acute. (R. 212.)

2. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The side AB}=\ 54^\circ.22'\\ \text{The side BC}=142^\circ.31'\end{array}\right\}\text{To find the angle A.}$$

Answer. A=136°.40′. obtuse. (R. 212.)

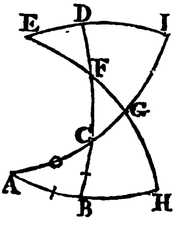
(H) CASE XIV. *Given the two sides, or legs, to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The side AB}=75^\circ.25'\\ \text{The side BC}=36^\circ.31'\end{array}\right\}\begin{array}{c}\text{To find the hypothe-}\\ \text{nuse AC.}\end{array}$$

BY NAPIER'S RULE.

The complement of the hypothenuse AC is the *middle part*, being separated from the sides by the angles A and C; and AB and BC are the *extremes disjunct*.



$$\text{Rad}\times\cos\overset{*}{\text{AC}}=\cos\text{AB}\times\cos\text{BC}.$$

$$\because(\log\cos\text{AB}+\log\cos\text{BC})-10=\log\cos\text{AC}.$$

or, Rad, sine of 90° - - 10·00000
: cos AB - =75°.25′ - 9·40103
∷ cos BC - =36°.31′ - 9·90509
: cos AC - =78°.20′ - 9·30612

Here AC *is acute.* (R. 212.)

BY RULE I.

Sine FB : sine FC ∷ sine BH : sine CG. But FB is radius, CF the complement of BC, BH the complement of AB, and CG the complement of AC. Hence,

Rad : cos BC ∷ cos AB : cos AC.

BY CONSTRUCTION.

See the construction of Case XIII. (G. 223.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given { The side AB = 50°.30′.30″ / The side BC = 40°.30′.20″ } To find the hypothenuse AC.

Answer. AC = 61°.4′.56″, acute. (R. 212.)

2. In the right-angled spherical triangle ABC.

Given { The side AB = 54°.22′ / The side BC = 142°.31′ } To find the hypothenuse AC.

Answer. AC = 117°.33′, obtuse. (R. 212.)

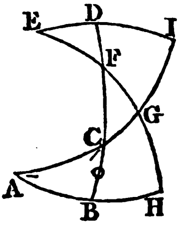
(I) CASE XV. *Given two angles, to find a side or leg.*

Example. In the right-angled spherical triangle ABC.

Given { The angle A = 37°.25′ / The angle C = 81°.12′ } Required the side BC.

BY NAPIER'S RULE.

Here the complement of the angle A is the *middle part*, and BC and the complement of C are the *extremes disjunct*, being separated from the middle part by the hypothenuse AC.



*

Rad × cos A = cos BC × sine C.

∵ (10 + log cos A) − log sine C = log cos BC.

or, sine ∠ C = 81°.12′ - - 9·99486

: rad sine of 90° - - - 10·00000

:: cos ∠ A = 37°.25′ - - 9·89995

: cos BC = 36°.31′ - - 9·90509

Here BC *is acute.* (S. 212.)

BY RULE I.

Sine ID : sine FG :: sine CD : sine CF. But ID is the measure of the angle C, FG the complement of HG which is the measure of the angle A, CD radius, and CF the complement of BC. Hence,

sine C : cos A :: rad : cos BC.

BY CONSTRUCTION. *(Plate V. Fig. 12.)*

1. With the chord of 60° describe the primitive circle, through the centre P draw CP*e*, and *rnm* at right angles to it.

2. Set one foot of your compasses on 90° in the line of semi-tangents, extend the other towards the left hand, till the degrees contained between the points of the compasses be equal to (81°12′) the ∠ C, and apply this extent from *m* to *n*.

3. Through the three points C*ne* draw a great circle, and find its pole *p*. (N. 159.)

4. With the tangent of the ∠ A 77°.25′, and centre P describe

an arc; with the secant of the ∠ A, and centre p crosses it in O; with O as a centre and radius Op, describe the great circle ABp.

5. Then ABC is the triangle required, right-angled at B.

To measure the required parts.

6. The hypothenuse AC, measured by a scale of chords, $=78°.20'$.

7. The sides BC and AC (C. 163.) $=36°.31'$, and $75°.25'$.

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

$$\text{Given}\begin{cases}\text{The angle A}=47°.54'.20''\\ \text{The angle C}=61°.50'.29''\end{cases}\text{Required the side AB}$$

Answer. AB $=50°.30'.30''$, acute. (S. 212.)

2. In the right-angled spherical triangle ABC.

$$\text{Given}\begin{cases}\text{The angle A}=104°.8'\\ \text{The angle C}=31°.51'\end{cases}\text{Required the side BC.}$$

Answer. BC $=117°.34'$, obtuse. (S. 212.)

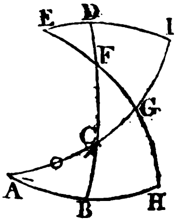
(K) CASE XVI. *Given two angles, to find the hypothenuse.*

Example. In the right-angled spherical triangle ABC.

$$\text{Given}\begin{cases}\text{The angle A}=37°.25'\\ \text{The angle C}=81°.12'\end{cases}\text{To find the hypothenuse AC.}$$

BY NAPIER'S RULE.

Here the complement of AC is the *middle part*, and the complements of A and C are the *extremes conjunct*.



$$\text{Rad}\times\cos\text{AC}=\cot\text{A}\times\cot\text{C}.$$

$\because$ (log cot A + log cot C) − 10 = log cos AC.

or, Rad, sine of 90° - - 10·00000

: cot ∠ A - $=37°.25'$ - 10·11633

:: cot ∠ C - $=81°.12'$ - 9·18979

: cos AC - $=78°.20'$ - 9·30612

Here AC *is acute.* (S. 212.)

BY RULE I.

Tang ID : tang FG :: sine CI : sine CG. But ID is the measure of the angle C, FG the complement of the angle A, CI radius, and CG the complement of AC. Hence,

tang ∠ C : cot ∠ :: rad : cos AC.*

BY CONSTRUCTION.

See the construction of CASE XV. (I. 225.)

PRACTICAL EXAMPLES.

1. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The angle } A=47^\circ.54'.20'' \\ \text{The angle } C=61^\circ.50'.29''\end{array}\right\}$ To find the hypothenuse AC.

Answer. $AC=61^\circ.4'.56''$, acute. (S. 212.)

2. In the right-angled spherical triangle ABC.

Given $\left\{\begin{array}{l}\text{The angle } A=31^\circ.51' \\ \text{The angle } C=104^\circ.8'\end{array}\right\}$ Required the hypothenuse AC.

Answer. $AC=113^\circ.55'$, obtuse. (S. 212.)

CHAP. VII.

II. PRACTICAL RULES FOR SOLVING THE DIFFERENT CASES OF RECTILATERAL, OR QUADRANTAL, SPHERICAL TRIANGLES, WITH THEIR APPLICATION BY LOGARITHMS.

(L) 1. When one side of a spherical triangle is 90°, or a quadrant, it is called a quadrantal triangle.

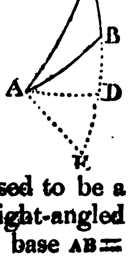
2. If two sides of a quadrantal triangle be each 90°, the triangle will be isosceles, each of the angles at the base will be 90°, and the base itself will be the measure of the vertical angle. If the three sides be each 90°, the three angles will be each 90°.

3. A quadrantal spherical triangle may be changed into a right-angled spherical triangle, and the contrary. (X. 138.)

(M) RULE I.

Subtract the angle opposite to the quadrantal side from 180°, and call the remainder the hypothenuse of a new triangle. The angle opposite to the quadrantal side must then be considered as a right angle, and the two remaining angles must be represented by the sides of the quadrantal triangle which are opposite to them.

The sides and angles of the right-angled spherical triangle, will represent the angles and sides of the quadrantal triangle.



Thus in the annexed figure, where AC is supposed to be a quadrant, the triangle ABC may be considered as a right-angled triangle, whose hypothenuse $AC=180^\circ-\angle B$, the base $AB=\angle C$; and the perpendicular $BC=$ the $\angle A$; et contra.

(N) RULE II.

1. In the triangle ABC, if one of the sides AC be a quadrant, and another side BC less than a quadrant.

Produce the side BC till it becomes a quadrant, and ADB will be a right-angled triangle; wherein AD is the measure of the angle C, DB is the complement of BC, and AE is the hypothenuse.

2. In the triangle AEC, where one of the sides AC is a quadrant, and another side CE greater than a quadrant.

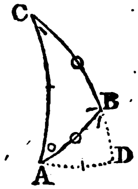
Subtract 90° from CE, and the remainder will be the side DE of the right-angled triangle ADE; AE is the hypothenuse, and AD is the measure of the angle C.

The same reasoning will apply whether AB and AE be less or greater than quadrants, for in either case ADB and ADE will be right-angled triangles.

(O) CASE I. *Given a quadrantal side, its adjacent and opposite angle, to find the rest.*

Given { The quadrantal side AC = 90°.00′
Its adjacent ∠ C = 42°.12′
Its opposite ∠ B = 115°.20′ } Required the sides AB, BC and the ∠ A.

BY RULE II. (N. 228.)



Because AC and CD are quadrants, the angles CAD and CDA are each of them a right angle (L. 227.), and AD is the measure of the angle C.

The angles ABC and ABD, being supplements to each other, may be considered as the same angle, since they have the same sine, tangent, &c.

1. *To find* AB, *by Case XII. right-angled spherics.*

sine ABD : sine AD = sine ∠ C : : rad : sine AB.

Here AB is acute = 48°.

2. *To find* BD *and hence* BC, *by Case X. right-angled spherics.*

rad : cot ABD : : tang AD = tang ∠ C : sine BD.

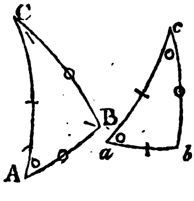
Here BD is acute = 25°.25′, and its complement = BC = 64°.35′.

3. *To find the* ∠ BAD, *by Case XI. right-angled spherics.*

cos AD = cos ∠ C : rad : : cos ABD = cos 64°.40′ : sine BAD.

Here BAD is acute = 35°.17′, and its complement = BAC = 54°.43′.

BY RULE I. (M. 227.)



Here the supplement of the angle B = 64°.40′, must represent the hypothenuse *ac*, the quadrantal side AC = the right angle *b*. The angles C and A must represent the sides *ab* and *bc*, and the sides BC and AB must represent the angles *a* and *c*.

1. *To find the* ∠ c, *in the triangle* abc, *by Case V. right-angled spherics.*

sine *ac* = sine suppt. ∠ B : rad :: sin *ab* = sin ∠ C : sin ∠ *c* = sin AB.
Here the ∠ *c* or AB is acute = 48°.

2. *To find the* ∠ a, *in the triangle* abc, *by Case IV. right-angled spherics.*

rad : cot *ac* = cot suppt. ∠ B :: tang *ab* = tang ∠ C : cos ∠ *a* = cos BC.
Here cosine of ∠ *a* = 64°.35′ acute.

3. *To find* bc, *in the triangle* abc, *by Case VI. right-angled spherics.*

cos *ab* = cos ∠ C : rad :: cos *ac* = cos suppt. ∠ B : cos *bc* = cos ∠ A.
Here *bc* is acute = 54°.43′.

BY CONSTRUCTION. (*Plate V. Fig.* 13.)

1. With the chord of 60 degrees describe the primitive circle; and through the centre P draw *a*PD, and CP*e* at right angles to it.

2. Set one foot of your compasses on 90 degrees on the line of semi-tangents, extend the other towards the beginning of the scale, till the degrees between them be equal to the angle C = 42°.12′, and apply this extent from D to A.

3. Through the three points CA*e* draw a great circle, then AC will represent the quadrantal side.

4. With the tangent of the complement of the angle B, and centre C, describe an arc; with the secant of the complement of the same angle, and centre A, cross it in *o*: with *o* as a centre, and radius *o*A, describe the great circle BA*b*. Then ABC is the triangle required.

To measure the required parts.

The sides AB and BC (C. 163.) will be 48° and 64°.35′,
And the angle A = 54°.43′. (G. 164.)

NOTE. The right-angled spherical triangle *abc* (Plate V. fig. 14.) into which the quadrantal triangle ABC is transformed, may be constructed and measured exactly in the same manner as Case IV. of right-angled spherical triangles was constructed and measured. See Plate V. fig. 9.

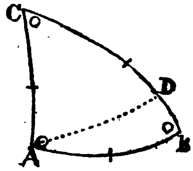
(P) CASE II. *Given a quadrantal side, and the other two sides, to find the rest.*

Given { The quadrantal side AC = 90°† ; The side AB = 115°. 9′ ; The side BC = 113°.18′ } Required the angles A, B, and C.

---

† This example was formed from Example 2, Case VI. of right-angled spherics. The ∠ B was made equal to the supplement of the hypothenuse, the two sides AB and BC were made equal to the angles of the right-angled triangle; and hence the legs of the right-angled triangle become angles in the quadrantal triangle. In a similar manner the other examples were made.

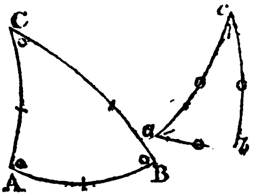
BY RULE II. (N. 228.)



Make CD=AC, then CAD, CDA, and ADB are each of them a right angle, and AD measures the angle C (L. 227, and M. 134.); hence, if we take 90° from BC we get 23°.18′=BD; and ADB is a right-angled triangle.

1. *To find the ∠B, by Case IV. right-angled spherics.*
   rad : cot AB : : tang BD : cos ∠B=78°.20′.
   But ∠B is obtuse (O. 211), and therefore=101°.40′.
2. *To find* AD, *by Case VI. right-angled spherics.*
   Cos BD : rad : : cos AB : cos AD=62°.26′.
   But AD is obtuse (O. 211), and therefore=117°.34′.
3. *To find the ∠DAB, by Case V. right-angled spherics.*
   Sine AB : rad : : sine BD : sine ∠DAB=25°.55′.
   Here ∠DAB is acute (O. 211.); and ∠DAB+∠CAD=∠CAB=25°.55′+90°=115°.55′.

BY RULE I. (M. 227.)



Here the quadrantal side AC must represent the right-angle *b*; AB the ∠*c*; and BC the ∠*a*; *ac* the supplement of the ∠B, *ab* the ∠C, and *bc* the ∠A.

1. *To find the hypoth.* ac, *by Case XVI. right-angled spherics.*
Rad : cot ∠*a*=cot BC : : cot ∠*c*=cot AB : cos *ac*=sup ∠B=78°.20′. Here *ac* or suppt. ∠B is acute (S. 212.); therefore ∠B is obtuse =101°.40′.

2. *To find* ab, *by Case XV. right-angled spherics.*
Sine ∠A=sin BC : rad : : cos ∠*c*=cos AB : cos *ab*=cos ∠C=62°.26′. Here *ab*=∠C is obtuse (S. 212.), and therefore=117°.34′.

3. *To find* bc, *by Case XV. right-angled spherics.*
Sine ∠*c*=sin AB : rad : : cos ∠*a*=cos BC : cos *bc*=cos ∠A=64°.5′. Here *bc*=∠A is obtuse (S. 212.), and therefore=115°.55′.

If the two preceding cases be thoroughly understood, there can be no difficulty in solving those which follow; for which reason, they are given as practical exercises.

(Q) CASE III. *Given a quadrantal side, and its two adjacent angles, to find the rest.*

Given { The quadrantal side AC=90° / The angle A=54°.43′ / The angle C=42°.12′ } Required AB, BC, and the angle B.

Answer. AB=48°, BC=64°.35′, and the angle B=115°.20′.

(R) CASE IV. *Given a quadrantal side, one of the other sides, and the angle comprehended between them, to find the rest.*

Given { The quadrantal side AC = 90°; The side AB = 115°. 9′; The angle CAB = 115°.55′ } Required BC, and the angles B and C.

Answer. BC = 113°.18′, the angle B = 101°.40′, C = 117°.34′.

(S) CASE V. *Given a quadrantal side, its adjacent angle, and a side opposite to that angle, to find the rest.*

Given { The quadrantal side AC = 90°; Its adjacent angle C = 42°.12′; And the opposite side AB = 48°.00′ } Required BC, and the angles A and B.

Answer. BC = 64°.35′, the angle A = 54°.43′. and B = 115°.20′.

But the required parts are ambiguous, and therefore are either acute or obtuse.

(T) CASE VI. *Given the quadrantal side, one of the other sides and an angle opposite to the quadrantal side, to find the rest.*

Given { The quadrantal side AC = 90°; The side AB = 115°. 9′; The angle B = 101°.40′ } Required BC, and the angles A and C.

Answer. BC = 113°.18′, the angle A = 115°.55′, and C = 117°.34′.

CHAP. VIII.

III. PRACTICAL RULES FOR SOLVING ALL THE CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES, WITH A PERPENDICULAR; AND THEIR APPLICATION BY LOGARITHMS.

RULE I.

(U) The rule Prop. xxv. (X. 173.) will solve ten cases; see also the rules P. 178.

RULE II.

(W) When the three sides are given, to find the angles.

Any one of the three sides may be called the base. Then,

The tangent of half the base,

Is to the tangent of half the sum of the sides,

As the tangent of half the difference between the sides,

Is to the tangent of the distance of a perpendicular from the middle of the base. (S. 179.)

According as this distance is less or greater than half the base, the perpendicular falls within or without the triangle. When the sum of the two sides is less than 180°, the perpendicular falls nearest to the less side, when greater than 180° it falls nearest to the greater side; consequently the greater segment is joined to the greater side in the former case, and to the less side in the latter. The sum of half the base, and the fourth term found by the above proportion, gives the greater segment, their difference gives the less.

The triangle being thus divided into two right-angled triangles, the remaining parts must be found by the proper rules.

RULE III.

(X) When the three angles are given, to find the sides.

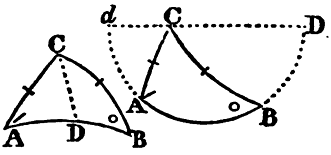
The co-tangent of half the sum of the angles at the base,
Is to the tangent of half their difference;
As the tangent of half the verticle angle,
Is to the tangent of the excess of the greater of the two vertical angles (formed by a perpendicular), above half the aforesaid vertical angle. (W. 180.)

If the sum of the base angles be less than 180°, the perpendicular and the less segment are nearest the greater base angle, if greater than 180° they are nearest the less base angle. The sum and difference of this fourth term, and half the vertical angle, gives the greater and less vertical angle formed by the perpendicular. The triangle being thus divided into two right-angled triangles, the remaining parts must be found by the preceding rules.

(Y) Case I. *Given two sides and an angle opposite to one of them, to find an angle opposite to the other.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=80°.19' \\ \text{The side BC}=63°.50' \\ \text{The } \angle \text{A}=51°.30'\end{array}\right\}\text{Required the } \angle \text{B}.$$

DETERMINATION OF THE SPECIES.



A perpendicular in this case is unnecessary.

If AC+BC, A+(B *acute*), and A+(B *obtuse*), be each of the same species with respect to 180°, B is *ambiguous:* — But if only two of these sums be of the same species, that value of B must universally be taken which agrees with the sum of the sides, in all such cases B is *not* ambiguous.

SOLUTION.

Sine BC : sine AC : : sine $\angle$ A : sine $\angle$ B = 59°.16′.

Here B is *ambiguous* = 59°.16′. or 120°.44′.

BY FORMULA† II. *page* 200.

Log sine B $=$ (*log* sine A $+$ *log* sine b) $-$ *log* sine $a = 59°.16'$.
This example is *ambiguous*, *vide* Table I. page 207.

EXAMPLE II.

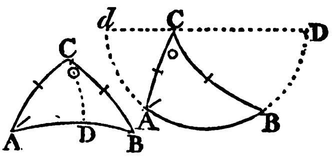
Given $\left\{\begin{array}{l}\text{The side AC} = \ \ 57°.30' \\ \text{The side BC} = 115°.20' \\ \text{The} \quad \angle \text{A} = 126°.37'\end{array}\right\}$ Required the $\angle$ B.

Answer. The $\angle$ B $= 48°.30'$, *not* ambiguous.

(Z) CASE II. *Given two sides and an angle opposite to one of them, to find the angle contained between these sides.*

Given $\left\{\begin{array}{l}\text{The side AC} = 80°.19' \\ \text{The side BC} = 63°.50' \\ \text{The} \quad \angle \text{A} = 51°.30'\end{array}\right\}$ Required the $\angle$ C.

DETERMINATION OF THE SPECIES.



1. If AC and the $\angle$ A be of the same species, the $\angle$ ACD is acute. The perpendicular CD is of the same species as the $\angle$ A. If BC and DC be of the same species, the $\angle$ BCD is acute.

2. If the $\angle$ BCD be less than the $\angle$ ACD, and their sum less than 180°, then the $\angle$ ACB is *ambiguous*; but if their sum be *not* less than 180°, their difference is the *true* value of the $\angle$ ACB, *not* ambiguous.

If the $\angle$ BCD be *not* less than the $\angle$ ACD, and at the same time their sum be less than 180°, this sum is the *true* value of the $\angle$ ACB, *not ambiguous.*

SOLUTION.

1. In the triangle ADC, find the $\angle$ ACD. *
Thus, rad $\times$ cos AC $=$ cot $\angle$ A $\times$ cot $\angle$ ACD.
Hence, cot $\angle$ A : rad :: cos AC : cot ACD $= 78°.4'$ acute.

*

2. In the triangle CDB, *rad* $\times$ cos BCD $=$ *tang* DC $\times$ cot BC.
In the triangle ADC, *rad* $\times$ cos ACD $=$ *tang* DC $\times$ cot AC.
Hence, cot AC : cot BC :: cos ACD : cos BCD $= 53°.28'$ acute.

Because the $\angle$ BCD is less than the $\angle$ ACD, and $\angle$ BCD $+$ $\angle$ ACD less than 180°, the $\angle$ ACB is *ambiguous*; *viz.* $\angle$ ACB $=$ $\angle$ ACD $+$ $\angle$ BCD $= 131°.32'$, or $\angle$ ACB $=$ $\angle$ ACD $-$ $\angle$ BCD $= 24°.36'$.

---

† In using the *Formulæ*, the three angles of the triangle are represented by A, B, C, and their opposite sides by a, b, c. The perpendicular is not regarded.

BY FORMULA III. *page* 201.

Log tan $\varphi=(log \cos b + log \text{ tang A})-10=11^\circ.56'$. acute.

Log sine $\varphi+\text{C}=(log \text{ tang } b + log \text{ sine } \varphi)-log \text{ tang } a=36^\circ.32'$.

Then $(\varphi+\text{C})-\varphi=36^\circ.32'-11^\circ.56'=24^\circ.36'=\angle\text{C}$.

This example is *ambiguous*, though not shewn by the formula, *vide* Table I. page 207.

OR,

Log cot $\varphi=(log \cos b + log \text{ tang A})-10=78^\circ.4'$ acute.

Log cos $(\varphi+\text{C})=(log \cot a + log \cos \varphi)-log \cot b=53^\circ.28'$.

Then $(\varphi+\text{C})\pm\varphi=53^\circ.28'+78^\circ.4'=131^\circ.32'=\angle\text{C}$.

EXAMPLE II.

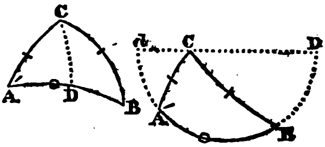
$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=\ \ 57^\circ.30'\\ \text{The side BC}=115^\circ.20'\\ \text{The}\qquad\angle\text{A}=126^\circ.37'\end{array}\right\}\text{Required the }\angle\text{C}.$$

Answer. $\angle$ ACD $=125^\circ.52'$ obtuse, $\angle$ BCD $=64^\circ.12'$ acute, and $\angle$ ACB $=61^\circ.40'$, *not* ambiguous.

(A) CASE III. *Given two sides and an angle opposite to one of them to find the other side.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=80^\circ.19'\\ \text{The side BC}=63^\circ.50'\\ \text{The}\qquad\angle\text{A}=51^\circ.30'\end{array}\right\}\text{Required the side AB.}$$

DETERMINATION OF THE SPECIES.



1. The segment AD is acute or obtuse, according as the $\angle$ A is of the same, or of different species with AC. DC is of the same species with the $\angle$ A.

DB is acute or obtuse, according as BC is of the same, or of different species, with DC.

2. If DB be less than AD, and their sum less than 180°, then AB is *ambiguous*; but if their sum be *not* less than 180°, their difference is the *true* value of AB, *not* ambiguous.

If DB be *not* less than AD, and at the same time their sum be less than 180°, this sum is the *true* value of AB, *not* ambiguous.

SOLUTION.

1. In the triangle ADC, find the side AD. *
   Thus, rad × cos $\angle$ A = cot AC × tang AD.
   Cot AC : rad : : cos $\angle$ A : tang AD $=74^\circ.40'$ acute. *
2. In the triangle CDB, *rad* × cos BC = *cos* DC × cos DB.
   In the triangle ADC, *rad* × cos AC = *cos* DC × cos AD.

Hence, cos AC : cos BC :: cos AD : cos DB $=46^\circ.6'$ acute.

Because DB is less than AD, and AD + DB less than 180°, AB is *ambiguous*; viz. AB = AD + DB $=120^\circ.46'$, or AB = AD − DB $=28^\circ.34'$.

BY FORMULA III. *page* 201.

Log tang $\varphi=(\log\cos A+\log\operatorname{tang} b)-10=74^\circ.40'$ acute.

Log cos $(c-\varphi)=(\log\cos a+\log\cos\varphi)-\log\cos b=46^\circ.6'$.

Then, $(c\mp\varphi)\pm\varphi=46^\circ.6'+74^\circ.40'=120^\circ.46'=$ AB.

This example is *ambiguous*, though not shewn by the formula, *vide* Table I. page 207.

EXAMPLE II.

$$\text{Given}\left\{\begin{array}{l}\text{The side AC}=\ 57^\circ.30'\\ \text{The side BC}=115^\circ.20'\\ \text{The}\quad\ \angle A=126^\circ.37'\end{array}\right\}\text{Required the side AB.}$$

Answer. AD $=136^\circ.53'$ obtuse, DB $=54^\circ.27'$ acute, and AB = AD − DB $=82^\circ.26'$ *not* ambiguous.

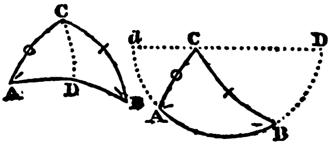
(B) CASE IV. *Given two angles, and a side opposite to one of them, to find the other opposite side.*

$$\text{Given}\left\{\begin{array}{l}\text{The}\quad\ \angle A=51^\circ.30'\\ \text{The}\quad\ \angle B=59^\circ.16'\\ \text{The side BC}=63^\circ.50'\end{array}\right\}\text{Required the side AC.}$$

DETERMINATION OF THE SPECIES.

A perpendicular in this case is unnecessary.

1. If BC + (AC *acute*), BC + (AC *obtuse*) and A + B, be each of the same species, with respect to 180°, AC is ambiguous; but, if only *two* of these sums be of the same kind, that value of AC must universally be taken which agrees with the sum of the angles; in all such cases AC is *not* ambiguous.



SOLUTION.

Sine∠A : sine BC :: sine∠B : sine AC $=80^\circ.19'$,

Because BC + (AC *acute*), BC + (AC *obtuse*) and A + B, are each less than 180°, AC is ambiguous, being either $80^\circ.19'$, or $99^\circ.41'$.

BY FORMULA II. *page* 201.

Log sine $b=(\log\text{sine } B+\log\text{sine } a)-\log\text{sine } A=80^\circ.19'$.

This example is *ambiguous*, *vide* Table II. page 207.

EXAMPLE II.

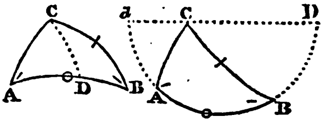
$$\text{Given}\left\{\begin{array}{l}\text{The}\angle A\qquad=126^\circ.37'\\ \text{The}\angle B\qquad=\ 48^\circ.30'\\ \text{The side BC}=115^\circ.20'\end{array}\right\}\text{Required the side AC.}$$

Answer. AC $=57^\circ.30'$ acute, and *not* ambiguous.

(C) Case V. *Given two angles, and a side opposite to one of them, to find the side adjacent to these angles.*

$$\text{Given}\left\{\begin{array}{l}\text{The } \angle \text{A}=51^\circ.30' \\ \text{The } \angle \text{B}=59^\circ.16' \\ \text{The side BC}=63^\circ.50'\end{array}\right\}\text{Required the side AB.}$$

DETERMINATION OF THE SPECIES.



1. DC is of the same species as the ∠A.

The species of AD cannot be determined.

DB is acute, or obtuse, according as BC and DC are of the same, or of different species.

2. When the angles A and B are of the *same* species; if DB+(AD *acute*), and DB+(AD *obtuse*), be each less than 180°, AB is *ambiguous*; but if only one of these sums be less than 180°, *that* sum is the *true* value of AB, *not* ambiguous.

When the angles A and B are of *different* species; if both AD *acute*, and AD *obtuse*, be greater than DB, then AB is ambiguous; but if only one of them be greater, *that* value diminished by DB leaves AB *not* ambiguous.

SOLUTION.

1. In the triangle BDC, find the segment DB.

Thus, $\text{rad} \times \cos \angle \text{B} = \cot \text{BC} \times \text{tang } \overset{*}{\text{DB}}$.

$\text{Cot BC} : \text{rad} :: \cos \angle \text{B} : \text{tang DB} = 46^\circ.7'$ acute.

2. In the triangle ADC, $\textit{rad} \times \text{sine } \overset{*}{\text{AD}} = \cot \angle \text{A} \times \textit{tang } \overset{*}{\text{DC}}$.

In the triangle BDC, $\textit{rad} \times \text{sine DB} = \cot \angle \text{B} \times \textit{tang } \text{DC}$.

$\cot \angle \text{B} : \cot \angle \text{A} :: \text{sine DB} : \text{sine AD} = 74^\circ.40'$, or $105^\circ.20'$,

Because A and B are of the same species, and DB+(AD *acute*), DB+(AD *obtuse*), are each less than 180°, AB is *ambiguous*, viz. $\text{AB}=\text{AD}+\text{DB}=120^\circ.47'$, or $151^\circ.27'$.

BY FORMULA III. *page* 202.

$\textit{Log} \text{ tang } \phi = (\textit{log} \cos \text{B} + \textit{log} \text{ tang } a) - 10 = 46^\circ.7'$ acute.

$\textit{Log} \text{ sine}(c-\phi) = (\textit{log} \text{ sine} \phi + \textit{log} \text{ tang B}) - \textit{log} \text{ tang A} = 74^\circ.39'$.

Then $(c-\phi)+\phi = 74^\circ.39' + 46^\circ.7' = 120^\circ.46' = \text{AB}$.

This example is *ambiguous*, though not shewn by the formula, *vide* Table II. page 207.

EXAMPLE II.

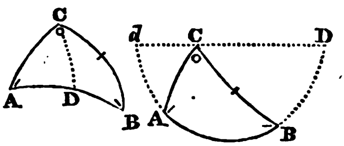
$$\text{Given}\left\{\begin{array}{l}\text{The } \angle \text{A}=126^\circ.37' \\ \text{The } \angle \text{B}=48^\circ.30' \\ \text{The side BC}=115^\circ.20'\end{array}\right\}\text{Required the side AB.}$$

Answer. $\text{DB}=54^\circ.27'$ acute, $\text{AD}=43^\circ.6'$, or $136^\circ.54'$, and $\text{AB}=82^\circ.26'$, *not* ambiguous.

(D) CASE VI. *Given two angles, and a side opposite to one of them, to find the third angle.*

$$\text{Given}\left\{\begin{array}{ll}\text{The} & \angle A = 51^\circ.30'\\ \text{The} & \angle B = 59^\circ.16'\\ \text{The side} & BC = 63^\circ.50'\end{array}\right\}\text{Required the angle C.}$$

DETERMINATION OF THE SPECIES.



1. DC is of the same species as the ∠ A.

The species of the ∠ ACD cannot be determined.

The ∠ BCD is acute, or obtuse, according as the side BC is of the same, or of different species with DC.

2. When the angles A and B are of the *same* species; if BCD + (ACD *acute*), and BCD + (ACD *obtuse*), be each less than 180°, ACB is *ambiguous*; but if only one of these sums be less than 180°, that sum is the *true* value of ACB, *not* ambiguous.

When the angles A and B are of *different* species; if both ACD *acute*, and ACD *obtuse*, be greater than BCD, then ACB is *ambiguous*; but if only one of them be greater, *that* value diminished by BCD leaves ACB, *not* ambiguous.

SOLUTION.

1. In the triangle BDC, find the ∠ BCD. *

Thus, $\text{rad} \times \cos BC = \cot \angle B \times \cot BCD$.

$\text{Cot} \angle B : \text{rad} :: \cos BC : \cot BCD = 53^\circ.26'$ acute. *

2. In the triangle ADC, $\textit{rad} \times \cos \angle A = \textit{cos}\ DC \times \text{sine}\ ACD$.

In the triangle BDC, $\textit{rad} \times \cos \angle B = \textit{cos}\ DC \times \text{sine}\ BCD$.

Hence, $\cos \angle B : \cos \angle A :: \text{sine}\ BCD : \text{sine}\ ACD = 78^\circ.4'$, or $101^\circ.56'$.

Because A and B are of the same species, and BCD + (ACD *acute*), BCD + (ACD *obtuse*), are each less than 180°; ACB is *ambiguous*, $= 131^\circ.30'$, or $155^\circ.22'$.

BY FORMULA III. *page* 202.

$\textit{Log} \cot \varphi = (\textit{log} \cos a + \textit{log}\ \text{tang}\ B) - 10 = 53^\circ.26'$ acute.

$\textit{Log}\ \text{sine}\ (C - \varphi) = (\textit{log} \cos A + \textit{log}\ \text{sine}\ \varphi) - \textit{log} \cos B = 78^\circ.3'$.

Then $(C - \varphi) + \varphi = 78^\circ.3' + 53^\circ.26' = 131^\circ.29' = \angle C$.

This example is *ambiguous*, though not shewn by the formula, *vide* Table II. page 207.

EXAMPLE II.

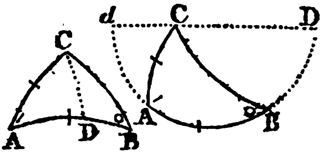
$$\text{Given}\left\{\begin{array}{ll}\text{The} & \angle A = 126^\circ.37'\\ \text{The} & \angle B = 48^\circ.30'\\ \text{The side} & BC = 115^\circ.20'\end{array}\right\}\text{Required the } \angle C.$$

Answer. The ∠ BCD = 64°.11′ acute, the ∠ ACD = 54°.8′, or 115°.52′ ambiguous, and the ∠ ACB = 61°.41′, *not* ambiguous.

(E) CASE VII. *Given two sides, and an angle contained between them, to find an opposite angle.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = \ \ 80^\circ.19' \\ \text{The side AB} = 120^\circ.47' \\ \text{The} \qquad \angle\text{A} = \ \ 51^\circ.30'\end{array}\right\}\text{Required the }\angle\text{ B.}$$

DETERMINATION OF THE SPECIES.



1. AD is acute, or obtuse, according as AC is of the same, or of different species with the ∠ A.

2. The ∠ B is of the *same*, or of *different* species with the ∠ A, according as AD is *less*, or *greater* than AB.

SOLUTION.

1. In the triangle ABC, find AD.
   Thus, rad × cos ∠ A = cot AC × tang AD.
   Cot AC : rad :: cos ∠ A : tang AD = 74 .40′ acute.
   Then AB − AD = 46°.7′ = DB.
2. In the triangle BDC, *rad* × sine DB = *tang* DC × cot ∠ B.
   In the triangle ADC, *rad* × sine AD = *tang* DC × cot ∠ A.

Hence, sine AD : sine DB :: cot ∠ A : cot ∠ B = 59°.16′ acute.

Because AD is less than AB, the ∠ B is of the *same* species with the ∠ A, and consequently it is acute.

BY FORMULA VII. *page* 203.

$$Log \text{ tang } \varphi = (log\ cos\ \text{A} + log \text{ tang } \underline{b) - 10} = 74^\circ.40' \text{ acute.}$$

$$Log \text{ cot B} = (\text{log cot A} + \text{log sine } \overline{c \sim \varphi)} - \text{sine } \varphi = 59^\circ.16'.$$

EXAMPLE II.

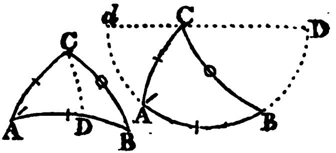
$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = \ \ 57^\circ.30' \\ \text{The side AB} = \ \ 82^\circ.27' \\ \text{The} \qquad \angle\text{A} = 126^\circ.37'\end{array}\right\}\text{Required the }\angle\text{ B.}$$

Answer. AD = 136°.54′ obtuse, DB = 54°.27′, and the ∠ B = 48°.30′ acute.

(F) CASE VIII. *Given two sides, and the angle contained between them, to find the third side.*

$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = \ \ 80^\circ.19' \\ \text{The side AB} = 120^\circ.47' \\ \text{The} \qquad \angle\text{A} = \ \ 51^\circ.30'\end{array}\right\}\text{Required the side BC.}$$

DETERMINATION OF THE SPECIES.



1. AD is acute or obtuse, according as AC is of the *same*, or of *different* species with the ∠ A.

2. CD is of the same species as the ∠ A; therefore, according as the ∠ A and DB are of the *same*, or of *different* species, BC is acute or obtuse.

SOLUTION.

1. In the triangle ADC, find AD.
Thus, rad × cos ∠ A = cot AC × tang AD.
Cot AC : rad : : cos ∠ A : tang AD = 74°.40′ acute.
Then AB − AD = 46°.7′ = DB acute.

2. In the triangle BDC, *rad* × cos BC = *cos* DC × cos DB.
In the triangle ADC, *rad* × cos AC = *cos* DC × cos AD.
Hence, cos AD : cos DB : : cos AC : cos BC = 63°.50′ acute.
Because DB and the ∠ A are both acute, BC is acute.

BY FORMULA I. *page* 203.

$Log\ \text{tang}\ \varphi = (log\ \cos \text{A} + log\ \text{tang}\ c) - 10 = 133°.45'$ obtuse.
$Log\ \cos a = (log\ \cos c + log\ \cos \overline{b \sim \varphi}) - \cos \varphi = 63°.50$.

OR,

$Log\ \text{tang}\ \varphi = (log\ \cos \text{A} + log\ \text{tang}\ b) - 10 = 74°.40'$ acute.
$Log\ \cos a = (log\ \cos b + log\ \cos \overline{c \sim \varphi}) - \cos \varphi = 63°.50'$.

EXAMPLE II.

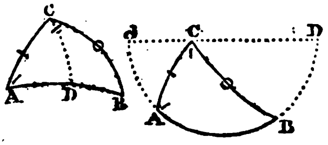
$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = \ 57°.30' \\ \text{The side AB} = \ 82°.27' \\ \text{The} \quad \angle \text{A} = 126°.37'\end{array}\right\}\text{Required the side BC.}$$

Answer. AD = 136°.54′ obtuse, AD − AB = DB = 54°.27′ acute, and BC = 115°.20′ obtuse.

(G) CASE IX. *Given two angles, and the side adjacent to both of them, to find a side opposite to one of the given angles.*

$$\text{Given}\left\{\begin{array}{l}\text{The} \quad \angle \text{A} = \ 51°.30' \\ \text{The} \ \angle \text{ACB} = 131°.30' \\ \text{The side AC} = \ 80°.19'\end{array}\right\}\text{Required the side BC.}$$

DETERMINATION OF THE SPECIES.



1. The ∠ ACD is acute or obtuse, according as the ∠ A is of the *same*, or of *different* species with the side AC.

2. The $\angle$ A is of the same species as CD; therefore, according as the $\angle$ A and the $\angle$ BCD are of the *same*, or of different species, BC is acute or obtuse.

SOLUTION.

1. In the triangle ADC, find the $\angle$ ACD.
   Thus, $\text{rad} \times \cos \text{AC} = \cot \angle \text{A} \times \cot \angle \text{ACD}$.
   $\cot \angle \text{A} : \text{rad} :: \cos \text{AC} : \cot \text{ACD} = 78^\circ.4'$ acute.
   $\angle \text{ACB} - \angle \text{ACD} = \angle \text{BCD} = 53^\circ.26'$ acute.
2. In the triangle BDC, $\textit{rad} \times \cos \text{BCD} = \textit{tang}\ \text{DC} \times \cot \text{BC}$.
   In the triangle ADC, $\textit{rad} \times \cos \text{ACD} = \textit{tang}\ \text{DC} \times \cot \text{AC}$.

Hence, $\cos \angle \text{ACD} : \cos \angle \text{BCD} :: \cot \text{AC} : \cot \text{BC} = 63^\circ.50'$ acute.

Because the $\angle$ A is of the same species as the $\angle$ BCD, the side BC is acute.

BY FORMULA VII. *page* 204.

$\textit{Log} \cot \varphi = (\textit{log} \tan \text{A} + \textit{log} \cos b) - 10 = 78^\circ.4'$ acute.

$\textit{Log} \cot a = (\textit{log} \cot b + \textit{log} \cos (\text{C} \sim \varphi)) - \textit{log} \cos \varphi = 63^\circ.50'$.

EXAMPLE II.

$$\text{Given} \left\{ \begin{array}{ll} \text{The} \angle \text{A} & = 126^\circ.37' \\ \text{The} \angle \text{ACB} & = 61^\circ.41' \\ \text{The side AC} & = 57^\circ.30' \end{array} \right\} \text{Required the side BC.}$$

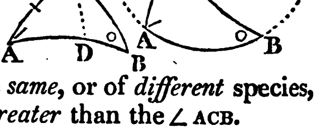
Answer. $\angle \text{ACD} = 125^\circ.52'$ obtuse, $\angle \text{ACD} - \angle \text{ACB} = \angle \text{BCD} = 64^\circ.11'$ acute, and the side $\text{BC} = 115^\circ.20'$ obtuse.

(H) CASE X. *Given two angles, and the sides adjacent to both of them, to find the other angle.*

$$\text{Given} \left\{ \begin{array}{ll} \text{The} \angle \text{A} & = 51^\circ.30' \\ \text{The} \angle \text{ACB} & = 131^\circ.30' \\ \text{The side AC} & = 80^\circ.19' \end{array} \right\} \text{Required the angle B.}$$

DETERMINATION OF THE SPECIES.

1. The $\angle$ ACD is acute or obtuse, according as the $\angle$ A is of the *same*, or of *different* species with the side AC.



2. The angles A and B are of the *same*, or of *different* species, according as the $\angle$ ACD is *less* or *greater* than the $\angle$ ACB.

SOLUTION.

1. In the triangle ADC, find the $\angle$ ACD.
   Thus, $\text{rad} \times \cos \text{AC} = \cot \angle \text{A} \times \cot \angle \text{ACD}$.
   $\cot \angle \text{A} : \text{rad} :: \cos \text{AC} : \cot \angle \text{ACD} = 78^\circ.4'$ acute.
   $\angle \text{ACB} - \angle \text{ACD} = \angle \text{BCD} = 53^\circ.26'$ acute.

2. In the triangle BDC, $rad \times \cos \mathrm{B} = \cos \mathrm{DC} \times \text{sine } \mathrm{BCD}$.
In the triangle ADC, $rad \times \cos \mathrm{A} = \cos \mathrm{DC} \times \text{sine } \mathrm{ACD}$.
Hence, $\text{sine} \angle \mathrm{ACD} : \text{sine} \angle \mathrm{BCD} :: \cos \angle \mathrm{A} : \cos \angle \mathrm{B} = 59°.16'$ acute.
Because the ∠ ACD is less than the ∠ ACB, the ∠ B is of the same species as the ∠ A, viz. acute.

BY FORMULA II. *page* 205.

$$\log \cot \varphi = (\log \text{tang } \mathrm{A} + \log \cos b) - 10 = 78°.4' \text{ acute.}$$
$$\log \cos \mathrm{B} = (\log \cos \mathrm{A} + \log \text{sine } \mathrm{C} \sim \varphi) - \log \text{sine } \varphi = 59'.16''.$$

EXAMPLE II.

$$\text{Given} \begin{cases} \text{The } \angle \mathrm{A} = 126°.37' \\ \text{The } \angle \mathrm{ACB} = 61°.41' \\ \text{The side } \mathrm{AC} = 57°.30' \end{cases} \text{Required the } \angle \mathrm{B}.$$

Answer. The $\angle \mathrm{ACD} = 125°.52'$ obtuse, $\angle \mathrm{ACD} - \angle \mathrm{ACB} = \mathrm{BCD} = 64°.11'$ acute, and the $\angle \mathrm{B} = 48°.30'$ acute.

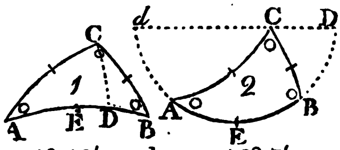
(I) CASE XI. *Given the three sides, to find the angles.*

$$\text{Given} \begin{cases} \text{The side } \mathrm{AC} = 80°.19' \\ \text{The side } \mathrm{BC} = 63°.50' \\ \text{The side } \mathrm{AB} = 120°.47' \end{cases} \text{Required the angles A, B, and C.}$$

The sum of the sides AC and BC being less than 180°, the perpendicular falls nearest to the less side BC; therefore if it falls without the triangle, the rule determines ED in fig. 2.

$\text{tang } \frac{1}{2} \mathrm{AB} = \text{tang } 60°.23'.30'' =$ - - 10·24544
$: \text{tang } \frac{1}{2} (\mathrm{AC} + \mathrm{BC}) = \text{tang } 72°.4'.30'' =$ 10·49016
$:: \text{tang } \frac{1}{2} (\mathrm{AC} - \mathrm{BC}) = \text{tang } 8°.14'.30'' =$ 9·16090
$: \text{tang } \mathrm{ED} = \text{tang } 14°.16'.35'' =$ - - 9·40562

Here ED is less than half the base, therefore the perpendicular falls *within* the triangle, and consequently the angles A and B are acute.



The segment $\mathrm{AD} = \frac{1}{2} \mathrm{AB} + \mathrm{ED} = 74°.40'$, and $\mathrm{DB} = 46°.7'$.

SECONDLY.

In the triangle ADC, the segment AD and the hypothenuse AC are given to find the $\angle \mathrm{A} = 51°.31'$, and the ∠ ACD.

In the triangle BDC, the segment BD and the hypothenuse BC are given to find the $\angle \mathrm{B} = 59°.17'$, and the ∠ BCD.

Lastly, $\angle \mathrm{ACD} + \angle \mathrm{BCD} = \angle \mathrm{ACB} = 131°.30'$.

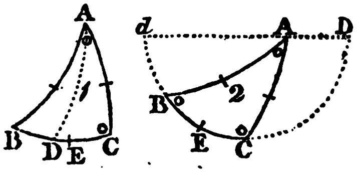
OR THUS, *Let* BC *be considered as the base.*

Then, the sum of the sides AB and AC being greater than 180°, the perpendicular falls nearest to the greater side AB;

therefore if it falls without the triangle, the rule determines Ed in fig. 2.

tang $\frac{1}{2}$ BC = tang 31°.55′. = - - 9·79438
: tang $\frac{1}{2}$ (AB + AC) = tang 100°.33′ = 10·72992
: : tang $\frac{1}{2}$ (AB − AC) = tang 20°.14′ = - . 9·56654
: tang ED or E*d* = tang 72°.32′ = - 9·50208

Here the fourth proportional is greater than half the base, therefore the perpendicular falls *without* the triangle.



The segment *d*B = E*d* − $\frac{1}{2}$ BC = 40°.37′; the segment BD = $\frac{1}{2}$ BC + supplement of E*d* = 139°.23′.

SECONDLY.

In the triangle A*d*B, or ADB, the ∠ B may be found = 59°.17′. And in the triangle ACD, or AC*d*, the ∠ C may be found = 131°.28′.

EXAMPLE II.

Given { The side AC = 57°.30′, The side BC = 115°.20′, The side AB = 82°.28′ } Required the angles A and B.

Answer. ∠ A = 126°.34′, and ∠ B = 48°.31′.

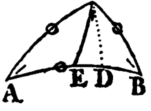
(K) Case XII. *Given the three angles to find the sides.*

Given { The ∠ A = 51°.30′, The ∠ B = 59°.16′, The ∠ C = 131°.30′ } Required the sides AC, BC, and AB.

Let the angles A and B, which are of the same species, be considered as base angles, then the perpendicular will fall within the triangle.

And since the sum of the two angles A and B is less than 180°, the sum of the sides AC and BC is less than 180°, therefore the perpendicular falls nearest to B, the greater angle.

Cot $\frac{1}{2}$ (A + B) = cot 55°.23′ = - - 9·83903
: tang $\frac{1}{2}$ (B − A) = tang 3°.53′ = - 8·83175
: : tang $\frac{1}{2}$ ∠ ACB = tang 65°.45′ = - 10·34634
: tang ∠ ECD = tang 12°.19′ = - 9·33906
Then, $\frac{1}{2}$ ∠ ACB + ∠ ECD = ∠ ACD = 78°.4′.
And $\frac{1}{2}$ ∠ ACB − ∠ ECD = ∠ BCD = 53°.26′.



SECONDLY.

In the triangle ADC, the ∠ A and the ∠ ACD are given, to find AC = 80°.19′, and AD = 74°.40′.

In the triangle BDC, the ∠ B and the ∠ BCD are given, to find BC = 63°.50′, and DB = 46°.7′.

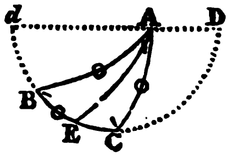
Lastly, AD + DB = AB = 120°.47′.

Or thus, *Let* BC *be considered as the base.*

The base angles being here of different species, the perpendicular falls without the triangle.

And because the sum of the two angles B and C is greater than 180°, the perpendicular falls nearest to the less angle B, or greatest side AB, consequently the rule determines the $\angle$ EA*d*.

$\text{Cot } \frac{1}{2}(\text{B}+\text{C}) = \cot 95°.23' =$ - 8·97421
: $\text{tang } \frac{1}{2}(\text{C}-\text{B}) = \text{tang } 36°.7' =$ 9·86312
:: $\text{tang } \frac{1}{2} \angle \text{BAC} = \text{tang } 25°.45' =$ 9·68336
: $\text{tang } \angle \text{EA}d = \text{tang } 75°.1' =$ 10·57227



Then $\frac{1}{2} \angle \text{BAC} + \angle \text{EA}d = \angle \text{CA}d = 100°.46'$; and hence the $\angle$ CAD, being the supplement, $= 79°.14'$ and $\angle \text{BAD} = 130°.44'$.

SECONDLY.

In the triangle A*d*C, or ADC, the side AC will be found $= 80°.19'$, and the side *d*C or DC may likewise be found.

In the triangle A*d*B, or ADB, the side AB will be found $= 120°.48'$, and the side *d*B, or DB, may likewise be found.

Lastly, $180° - (\text{DC} + d\text{B})$, or $d\text{C} - d\text{B}$, or $\text{DB} - \text{DC} = \text{BC} = 63°.50'$.

EXAMPLE II.

$$\text{Given} \left\{ \begin{array}{l} \text{The } \angle \text{A} = 48°.31' \\ \text{The } \angle \text{B} = 62°.52' \\ \text{The } \angle \text{C} = 125°.20' \end{array} \right\} \text{Required the sides AC, BC, and AB.}$$

Answer. $\text{AC} = 83°.8'.32''$, $\text{BC} = 56°.41'.40''$, and $\text{AB} = 114°.28'.45''$.

CHAP. IX.

PRACTICAL RULES FOR SOLVING ALL THE DIFFERENT CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES, WITHOUT A PERPENDICULAR, WITH THEIR APPLICATION BY LOGARITHMS.

CASE I.

(L) When two sides and an angle opposite to one of them are given, to find the rest.

RULE.

1. *To find the other opposite angle.*

Sine of the side opposite to the given angle,
Is to sine of the given angle;
As sine of the other given side,
Is to sine of its opposite angle.

To the angle found by this proportion, and its supplement, add the given angle. Then, if each of these sums be of the same species with respect to 180°, as the sum of the given sides, the problem is ambiguous; that is, the angle thus found may be either acute or obtuse.

But, if only *one* of these sums be of the same species with the sum of the sides, that value of the angle, found by this proportion, must be taken, whether it be acute or obtuse, which when added to the given angle agrees with the sum of the sides. In this case the problem is not ambiguous.

2. *To find the angle contained between the given sides.*

Find the angle opposite to the other given side, by the first part of the rule, and note whether it be acute, or obtuse, or ambiguous.

Then, Sine of half the difference between the two given sides,
Is to sine of half their sum;
As tangent of half the difference between their opposite angles,
Is to cotangent of half the angle* contained between the given sides. (M. 188.)

3. *To find the third side.*

Find the angle opposite to the other given side, by the first part of the rule, and note whether it be acute, obtuse, or ambiguous.

Then, Sine of half the difference between the two angles,
Is to sine of half their sum;
As tangent of half the difference between the two given sides,
Is to tangent of half the required side. (N. 189.)

CASE II.

(M) When two angles, of an oblique-angled spherical triangle, and a side opposite to one of them are given, to find the rest.

---

* Since a side, or an angle, of any spherical triangle is always less than 180°; the *half* of any side or angle must *always* be acute. The ambiguity therefore ascribed to Case I. and II. arises from the first proportion in each case; if the angle, or side, found by these proportions be ambiguous, the remaining parts of the triangle will necessarily be ambiguous, but if the angle, or side, found by these proportions be determinate, the remaining parts of the triangle will also be determinate.

The ambiguous parts derived from the first proportion, in Case I. or II. are always supplements of each other; but the remaining parts of the triangle, when ambiguous, are not supplements of each other, as is obvious both from the constructions and calculations following.

RULE.

1. *To find the other opposite side.*

Sine of the angle opposite to the given side,
Is to sine of the given side;
As the sine of the other given angle,
Is to the sine of its opposite side.

To the side found by this proportion, and its supplement, add the *given* side. Then if each of these sums be of the same species with respect to 180°, as the sum of the given angles, the problem is ambiguous; that is, the side thus found may be either acute or obtuse.

But, if only *one* of these sums be of the same species as the sum of the given angles, that value of the side, found by this proportion, must be taken, which when added to the given side agrees with the sum of the angles. In this case the problem is not ambiguous.

2. *To find the side adjacent to the two given angles.*

Find the side opposite to the other given angle, by the first part of the rule, and note whether it be acute, obtuse, or ambiguous.

Then, Sine of half the difference between the two given angles,
Is to sine of half their sum;
As tangent of half the difference between the two sides,
Is to tangent of half the third side. (N. 189.)

3. *To find the third angle.*

Find the side opposite to the other given angle, by the first part of the rule, and note whether it be acute, obtuse, or ambiguous.

Then, Sine of half the difference between the two sides containing the required angle,
Is to sine of half their sum;
As tangent of half the difference between the other two angles,
Is to cotangent of half the required angle. (M. 188.)

CASE III.

(N) When two sides and the included angle, of an oblique angled spherical triangle, are given, to find the rest.

RULE.

1. *To find the other two angles.*

Cosine of half the sum of the two given sides,
Is to cosine of half their difference;
As cotangent of half the included angle,
Is to tangent of half the sum of the other two angles.

Half the sum of these two angles must be of the same species as half the sum of the given sides.

Secondly,

Sine of half the sum of the two given sides,
Is to sine of half their difference;
As cotangent of half the included angle,
Is to tangent of half the difference between the other two angles. (M. 188.)

Half the difference between these angles is always acute.

Lastly,

Half the sum of the two angles increased by half their difference, gives the angle opposite to the greater side, and diminished by the same, leaves the angle opposite to the less side. (C. 35.)

2. *To find the third side.*

Find the two required angles by the first part of the rule.

Then, Sine of half the difference between these angles,
Is to sine of half their sum;
As tangent of half the difference between the given sides,
Is to tangent of half the third side. (N. 189.)

OR, *without finding the other two angles.*

To the sum of the logarithmical sines of the given sides, add double the logarithmical sine of half the contained angle, and reject 30 from the index.

Look for the remainder in the table of logarithmical sines, and take the degrees and minutes answering to it. Then take the difference between twice the natural sine of those degrees, and the natural cosine of the difference between the given sides; the remainder will be the natural cosine of the side required. This side is acute or obtuse, according as the double natural sine is less, or greater, than the natural cosine of the difference between the given sides. (X. 192.)

[illegible] When two [illegible] angle, and the side [illegible] the rest.

[illegible]

Cosine of half [illegible]
Is to cosine of half their [illegible]
As tangent of half the [illegible]
Is to tangent of half the [illegible]

Half the sum of these sides, [illegible] half the sum of the given angles.

[illegible]

Sine of half the sum of the [illegible]
Is to sine of half their difference [illegible]
As tangent of half the adjacent side,
Is to tangent of half the difference between [illegible] two sides.

Half the difference between these sides [illegible] (N. 189 [illegible]

[illegible]

Half the sum of the two sides [illegible] ence, gives the side opposite to the greater [illegible] nished by the same, leaves the side [illegible]

[illegible]

Find the two required sides [illegible]
Then, Sine of half the difference [illegible]
Is to sine of half their [illegible]
As tangent of half the difference [illegible] angles,
Is to [illegible]

OR, [illegible]

To the sum of the [illegible] add double the [illegible] reject 30 from the index.

Look for the [illegible] and take the degrees [illegible] the difference between [illegible] &c. and the natural [illegible]

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CASE V.

(P) When the three sides, of an oblique-angled spherical triangle, are given to find the angles.

RULE I.

From half the sum of the three sides subtract the side opposite to the required angle, and note the half sum and remainder. Then add together,

The logarithmical co-secants of each of the sides containing the required angle, rejecting the indices; and the sines of the above half sum and remainder: half the sum of these four logarithms is the logarithmical cosine of half the angle sought. (G. 185.)

OR, RULE II.

Add all the three sides together, from the half sum subtract each side containing the required angle, and note the remainders. Then add together,

The logarithmical co-secants of each of the sides containing the required angle, rejecting the indices; and the sines of the above-noted remainders: half the sum of these four logarithms, is the logarithmical sine of half the angle sought. (F. 184.)

OR, RULE III.

From half the sum of the three sides subtract each side separately. Then add together,

The logarithmical co-secants of half the sum of the sides, and of the difference between that half sum and the side opposite to the angle required, rejecting the indices; the logarithmical sines of the difference between the half sum and each side containing the required angle, half the sum of these four logarithms is the logarithmical tangent of half the angle sought. (H. 186.)

CASE VI.

(Q) When the three angles, of an oblique-angled spherical triangle, are given to find the sides.

RULE I.

Add all the three angles together, take the difference between the half sum and the angle opposite to the side sought, and note the half sum and remainder. Then add together, The logarithmical co-secants of each of the angles adjacent to the required side, rejecting the indices, and the cosines of the above half sum and remainder; half the sum of these four logarithms is the logarithmical sine of half the side sought. (I. 186.)

OR, RULE II.

Take the supplements of each of the angles, and use the remainders as sides in a new triangle.

Find the angles of this triangle, by any of the rules in Case V. the supplements of which will be the sides sought. (U. 137.)

(R) CASE I. *Given two sides of an oblique spherical triangle, and an angle opposite to one of them, to find the rest.*

In the oblique spherical triangle ABC.

Given { The side AC = 80°.19′
The side BC = 63°.50′
The ∠ A = 51°.30′ } Required the rest.

BY CONSTRUCTION. (*Plate V. Fig.* 15.)

1. With the chord of 60 degrees describe the primitive circle; through the centre P draw CP*e*, and *a*P*r* at right angles to it.
2. Set off the side AC = 80°.19′ from C to A, by the scale of chords.
3. Through A draw the great circle A*b*B*n*, making an angle of 51°.30′ with the primitive. (P. 160.)
4. Set off the side BC = 63°.50′ by a scale of chords, from C to *m*, and draw the parallel circle *mb*B*m*. (Z. 162.) Through the points *b*, B, where it cuts the oblique circle A*b*B*n*, and the point C, draw the great circles C*be*, CB*e*.
5. Then, A*b*C or ABC is the triangle required, each having the same *data*, which shews this example to be *ambiguous*.

To measure the required parts.

6. The side A*b* (C. 163.) = 28°.33′, and AB = 120°.47′.
7. The ∠ AC*b* (G. 164.) = 24°.37′, and ∠ ACB = 131°.29′; the ∠ A*b*C = 120°.44′, and the ∠ ABC = 59°.16′.

BY CALCULATION.

I. *To find the* ∠ B.

sine BC = 63°.50′ - - 9·95304
: sine ∠ A = 51°.30′ - - 9·89354
: : sine AC = 80°.19′ - - 9·99377
: sine ∠ B = 59°.16′ - = 9·93427

Because AC + BC, A + (B *acute*), and A + (B *obtuse*) are each of the same species with respect to 180°, the ∠ B is *ambiguous* (Y. 232.) being = 59°.16′ or its supplement 120°.44′.

II. *To find the* ∠ C.

sine ½ (AC ∼ BC) = 8°.14′.30″ 9·15639
: sine ½ (AC + BC) = 72°.4′.30″ 9·97839
:: tang ½ (A ∼ B) = 3°.53′ - 8·83175
: cot ½ ∠ C = 65°.44′.50″ 9·65375
2

131°.29′.40″ = ∠ C.

Had the obtuse value of the ∠ B been used, viz. 120°.44′, the ∠ C by the same method would have been 24°.37′.20″.

III. *To find the side* AB.

sine ½ (A ∼ B) = 3°.53′ - 8·83075
: sine ½ (A + B) = 55°.23′ - 9·91538
:: tang ½ (AC ∼ BC) = 8°.14′.30″ 9·16096
: tang ½ AB = 60°.23′.50″ - 10·24553
2

120°.47′.40″ = AB.

If the obtuse value of the ∠ B had been used, the side AB ~~by the same~~ process would have been 28°.33′.

EXAMPLE II.

In the oblique spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The side AC} = 57^\circ.30'\\ \text{The side BC} = 115^\circ.20'\\ \text{The } \angle\text{A} = 126^\circ.37'\end{array}\right\}\text{Required the rest.}$$

BY CONSTRUCTION. (*Plate V. Fig.* 16.)

1. With the chord of 60° describe the primitive circle, through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Set off the side AC = 57°.30′ from C to A, by the scale of chords.

3. Through A draw the great circle AB*n*, making an angle of 126°.37′ with the primitive. (P. 160.)

4. Set off the side BC = 115°.20′, by a scale of chords, from C to *m*, and draw the parallel circle *m*B*m*. (Z. 162.) Through the point B, where it cuts the oblique circle AB*n*, and the point C, draw the great circle CB*e*.

5. Then ABC is the triangle required; and though it has exactly the same *data* as the former example, *none of the parts are ambiguous.*

To measure the required parts.

6. The side AB (C. 163.) = 82°.26′.

7. The ∠ B (G. 164.) = 48°.30′, and the ∠ C = 61°.40′.

BY CALCULATION.

I. *To find the* ∠ B.

sine BC = 115°.20′ - - 9·95609
: sine ∠ A = 126°.37′ - 9·90452
:: sine AC = 57°.30′ - 9·92603
: sine ∠ B = 48°.30′ - - 9·87446

Because AC + BC, and A + (B *acute*) only, are of the same species with respect to 180°, the ∠ B is acute, and *not* ambiguous. (Y. 232.)

II. *To find the* ∠ C.

sine ½ (AC ∼ BC) = 28°.55′ - 9·68443
: sine ½ (AC + BC) = 86°.25′ 9·99915
:: tang ½ (A ∼ B) = 39°.3′ 30″ 9·90927
: cot ½ ∠ C = 30°.50′.20″ - 10·22399
2

61°.40′.40″ = ∠ C.

III. *To find the side* AB.

sine ½ (A ∼ B) = 39°.3′.30″ - 9·79942
: sine ½ (A + B) = 87°.33′.30″ 9·99960
:: tang ½ (AC ∼ BC) = 28°.55′ 9·74226
: tang ½ AB = 41°.12′.50″ 9·94244
2

82°.25′.40″ = AB.

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given { The side AC = 40°
The side BC = 70°
The ∠A = 130°.3′.11″

Answer { ∠B = 31°.34′.26″
∠C = 30°.28′.20″
AB = 38°.30′.

Required the other parts.

2. In the oblique spherical triangle ABC.

Given { The side AC = 119°. 5′
The side BC = 79°.13′
The ∠A = 58°. 8′

Answer { ∠B = 130°.56′
∠C = 50°.11′.40″
AB = 62°.42′.

Required the other parts.

3. In the oblique spherical triangle ABC.

Given { The side AC = 30°
The side BC = 24°.4′
The ∠A = 36°.8′

Ans. { ∠B = 46°.18′, or 133°.42′.
∠C = 104°, or 11°.23′.
AB = 42°.9′, or 7°.51′.

Required the other parts.

(S) CASE II. *Given two angles of an oblique spherical triangle, and a side opposite to one of them, to find the rest.*

In the oblique spherical triangle ABC.

Given { The ∠A = 51°.30′
The ∠B = 59°.16′
The side BC = 63°.50′ } Required the rest.

BY CONSTRUCTION. (*Plate V. Fig.* 17.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw BP*e*, and DPE at right angles to it.

2. Set one foot of your compasses on 90 degrees, on the line of semi-tangents, extend the other towards the beginning of the scale, till the degrees between them be equal to the angle B = 59°.16′, and apply this extent from E to *n* (P. 160.); and through the three points B*ne* draw a great circle.

3. Set off the side BC = 63°.50′. taken from a scale of chords, from B to *m*, and draw the parallel circle *mcm*, cutting the oblique circle, B*ne* in C.

4. With the tangent of the angle A = 51°.30′ and P as a centre, describe an arc; and with the secant of the same angle, and C as a centre, cross it in *o*.

5. With the centre *o*, and radius *o*C, draw the great circle *a*CA. Then ABC is the triangle required.

To measure the required parts.

6. AB measured by a scale of chords will be 120°.47′, or 151°.27′ *ambiguous*.

7. AC will be 80°.19′, or 99°.41′ (C. 163.) *ambiguous*.

8. The ∠C = 131°.30′, or 155°.22′ (G. 164.) *ambiguous*.

BY CALCULATION.

I. *To find the side* AC.

sine ∠A = 51°.30′ - - 9·89354
: sine BC = 63°.50′ - - 9·95304
: : sine ∠B = 59°.16′ - - 9·93427
: sine AC = 80°.19′ - - 9·99377

Because BC + (AC *acute*), BC + (AC *obtuse*), and A + B, are each of the same species, with respect to 180°, AC is *ambiguous*, (B.235.) being either 80°.19′, or 99°.41′.

II. *To find the side* AB.

sine ½ (A ∼ B) = 3°.53′ - 8·83075
: sine ½ (A + B) = 55°.23′ - 9·91538
: : tang ½ (AC ∼ BC) = 8°.14′.30″ 9·16090
: tang ½ AB = 60°.23′.50″ - 10·24553
2

120°.47′.40″ = AB.

If the obtuse value of AC (= 99°.41′) had been used, the side AB, by the same process, would have been = 151°.27′.

III. *To find the* ∠C.

sine ½ (AC ∼ BC) = 8°.14′.30″ 9·15639
: sine ½ (AC + BC) = 72°.4′.30″ 9·97839
: : tang ½ (A ∼ B) = 3°.53′ - 8·83174
: cot ½ ∠C = 65°.44′.50″ - 9·65374
2

131°.29′.40″ = ∠C.

If the obtuse value of the side AC had been used, the ∠C, by the same method, would have been = 155°.22′.40″.

EXAMPLE II.

In the oblique spherical triangle ABC.

Given { The ∠A = 126°.37′
The ∠B = 48°.30′
The side BC = 115°.20′ } Required the rest.

BY CONSTRUCTION. (*Plate V. Fig.* 18.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw BP*e*, and DPE at right angles to it.

2. Draw the great circle BC*e* making an angle of 48°.30′ with the primitive. (P. 160.)

3. Set off the side BC = 115°.20′ from B to *m*, and draw the parallel circle *mcm* (Z. 162.), cutting the oblique circle BC*e* in C.

4. With the tangent of the complement of the angle A = 53°.23′, and centre P, describe an arc; and with the secant of the same angle and centre C, cross it in *o*.

5. With the centre *o*, and radius *o*C, draw the great circles A*cb*, *acb*. Then ABC is the triangle required; and *none* of the parts are ambiguous.

To measure the required parts.

6. AB, measured by a scale of chords, = 82°.26′.
7. AC (C. 163.) = 57°.30′.
8. The ∠C (G. 164.) = 61°.41′.

BY CALCULATION.

I. *To find the side* AC.

sine ∠A = 126°.37′ - - 9·90452
: sine BC = 115°.20′ - - 9·95609
: : sine ∠B = 48°.30′ - - 9·87446
: sine AC = 57°.30′ - - 9·92603

Because BC + (AC *acute*,) and A + B only, are of the same species with respect to 180°, the side AC is acute, and *not* ambiguous. (B. 235.)

II. *To find the side* AB.

sine ½ (A ~ B)	= 39°.3′.30″ -	9·79942
: sine ½ (A + B)	= 87°.33′.30″	9·99960
:: tang ½ (AC ~ BC)	= 28°.55′	9·74226
: tang ½ AB	= 41°.13′ -	9·94244
	2	
	82°.26′ = AB.	

III. *To find the* ∠ C.

sine ½ (AC ~ BC)	= 28°.55′ -	9·68443
: sine ½ (AC + BC)	= 86°.25′	9·99915
:: tang ½ (A ~ B)	= 39°.3′.30″	9·90927
: cot ½ ∠ C	= 30°.50′.20″ -	10·22399
	2	
	61°.40′.40″ = ∠ C.	

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given		Answer.
The ∠ A = 121°.36′.20″		AC = 50°.10′.30″.
The ∠ B = 42°.15′.13″		AB = 40°.0′.10″.
The side BC = 76°.35′.36″		∠ C = 34°.15′.3″.

Required the other parts.

2. In the oblique spherical triangle ABC.

Given		Answer.
The ∠ A = 34°.15′. 5″		AC = 50°.10′.30″, or 129°.49′.30″.
The ∠ B = 42°.15′.13″		AB = 76°.35′.40″, or 167°. 6′.
The side BC = 40°. 0′.10″		∠ C = 121°.36′.20″, or 168°.43′.40″.

Required the other parts.

3. In the oblique spherical triangle ABC.

Given		Ans.
The ∠ A = 36°. 8′		AC = 30°, or 150°.
The ∠ B = 133°.42′		AB = 7°.51′, or 137°.51′.
The side BC = 24°. 4′		∠ C = 11°.23′, or 104°.

Required the other parts.

(T) CASE III. *Given two sides of an oblique spherical triangle, and the angle contained between them, to find the rest.*

Given	
The side AC = 80°.19′	Required the rest.
The side AB = 120°.47′	
The ∠ A = 51°.30′	

BY CONSTRUCTION. (*Plate V. Fig.* 19.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw CP*c*, and *a*P*r* at right angles to it.

2. Set off the side AC = 80°.19′ by a scale of chords, and through the point A, draw the great circle AB*n*, making an angle of 51°.30′ with the primitive. (P. 160.)

3. Set off the supplement of AB = 59°.13′ from *n* to *m*, and draw the parallel circle *m*B*m*′ (Z. 162.) cutting the oblique circle AB*n* in B. Through the three points C, B, *e*, draw a great circle, then ABC is the triangle required.*

---

* A perpendicular may be drawn from the vertical angle C upon the base AB, by finding *p* the pole of the oblique circle AB*n*, (N. 159.) and drawing a great circle CD*p*, through *p* and the point C. (W. 161.)

To measure the required parts.

4. BC (C. 163.) $= 63^\circ.50'$.
5. The angles B and C (G. 164.) $= 59^\circ.16'$ and $131^\circ.30'$.

BY CALCULATION.

I. *To find the angles* B *and* C.

$\cos\frac{1}{2}(AB+AC) = 100^\circ.33'$ 9·26267
$: \cos\frac{1}{2}(AB\sim AC) = 20^\circ.14'$ 9·97234
$:: \cos\frac{1}{2}\angle A = 25^\circ.45'$ 10·31664
$: \text{tang}\,\frac{1}{2}(B+C) = 84^\circ.37'.23''$ 11·02631

$\text{sine}\,\frac{1}{2}(AB+AC) = 100^\circ.33'$ 9·99259
$: \text{sine}\,\frac{1}{2}(AB\sim AC) = 20^\circ.14'$ 9·53888
$:: \cot\frac{1}{2}\angle A = 25^\circ.45'$ - 10·31664
$: \text{tang}\,\frac{1}{2}(B\sim C) = 36^\circ.6'.20''$ 9·86293

Because $\frac{1}{2}(AB+AC)$ is obtuse $\frac{1}{2}(B+C)$ must be obtuse $= 95^\circ.22'.37''$. Hence $95^\circ.22'.37'' + 36^\circ.6'.20'' = 131^\circ.28'.57'' = \angle C$, and
$95^\circ.22'.37'' - 36^\circ.6'.20'' = 52^\circ.16'.17'' = \angle B$.

II. *To find* BC.

1. *By using the angles* B *and* C.

$\text{sine}\,\frac{1}{2}(B\sim C) = 36^\circ.6'.20''$ - 9·77032
$: \text{sine}\,\frac{1}{2}(B+C) = 84^\circ.37'.23''$ 9·99808
$:: \text{tang}\,\frac{1}{2}(AC\sim AB) = 20^\circ.14'$ 9·56654
$: \text{tang}\,\frac{1}{2}BC = 31^\circ.54'.40''$ - 9·79430
2

$63^\circ.49'.20'' = BC$.

2. *Without the angles* B *and* C.

log sine $AC = 80'.19''$ - 9·99377
log sine $AB = 120^\circ.47'$ - 9·93405
log sine $\frac{1}{2}\angle A = 25^\circ.45'$ $\begin{cases} 9.63794 \\ 9{\cdot}63794 \end{cases}$

log sine $9^\circ.12'$ - Sum 9·20370

nat. sine $9^\circ.12' = {\cdot}15988$; $\times 2 = {\cdot}31976$
nat. $\cos(AB\sim AC) = 40^\circ.28'$ - ·76078

nat. cos $63^\circ.50'$ diff. ·44102

Here BC is acute.

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given { The side AC $= 57^\circ.30'$
The side BC $= 82^\circ.27'$
The $\angle A = 126^\circ.37'$
Answer. { $\angle B = 48^\circ.30'$.
$\angle C = 61^\circ.41'$.
$BC = 115^\circ.21'$.

Required the other parts.

2. In the oblique spherical triangle ABC.

Given { The side AC $= 30^\circ$
The side AB $= 42^\circ.9'$
The $\angle A = 36^\circ.8'$
Answer. { $\angle B = 46^\circ.18'$.
$\angle C = 104^\circ$
$BC = 24^\circ.4'$.

Required the other parts.

3. In the oblique spherical triangle ABC.

Given { The side AC $= 50^\circ.10'.30''$
The side AB $= 40^\circ.\ 0'.10''$
The $\angle A = 121^\circ.36'.20''$.
Ans. { $\angle B = 42^\circ.15'.13''$.
$\angle C = 34^\circ.15'.\ 3''$.
$BC = 76^\circ.35'.36''$.

Required the other parts.

(U) Case IV. *Given two angles, and the side adjacent to both of them, to find the rest.*

$$\text{Given} \left\{\begin{array}{ll}\text{The} & \angle A = 51°.30' \\ \text{The} & \angle C = 131°.30' \\ \text{The side} & AC = 80°.19'\end{array}\right\} \text{Required the rest.}$$

BY CONSTRUCTION. (*Plate V. Fig.* 20.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Set off the side AC=80°.19′ by a scale of chords, and through the point A, draw the great circle AB*n*, making an angle of 51°.30′ with the primitive. (P. 160.) In the same manner draw the great circle CB*e* through C, making an angle BC*r*=48°. 30′, the supplement of C. Then ABC is the triangle required.*

To measure the required parts.

3. AB and BC (C. 163.)=120°.47′, and 63°. 50′.

4. The ∠B (G. 164.)= 59°.16′.

BY CALCULATION.

I. *To find the sides* AB *and* BC.

$\cos \frac{1}{2}(A + C) = 91°.30'$ -	8·41792
$: \cos \frac{1}{2}(A \sim C) = 40°$ ..	9·88425
$:: \text{tang} \frac{1}{2} AC = 40°.9'.30''$ -	9·92625
$: \text{tang} \frac{1}{2}(AB + BC) = 87°·41'$	11·39258

$\text{sine} \frac{1}{2}(A + C) = 91°.30'$ -	9·99985
$: \text{sine} \frac{1}{2}(A \sim C) = 40°$ -	9 80807
$:: \text{tang} \frac{1}{2} AC = 40°.9'.30''$ -	9·92625
$: \text{tang} \frac{1}{2}(AB \sim BC) = 28°.29'$	9·73447

Because $\frac{1}{2}(\angle A + \angle C)$ is obtuse, $\frac{1}{2}(AB + BC)$ must be obtuse= 92°. 19′.

Hence 92°.19′+28°.29′=120°.48′=AB. And
92°.19′−28°.29′= 63°.50′=BC.

II. *To find the* ∠B.

1. *By using the sides* AB *and* BC.

$\text{sine} \frac{1}{2}(AB \sim BC) = 28°.29'$ -	9·67843
$; \text{sine} \frac{1}{2}(AB + BC) = 92°.19'$	9·99964
$:: \text{tang} \frac{1}{2}(A \sim C) = 40°$ -	9·92381
$: \cot \frac{1}{2} \angle B = 29°.38'$ - -	10·24502
	2
$59°.16'' = \angle B.$	

2. *Without the sides* AB *and* BC.

log sine A=51°.30′ - -	9·89354
log sine C=131°.30′ -	9·87446
log cos $\frac{1}{2}$ AC=40°.9′ -	9·88325 9·88325
log sine 20°.1′.17″ -	sum 9·53450
nat.sine20°1′.17′=·34236; + 2 ×	·68472
nat. cos (A∼C) 80° -	·17365
nat. cos ∠B=59°.16′ -	diff. ·51107

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

$$\text{Given} \left\{\begin{array}{ll}\text{The} & \angle A = 126°.37' \\ \text{The} & \angle C = 61°.41' \\ \text{The side} & AC = 57°.30'\end{array}\right. \textit{Answer.} \left\{\begin{array}{l} AB = 82°.27'. \\ BC = 115°.20'. \\ \angle B = 48°.30'.\end{array}\right.$$

Required the other parts.

---

* A perpendicular CD may be drawn, if required, by the note Cas

2. In the oblique spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The } \angle A = 121^\circ.36'.20'' \\ \text{The } \angle C = 42^\circ.15'.13'' \\ \text{The side } AC = 40^\circ.\ 0'.10''\end{array}\right. \quad \textit{Ans.}\left\{\begin{array}{l} AB = 50^\circ.10'.30''. \\ BC = 76^\circ.35'.36''. \\ \angle B = 34^\circ.15'.3''.\end{array}\right.$$

Required the other parts.

3. In the oblique spherical triangle ABC.

$$\text{Given}\left\{\begin{array}{l}\text{The } \angle A = 36^\circ.8'. \\ \text{The } \angle C = 104^\circ \\ \text{The side } AC = 30^\circ\end{array}\right. \quad \textit{Answer.}\left\{\begin{array}{l} AB = 42^\circ.9'. \\ BC = 24^\circ.4'. \\ \angle B = 46^\circ.19'.\end{array}\right.$$

Required the other parts.

(W) CASE V. *Given the three sides to find an angle.*

$$\text{Given}\left\{\begin{array}{l}\text{The side } AC = 80^\circ.19' \\ \text{The side } BC = 63^\circ.50' \\ \text{The side } AB = 120^\circ.47'\end{array}\right\}\text{Required the angles.}$$

BY CONSTRUCTION. (*Plate V. Fig.* 21.)

1. With the chord of 60 degrees describe the primitive circle, through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Set off the side BC=63°.50′ from the scale of chords, and draw BPE.

3. Set off the side AC=80°.19′, by a scale of chords, from C to *m*, and draw the parallel circle *m*A*m*. (Z. 162.)

4. Set off the supplement of AB=59°.13′, by a scale of chords, from E to *n*, and draw the parallel circle *n*A*n*. (Z. 162.)

5. Through C and A, and B and A, draw the great circles CA*e* and BAE. Then ABC is the triangle required.[*]

To measure the required parts.

6. The angles A, B, and C (G. 164.) will be 51°.30′; 59°.16′; and 131°.30′.

BY CALCULATION.

To find the ∠ A *by Rule* I.

Cosec AC	= 80°.19′	·00623	reject indices †.
Cosec AB	= 120°.47′	·06595	
Sine ½ (AC+BC+AB)	= 132°.28′	9.86786	
Sine ½ (AC+BC+AB) − BC	= 68°.38′	9.96907	
		2)19.90911	
Cos ½ ∠ A	= 25°.45′	9.95456	
	2		
	51.30′ = ∠ A.		

---

[*] A perpendicular CD may be drawn from the vertical ∠ C upon the base AB, by the note to Case III.

[†] Or take the sines of AC and AB, add them together and subtract the sum from 20; the remainder will be the same as the sum of these cosecants without the indices.

To find the ∠ B *by Rule II.*

Cosec BC - - = 63°.50′ - ·04696 } reject
Cosec AB - - =120°.47′ - ·06595 } indices.
Sine½(AC+BC+AB)—BC= 68°.38′ - 9·96907
Sine½(AC+BC+AB)—AB= 11°.41′ - 9·30643

2)19·38841

Sine ½ ∠ B=29°.38′.24″ - 9·69420
2

59°.16′.48″=∠ B

To find the ∠ C *by Rule III.*

Cosec½ (AC+BC+AB) - =132°.28′ - ·13214 } reject
Cosec½ (AC+BC+AB)—AB = 11°.41′ - ·69357 } indices.
Sine ½ (AC+BC+AB)—BC = 68°.38′ - 9·96907
Sine ½ (AC+BC+AB)—AC = 52°. 9′ - 9·89742

2)20·69220

Tang ½ ∠ C=65°.44′.17″ - 10·34610
2

131°.28′.34″=∠ C.

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given { The side AC= 57°.30′; The side BC=115°.20′; The side AB= 82°.28′ } *Answer.* { ∠ A=126°.37′; ∠ B= 48°.30′; ∠ C= 61°.41′.

Required the angles.

BY CONSTRUCTION.* (*Plate V. Fig.* 22.)

1. Describe the primitive circle with the chord of 60°, on which set off AB=82°.28′, and from A and B through the centre P draw AP*e* and BP*r*.

2. Set off the side AC=57°.30′, by a scale of chords, from A to *m*, and draw the parallel circle *mcm*. (Z. 162.)

---

* The construction of the figure to this example is not essentially different from the construction of the figure to the preceding example, and is introduced here to shew in what manner the general figures, Case XI. &c. of oblique spherics with a perpendicular, were formed.—Compare the two figures in this Case (V) with those in Case XI. referred to above.

3. Set off the supplement of the side BC=64°.40′ from *r* to *n*, and draw the parallel circle *ncn*. (Z. 162.)

4. Through A, and the point of intersection c, draw the oblique circle, A*ce*; and through B, and the same point, draw B*cr*.

5. Then ABC is the triangle required; and *cd*, or CD, are perpendiculars on the base AB produced.

2. In the oblique spherical triangle ABC.

Given	The side AC=50°.10′.30″	*Ans.*	∠A= 34°.15′. 3″.
	The side BC=40°. 0′.10″		∠B= 42°.15′.13″.
	The side AB=76°.35′.36″		∠C=121°.36′.20″.

Required the angles.

3. In the oblique spherical triangle ABC.

Given	The side AC=30°	*Answer.*	∠A= 36°. 8′.
	The side BC=24°.4′		∠B= 46°.19′.
	The side AB=42°.9′		∠C=104°.

Required the angles.

(X) CASE VI. *Given the three angles to find the sides.*

Given	The ∠A= 51°.30′	Required the sides.
	The ∠B= 59°.16′	
	The ∠C=131°.30′	

BY CONSTRUCTION. (*Plate V. Fig.* 23.)

1. With the chord of 60° describe the primitive circle, through the centre P draw CP*e*, and *a*P*r* at right angles to it.

2. Draw the great circle CB*e*, making an angle of 48°.30′ (the supplement of C) with the primitive (P. 160.) and find its pole *p*. (N. 159.)

3. With the semi-tangent of the ∠A=51°.30′ and centre P, describe the small circle *oso*.

4. Through *e* and *p* draw *epm*; make *mw*=59°.16′, the ∠B; draw *ew* cutting *ar* in *n*; bisect *nn* in *v*; with *v* as a centre, and radius *vn*, draw the small circle *nsn*, cutting *oso* in *s*.

5. Through *s* and P draw *bs* P*b*, and AP*d* at right angles to it; through A and *s* draw A*sx*, make *xy* an arc of 90°, and draw A*y* cutting *bs* P*b* in *t*.

6. Through the three points A*td* draw a great circle. Then ABC is the triangle required.

To measure the required parts.

7. AC=80°.19′; BC=63°.50′ and AB=120°.47′. (C. 163.)

BY CALCULATION.

To find the side AC, *by Rule I.*

Cosec $\angle$ C - - = 131°.30′ - ·12554 } reject indices.
Cosec $\angle$ A - - = 51°.30′ - ·10646 }
Cos $\frac{1}{2}$ (A + B + C) - = 121°. 8′ - 9·71352
Cos $\frac{1}{2}$ (A + B + C) − B = 61°.52′ - 9·67350

2)19·61902

Sine $\frac{1}{2}$ AC = 40°.9′.36″ 9·80951
2

80°.19′.12″ = AC.

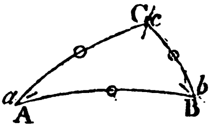
By the same rule the other sides may be found.

OR, *To find the side* BC, *by Rule II.*

180° − $\angle$ A = *bc*; 180° − $\angle$ B = *ac*; and 180° − $\angle$ C = *ab*.

Cosec *ab* - - = 48°.30′ ·12554 } reject indices.
Cosec *ac* - - = 120°.44′ ·06573 }
Sine $\frac{1}{2}$(*bc* + *ac* + *ab*) = 148°.52′ 9·71352
Sine $\frac{1}{2}$(*bc* + *ac* + *ab*) − *bc* = 20°.22′ 9·54161

2)19·44640



Cos $\frac{1}{2}$ $\angle$ *a* = 58°.5′ 9·72320
2

$\angle$ *a* = 116°.10′ its suppt. = 63°.50′ = BC.

To find the side AB, *by Rule II.*

Cosec $\frac{1}{2}$ (*bc* + *ac* + *ab*) - = 148°.52′ ·28648 } reject indices.
Cosec $\frac{1}{2}$ (*bc* + *ac* + *ab*) − *ab* = 100°.22′ ·00715 }
Sine $\frac{1}{2}$ (*bc* + *ac* + *ab*) − *bc* = 20°.22′ 9·54161
Sine $\frac{1}{2}$ (*bc* + *ac* + *ab*) − *ac* = 28°. 8′ 9·67350

2)19·50874

Tang $\frac{1}{2}$ $\angle$ *c* = 29°.35′.50″ - 9·75437
2

$\angle$ *c* = 59°.11′.40″ its suppt. = 120°.48′.20″ = AB.

PRACTICAL EXAMPLES.

1. In the oblique spherical triangle ABC.

Given	The ∠A=126°.37′	*Answer.*	AC= 57°.30′.
	The ∠B= 48°.30′		BC=115°.20′.
	The ∠C= 61°.41′		AB= 82°.28′.

Required the sides.

2. In the oblique spherical triangle ABC.

Given	The ∠A= 34°.15′. 3″	*Answer.*	AC=50°.10′.30″.
	The ∠B= 42°.15′.13″		BC=40°. 0′.10″.
	The ∠C=121°.36′.20″		AB=76°.35′.36″.

Required the sides.

3. In the oblique spherical triangle ABC.

Given	The ∠A= 36°. 8′	*Answer.*	AC=30°.
	The ∠B= 46°.19′		BC=24°.4′.
	The ∠C=104°		AB=42°.9′.

Required the sides.

CHAP. X.

ASTRONOMICAL DEFINITIONS AND INTRODUCTORY PROBLEMS.

1. *Astronomical Definitions.*

(A) *The Celestial Sphere*, is that apparent concave in which the sun, moon, stars, and all the heavenly bodies seem to be situated.

(B) *The axis* of the celestial sphere is an imaginary line passing through the centre of the earth, about which all the heavenly bodies appear to have a diurnal revolution.*

(C) *The poles* of the celestial sphere are the extremities of its axis, the one called the north pole, the other the south pole.†

---

* Although the earth's *real* motion on its axis from west to east, is the cause of day and night; and its motion in its orbit, or path round the sun, is the cause of the variation of the seasons of the year: yet as all appearances and places of the celestial bodies will be the same, whether the earth moves and the celestial sphere is at rest; or the earth is at rest, and the celestial sphere in motion; astronomers, for the ease of calculation, assume the earth as a point at rest in the centre of the celestial sphere, and ascribe to the heavenly bodies that motion which they appear to have to a spectator on the earth.

† The polar star is a star of the second magnitude, near the north pole, in the tail of the little bear. Its mean right ascension, for the beginning of the year 1820, was 14°.13′.7″, and its declination 88°.20′.55″ north. *Connaissance des Tems* for 1820, page 168.

In the Nautical Almanac for 1820 the north polar distance of the polar star is stated to be 1°.39′.44″.50 for the year 1818, and its annual variation—19″.45.

(D) *The equinoctial* is a great circle which divides the heavens into two hemispheres, the northern and southern; it is called the equinoctial, because, when the sun appears in it, the days and nights all over the world are equal, viz. 12 hours each. This happens twice in the year, about the 21st of March and the 23d of September; the former is called the *vernal* equinox, the latter the *autumnal* equinox.

(E) *The ecliptic* is a great circle in which the sun makes his apparent annual progress; it cuts the equinoctial in an angle of 23°.28′[*], called the obliquity of the ecliptic; and the points of intersection are called the equinoctial points.

The ecliptic is divided into twelve equal parts called signs, each sign contains 30 degrees. Their names and characters are as follow:

♈	Aries	♋	Cancer	♎	Libra	♑	Capricornus
♉	Taurus	♌	Leo	♏	Scorpio	♒	Aquarius
♊	Gemini	♍	Virgo	♐	Sagittarius	♓	Pisces.

The first six signs lie on the north of the equinoctial, and are called northern signs, the six following lie on the south side of the equinoctial, and are called southern signs. The sun continues about a month in one of these signs, and goes through nearly a degree in a day.

(F) *The Zodiac* is a space which extends about 8 degrees on each side of the ecliptic, like a belt or girdle, within which the motions of all the planets are performed.

(G) *The Nodes* are the points where the orbits or paths of the planets round the sun intersect the ecliptic. That where the planet ascends from the south towards the north of the ecliptic is called the north or ascending node, and is marked thus ☊; the other the south or descending node, and is marked thus ☋. The names and characters of the planets are as follow:

☉	The Sun	☽	The Moon	⚳	Ceres	♄	Saturn
☿	Mercury	♂	Mars	⚴	Pallas	♅	Herschel,
♀	Venus	⚶	Vesta	♃	Jupiter		or Georgian.
⊕	The Earth	⚵	Juno				

When two planets are referred to the same point of the ecliptic, they are said to be in *conjunction;* and those that are referred to opposite points of the ecliptic are said to be in *opposition*, or 180 degrees apart. If they be three signs or 90° distant, they are in a *quartile* aspect. If two sines or 60 degrees, a *sextile* aspect. The astronomical marks are as follow:

---

[*] The angle which the ecliptic makes with the equinoctial is a variable quantity, and is equal to half the difference between the greatest and least meridian altitude of the sun at any place, supposing the sun to have the greatest declination when on the meridian.

☌ Conjunction when planets are in the same point of the ecliptic.
⚹ Sextile when 2 Signs dist.
□ Quartile when 3 Signs dist.
△ Trine when 4 Signs dist.
☍ Opposition when 6 Signs dist.

The conjunction and opposition are called the *syzygies*, and the quartile aspects the *quadratures*; these terms are applied chiefly to the moon.

(H) *The horizon* is a great circle which separates the visible half of the heavens from the invisible.

This horizon is distinguished by the *sensible* and *rational* horizon, when applied to the earth. The *sensible horizon* is the boundary of the spectator's view at sea or land; and a plane parallel to this circle, passing through the earth's centre, is called the rational horizon.

(I) *The cardinal points* are the *east*, *west*, *north* and *south*, points of the horizon. The mariner's compass, which is divided into 32 points, each 11°.15′, (F. 74.) is a representation of the horizon.

(K) *The Zenith* is a point in the celestial sphere directly over the head of the spectator, being the elevated pole of the horizon.

(L) *The Nadir* is a point in the celestial sphere directly under the feet of the spectator, and is diametrically opposite to the zenith; being the depressed pole of the horizon.

(M) *Azimuth*, or *vertical circles*, are great circles passing through the zenith and nadir. They cut the horizon at right angles. The altitudes of the heavenly bodies are measured on these circles.

(N) *The prime vertical* is that azimuth circle which passes through the east and west points in the horizon.

(O) *Meridians* are great circles passing through the poles of the world, and cutting the equinoctial at right angles. They are also called hour circles; and upon the terrestrial sphere, circles of longitude.

(P) *Circles of celestial longitude* are great circles passing through the poles of the ecliptic, and cutting it at right angles.

(Q) *The latitude* of any object in the heavens, is an arc of a circle of longitude contained between the centre of that object and the ecliptic.

(R) *The latitude* of any place on the earth, is the elevation of the pole above the horizon, and the *complement* of the latitude, is the distance of the pole from the zenith. Or the latitude is the distance of the zenith of the place from the equinoctial, on the celestial sphere.

(S) *The declination* of any celestial object, is an arc of a meridian contained between the centre of that object and the equinoctial.

(T) *Parallels of declination* are small circles parallel to the equinoctial.

(U) *The altitude* of any object in the heavens, is an arc of an azimuth or vertical circle, contained between the centre of the object and the horizon.

(W) *Parallels of altitude* are small circles parallel to the horizon.

(X) *Parallels of celestial latitude* are small circles parallel to the ecliptic.

(Y) *The tropics* are small circles parallel to the equinoctial, at 23°.28′ from it, and touch the ecliptic in the points of cancer and capricorn; they are the limits of the sun's progress to the north and south of the equinoctial.

(Z) *The zenith distance* of any celestial object is the arc of a vertical circle, contained between the centre of that object and the zenith, being the complement of the altitude.

(A) *The polar distance* of any object in the heavens, is an arc of a meridian contained between the centre of that object and the pole of the equinoctial.

(B) *The amplitude* of any celestial object is an arc of the horizon, contained between the centre of the object when rising or setting, and the east or west point of the horizon.

(C) *The azimuth* of any object in the heavens, is an arc of the horizon, contained between an azimuth or vertical circle, (passing through the object,) and the north or south point of the horizon.

(D) *The right ascension* of an object, is the distance between the point aries and a meridian passing through the object, reckoned on the equinoctial. It is so called, because, in a right sphere, this meridian will coincide with the horizon when the object is rising. Or, we may define it to be the angle at the pole, formed between a meridian passing through aries, and a meridian passing through the object.

(E) *The oblique ascension* of an object, is the distance of the equinoctial point aries from the horizon when the object is rising. Or, it is that degree of the equinoctial which rises with the object in an oblique sphere.

(F) *The oblique descension* is the distance of the point aries from the horizon when the object is setting. Or, it is that degree of the equinoctial which sets with the object in an oblique sphere.

(G) *The ascensional*, or *descensional difference*, is the difference between the right and oblique ascension or descension, and with respect to the sun, it is the time he rises before six, when his declination is of the same name as the latitude, or sets before six, when the declination and latitude have contrary names.

(H) *The equinoctial colure* is a great circle passing through the pole and the equinoctial points aries and libra.

(I) *The solstitial colure* is a great circle passing through the pole and the points ♋ and ♑; called solstitial points, because when the sun is near these points he seems to have nearly the same altitude at noon, for several days, and therefore apparently stops or stands still.

(K) *The arctic circle* is a parallel of declination at the distance of 23°.28′ from the north pole, or 66°.32′ from the equinoctial. It is generally called the north polar circle.

(L) *The antarctic circle*, called likewise the south polar circle, is the same distance from the south pole as the arctic circle is from the north pole.

(M) *Apparent noon*, the time when the sun comes to the meridian, or 12 o'clock, as shewn by a sun-dial.

(N) *True*, or *mean noon*, twelve o'clock as shewn by a well regulated chronometer, so adjusted as to go 24 hours in a mean solar day.*

(O) *The equation of time* at noon, is the interval between the true and apparent noon.

(P) *A sidereal year* is the interval of time from the sun's leaving any fixed star till he returns to it again, and consists of 365d. 6h. 9m. 12sec. of mean solar time.

(Q) *A tropical* or *solar year* is the interval of time from the sun's leaving one tropic, or equinox, till he returns to it again, and consists of 365d. 5h. 48m. 48sec. of mean solar time.

---

* A *mean solar day* is a period not marked out by any observable phenomena, but an artificial interval of time. The time elapsed from the sun's leaving the meridian on any day till it returns to the same meridian the next day is called a *true solar day*, and is subject to a continual variation, arising from the obliquity of the ecliptic, and the unequal motion of the earth in its orbit.

A clock or chronometer, therefore, which measures time by equal motion, cannot be so adjusted as to keep time exactly with the sun, or always to shew 12 o'clock when the sun is on the meridian; to correct these irregularities, the year is divided into as many imaginary days, each of 24 hours in length, as there are real days in the year measured by the sun's return to the meridian; one of these imaginary days is called a *mean solar day*, and a clock adjusted so as to go 24 hours in one of these days, is said to be regulated to *mean solar time*.

The year thus consists of as many *mean solar days* as *true solar days;* the clock being just as much before the sun, on some days of the year, as the sun is before the clock on others. The difference is given in page II. of the Nautical Almanac for every day in the year. The time shewn by the clock is called *true* or *mean* time, and the time shewn by the sun is called *apparent* time.

If a clock be adjusted to go 24 hours, from the passage of any fixed star over the meridian till it returns to it again, its rate of going at any time may be determined by comparing it with the transit of that fixed star. A clock thus regulated is said to be adjusted to *sidereal* time. Here nature affords a standard exceeding in exactness any imitation that can be produced by art, there is no irregularity in the earth's diurnal motion, its diurnal revolution on its axis being uniformly performed in 24 hours of sidereal time = 23h. 56m. 4sec. of mean solar time.

(R) *Nonagesimal* degree of the ecliptic, is that point which is the most elevated above the horizon; and is measured by the angle which the ecliptic makes with the horizon at any elevation of the pole; or, it is the distance between the zenith, and the pole of the ecliptic. This angle is frequently used in the calculation of solar eclipses.

(S) *The medium Cœli*, or *mid-heaven*, is that point of the ecliptic which culminates, or is on the meridian at any given time.

(T) *The Crepusculum*, or *twilight*, is that faint light which we perceive before the sun rises, and after he sets. It is produced by the rays of light being refracted in their passage through the earth's atmosphere, and reflected from the different particles thereof.

(U) *A constellation* is a collection of stars on the surface of the celestial sphere, circumscribed by the outlines of some assumed figure, as a *ram*, a *dragon*, a *bear*, &c.

This division is necessary, in order to direct a person to any part of the heavens, where any particular star is situated.

(W) *The diurnal and nocturnal arcs.* In all places of the earth, except the two poles, the horizon cuts the equinoctial into two equal parts. In all places situated on the equator, the horizon cuts all the parallels of declination into two equal parts, and here the sun and all the stars are 12 hours above the horizon, and 12 hours below. In places between the equator and the elevated pole, the parallels of declination are unequally divided; the greater arc being above the horizon, and the less arc below. In all places between the equator and the depressed pole, the parallels of declination are unequally divided; the greater arc being below the horizon, and the less arc above.

In all cases, the arcs which are above the horizon are called diurnal arcs, and those below, nocturnal arcs. Or, the parallel, which the sun, moon, or stars, describe from their rising to setting, is called the *diurnal* arc; and that parallel which each of them describes, from the setting to the rising, is called the *nocturnal* arc.

II. *Introductory Astronomical Problems.*[*]

PROBLEM I.

(X) *To turn degrees, or parts of the equator into time.*

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[*] These are the same as in the former editions, being extracted from the general examples, and from the notes upon them. Those which depend upon the Nautical Almanac have been recalculated and adapted to the year 1822.

Rule. Multiply the number of degrees by 4, and the product will be the corresponding time.

Note. Seconds multiplied by 4 produce thirds of time.
Minutes multiplied by 4 produce seconds of time.
Degrees multiplied by 4 produce minutes of time.

EXAMPLE.

Turn $25^\circ.15'.16''$ of the equator into time.

$$
\begin{array}{r}
25^\circ.\ 15'.\ 16''. \\
4 \\
\hline
\textit{Answer.}\ 1^h.\ 41'.\ 1''.\ 4'''. \\
\hline
\end{array}
$$

Also, $77^\circ.2'.10''$ of longitude$=5^h.8'.8''.40'''$ of time, and $124^\circ.16'.30''$ of the equator$=8^h.17'.6''$ of time.

PROBLEM II.

(Y) *To turn time into degrees.*

Rule. Multiply the hours by 60, and add the odd minutes, if any, to the product, one-fourth of which will be degrees; multiply the remainder by 60, and add the odd seconds, if any, to the product, one-fourth of which will be minutes, &c.

EXAMPLE.

Find the number of degrees, &c. corresponding to $1^h.41'.1''.4'''$.

$$
\begin{array}{l}
1^h.\ 41'.\ 1''.\ 4'''. \\
\quad 60 \\
\hline
4)101 \\
\hline
25^\circ \text{——} 1\ \textit{rem.} \\
\qquad 60 \\
\hline
\qquad 4)61 \\
\hline
\qquad 15' \text{——} 1\ \textit{rem.} \\
\qquad\qquad 60 \\
\hline
\qquad\qquad 4)64 \\
\hline
\qquad\qquad 16''
\end{array}
$$

Answer $25^\circ.15'.16''$

Also, $3^h.4'.28''$ of time$=46^\circ.7'$ of longitude, and $8_h.17'.6''$ of time$=124^\circ.16'.30''$ of longitude.

PROBLEM III.

(Z) *Given the time under any known meridian to find the corresponding time at Greenwich.*[*]

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[*] Since the earth makes one revolution on its axis from west to east in 24 hours, the sun must *apparently* make one revolution round the earth from east to west in

RULE. Turn the longitude of the place under the known meridian into time (X. 265.): add this time to the time at the given place if the longitude be west, or subtract it if east, and the sum or remainder will be the time at Greenwich. If the sum exceed 24 hours, subtract 24 hours from it, the remainder will shew the time at Greenwich on the *following* day: if the longitude, when turned into time, cannot be subtracted from the time at the given place, add 24 hours to the time at the given place before you subtract, the remainder will shew the time on the *preceding* day.

EXAMPLE I.

Find the time at Greenwich, on the 12th of August, when it is $7^h.25'$ at a place in longitude 97°.45′ west.

Time at the given place $7^h.25'$
Long. 97°.45′, in time - - = 6 .31 W.

Time at Greenwich - - 13 . 56. or 56 minutes past 1 in the morning on the 13th of August.*

EXAMPLE II.

Find the time at Greenwich, on the 1st of May, when it is $22^h.40'$ at a place in longitude 160° W.

Time at the given place $22^h.40'$.
Long. 160°, in time =10 .40 W.

Sum 33 .20
24

Time at Greenwich 9.20, on May 2d.

EXAMPLE III.

Find the time at Greenwich, on the 8th of April, when it is 16 .26′ at a place in longitude 98°.45′ East.

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the same time. Now, the longitudes of all places on the earth are reckoned on the equator, which is divided into 360 degrees, and the whole of it passes the sun in 24 hours; it follows that every 15° of motion is *one* hour in time, every degree 4 minutes, &c. (as in Prob. I. and II.) Hence, a place one degree eastward of Greenwich will have noon, and every hour of the day, *four* minutes sooner than at Greenwich; and a place one degree westward of Greenwich will have noon, and every hour of the day, *four* minutes later.

* The astronomical day begins at noon, and is counted forward to 24 hours, or the succeeding noon, when the next day begins, being 12 hours later than the civil day, which commences at the preceding midnight; thus August 12th, at $13^h.56'$ astronomical time, is August 13th at $1^h.56'$ in the morning, according to civil reckoning.

Time at the given place 16h. 26′
Long. 98°.45′, in time = 6 . 35 East

Time at Greenwich 9 . 51 on the 8th of April.

EXAMPLE IV.

Find the time at Greenwich, on the 4th of June, when it is 5h. 26′ at a place in longitude 120° East.

Time at the given place + 24h. = 29h.26′
Long. 120°, in time - - - - = 8 .— E.

Time at Greenwich - - - - 21 .26, on the 3d of June.

PRACTICAL EXAMPLES.

1. What Greenwich time answers to noon at a place in 60° East longitude?

Answer. 20 hours, on the preceding day.

2. What Greenwich time answers to noon at a place in longitude 60° West?

Answer. 4 hours.

3. Find the time at Greenwich when it is 19h.42′ at a place in 28°.30′ E. longitude.

Answer. 17h.48′.

PROBLEM IV.

(A) *Given the time at Greenwich to find the corresponding time under any known meridian.*

RULE. Turn the longitude of the place under the known meridian into time (X. 265.): add this time to the time at Greenwich if the longitude be east, or subtract it if west, and the sum or remainder will be the time under the known meridian. If the sum exceed 24 hours, subtract 24 hours from it, the remainder will shew the time at the given meridian on the *following* day: if the longitude, when turned into time, cannot be subtracted from the given time at Greenwich, add 24 hours to the time at Greenwich before you subtract, the remainder will shew the time on the *preceding* day.

EXAMPLE I.

When it is 9h.51′ at Greenwich, on the 8th of April, what hour is it in longitude 98°.45′ East?

Time at Greenwich 9h.51′
Long. 98°.45′, in time - = 6 . 35 E.

Time in long. 98°.45′ E. = 16 . 26, on April 8th.

EXAMPLE II.

When it is 21^{h}.26′ at Greenwich, on the 3d of June, what hour is it at a place in 120° East longitude?

Time at Greenwich - 21^{h}.26′
Longitude 120°, in time = 8 . — East.

Sum 29 . 26
24 .

Time in longitude 120° E. = 5 . 26, on June 4th.

EXAMPLE III.

When it is 13^{h}. 56′ at Greenwich, on the 12th of August, find what hour it is at a place in longitude 97°.45′ West.

Time at Greenwich 13^{h}. 56′
Longitude 97°.45′, in time = 6 . 31 W.

Time in longitude 97°.45′ W. = 7 . 25, on the 12th of August.

EXAMPLE IV.

When it is 9^{h}.20′ at Greenwich, on the 2d of May, what hour is it at a place in longitude 160° West?

Time at Greenwich - - +24^{h}. = 33^{h}. 20′
Longitude 160°, in time - - - - = 10 . 40 W.

Time in longitude 160° W. - - = 22 . 40, on the 1st of May, or 40 minutes past 10 in the morning on the 30th of April.

PRACTICAL EXAMPLES.

1. When may an emersion of the first satellite of Jupiter be observed at Bombay, in longitude 72°.54′.30″ E. which, by the Nautical Almanac, happens at Greenwich on the fourteenth of January 1822, at 6^{h}.23′.33″.
Answer. 11^{h}.15′.11″.

2. What is the expected time of the beginning of the Lunar eclipse, which happens on August 2d, 1822, at 10^{h}.51′.40″ at Greenwich, in longitude 76°.49′.30″ West?
Answer. 5.44′.22″.

PROBLEM V.

(B) *To reduce the declination of the sun, as given in the Nautical Almanac, to any other meridian, and to any given time of the day.*

RULE. The corresponding time at Greenwich being ascertained (Z. 266.), find the change of the sun's declination in 24 hours from the Nautical Almanac: Then, 24 hours : this change : : the time from noon at Greenwich : the variation of the sun's declination in that time.

This variation must be added to the sun's declination at noon*, or subtracted from it, according as the declination is increasing or decreasing.

NOTE. By a similar process the change of the sun's longitude, or of right ascension, may be determined for any given time, or at any given place; and also the declination of a planet.

EXAMPLE I.

Required the sun's declination at noon, on the 12th of October 1822, at Glasgow, longitude 4°.15′ W.

First, 4°.15′=17minutes, the time by which the clocks at Glasgow are *slower* than at Greenwich; hence when it is noon at Glasgow, it is 0^{h}·17′ at Greenwich.

⊙'s declination at noon October 12th, Naut. Alm. is 7°.17′.34″
⊙'s declination at noon October 13th, Naut. Alm. is 7°.40′. 8″

Increase of declination in 24 hours - - 22′.34″

Then 24^{h} : 22′.34″ : : 17′ : 16″ the increase of the sun's declination in 17 minutes of time; consequently when it is noon at Glasgow, the sun's declination is (7°.17′.34″ + 16″=)7°.17′.50″ South.

EXAMPLE II.

What is the sun's right ascension, June 5th 1822, at 13^{h}.48′. in longitude 63°.10′ E.?

Time at the given place - - - 13^{h}. 48′.
Longitude 63°.10′, in time - - = 4 . 12 . 40″ E.

Time at Greenwich - - - - 9 . 35 . 20

⊙'s right ascension at noon June 5th, Naut. Alm. is 4^{h}.51′.21″.7
⊙'s right ascension at noon June 6th, Naut. Alm. is 4^{h}.55′ 28″.6

Increase of right ascension in 24 hours - - 4′. 6″.9

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[*] The sun's longitude, right ascension in time, and declination are given, in the IId page of the Nautical Almanac, for every day in the year, at noon, calculated for the meridian of Greenwich.

Then 24^h : 4′.6″.9 : : 9^h.35′.20″ : 1′.38″.6
☉'s right ascension at noon, June 5th, - - 4^h.51′.21″.7
Variation of the right ascension, add 1.38.6

☉'s right ascension at 13^h.48′ in long. 63°.10′ E. = 4^h.53′. 0″.3

PRACTICAL EXAMPLES.

1. Required the sun's declination on the 25th of August 1822, at 8^h.20′, in longitude 48° West.

The declination at Greenwich, at noon, (by the *Nautical Almanac*) being 10°.53′.35″ N. and on the 26th, 10°.32′.50″ N.

Answer. 10°.43′.37″. N.

2. Required the sun's right ascension at noon, on the 25th of May 1822, in longitude 124° East.

The right ascension at Greenwich (*Naut. Alm.*) being 4^h.6′.31″, and on the 24th of May, 4^h.2′.29″.

Answer. 4^h.5′.29″.

3. Required the sun's declination January 24th, 1822, at 18^h.40′ in longitude 132° East.

The declination at Greenwich, at noon, (*Naut. Alm.*) being 19°.16′.13″ South, and on the 25th, 19°.1′.38″ S.

Answer. 19°.10′.14″ S.

4. Required the sun's right ascension on the 16th January 1822, at 18^h.48′, in longitude 68° West.

The right ascension at Greenwich, at noon, (*Naut. Alm.*) being 19^h.51′.30″, and on the 17th, 19^h.55′.47″.

Answer. 19^h. 54′.2″.

PROBLEM VI.

(C) *To reduce the declination of the moon, as given in the Nautical Almanac, to any other meridian, and to any given time of the day.*

RULE. Find the time at Greenwich corresponding to the time at the given place. (Z. 266.) Take the change of the moon's declination in 12 hours from the Nautical Almanac. Then, 12 hours : this change : : the time at Greenwich : the variation of the moon's declination in that time. This variation must be added to the moon's declination (at noon or midnight) if the declination be increasing, or subtracted if the declination be decreasing.

NOTE. By a similar process the change of the moon's right ascension*, semidiameter, and horizontal parallax may be ascertained for any given time, or at any given place.

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* The moon's age and time of passage over the meridian of Greenwich are

EXAMPLE I.

1. Required the moon's declination March 17th 1822, at $7^h.22'$ in longitude 57° W.

The moon's declination at noon (*Naut. Alm.*) at Greenwich, being $26°.12'$ S. and at midnight $24°.58'$ S.

Time at the given place - - $7^h.22'$
Longitude 57° W. in time - $= 3\,.\,48'$ W.

Time at Greenwich - $= 11^h.\,10'$

☽'s declination at noon* $26°.12'$ S.
☽'s declination at midnight $24°.58'$ S.

Decrease in 12 hours - $= 1°.14'$

$12^h : 1°.\,14' :: 11^h.\,10' : 1°.\,9'$ variation of the ☽'s declination in $11^h.\,10'$.

Hence $26°.12' - 1°.9' = 25°.3'$ S. the moon's declination at the time and place required.

EXAMPLE II.

Required the moon's semidiameter and horizontal parallax on the 26th of January 1822, in longitude $11°.45'$ West, at $15^h.45'$ apparent time.

The semidiameter, at Greenwich, at midnight, (*Naut. Alm.*) being $16'.1''$ and at noon on the 27th, $16'.3''$, also the horizontal parallax at the same time $58'.32''$, and $58'.41''$.

Time at the given place - $15^h.45'$
Longitude $11°.45'$ West, in time - - $=$ 0 47 W.

Time at Greenwich - - - $= 16^h.32$, or $4^h.32'$ past midnight.

☽'s semi-dia. at midnight, 26th, $= 15'.57''$; hor. paral. - - $= 58'.32''$
☽'s semi-dia. at noon, - 27th, $= 15'.59''$; hor. paral. - - $= 58'.41''$

Increase in 12 hours, - - $= 0'.2''$; increase in 12 hours - $0'.9''$

$12^h : 2'' :: 4^h.32' :$ - - $0''.$
☽'s semi-dia. at midnight $= 15'.57''$.

☽'s ditto in long. $11°.45'$ W. $= 15'.57''.7$

$12_h : 9'' :: 4^h.32' : 3''4$
hor. par. at midnight - $= 58'\,32''$

hor. par. in long. $11°.45'$W. $= 58'.35''.4$

Hence the ☽'s semi-diameter at $15^h.45'$ in long. $11°.45'$ W. is $15'.57''.7$, and the horizontal parallax is $58'.35''.4$.

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given in the VIth page of the Nautical Almanac: and her latitude, longitude, right ascension, declination, semidiameter, and horizontal parallax, are given for noon and midnight at Greenwich in pages Vth, VIth, and VIIth, for each month.

* If the time at Greenwich had exceeded 12 hours, the moon's declination must have been taken out for midnight and the noon of the next day; and the variation applied to the midnight declination.

PRACTICAL EXAMPLES.

1. Required the moon's declination on the 11th of January, 1822, at $17^h.47'$ in longitude 162° West?

The moon's declination at noon at Greenwich (*Naut. Alm.*) on the 12th of January, being $1°.19'$ N. and at midnight $1°.35'$ S.

Answer. The time at Greenwich is $4^h.35'$ on the 12th of January, and the moon's declination is $0°.13'$ North.

2. Required the moon's semidiameter and horizontal parallax on the 19th of May, 1822, in longitude $38°.40'$ E. at $11^h.15'$ apparent time.

The moon's semidiameter at Greenwich (*Naut. Alm.*) at noon and midnight being $16'.38''$ and $16'.41''$; and the horizontal parallax at the same time $61'.3''$ and $61'.13''$.

Answer. The time at Greenwich is $8^h.\ 40'.20''$, ☽'s semidiameter$=16'.40''$ and horizontal parallax$=61'.10''$.

3. Required the moon's declination on the 13th of May, 1822, at 19^h in longitude 67° East.

The moon's declination at Greenwich (*Naut. Alm.*) at midnight being $15°.5'$ S. and at noon on the 14th $12°.27'$ S.

Answer. The time at Greenwich is $14^h.32'$, ☽'s declination $14°.32'$ S.

4. What is the moon's declination on July 19th, 1822, at $4^h.49'$ in longitude 114° East?

The moon's declination at Greenwich (*Naut. Alm.*) at midnight on July 18th being $21°.12'$ N. and on July the 19th at noon $18°.51'$ N.

Answer. The time at Greenwich is $21^h.13'$ July 18th, and the moon's declination $19°.24'$ N.

PROBLEM VII.

(D) *To find the time of a star's culminating, or coming to the meridian of Greenwich.*

RULE. Subtract the right ascension of the sun for the given day from the right ascension of the star, and the remainder will be the time of the star's culminating nearly.—If the sun's right ascension exceed the star's, add 24 hours to the star's before you subtract.

Take the increase of the sun's right ascension in 24 hours, and add it to 24 hours. Then,

This sum is to 24 hours as the star's right ascension diminished by the sun's, is to the time of the star's culminating.

NOTE. If the time of culminating be required for any other meridian than that of Greenwich; allow 10 seconds* of time for every 15° of longitude, which subtract from the time at Greenwich for places in west longitude, or add to that time for places in east longitude, and the result will shew the time of culminating at the given meridian.

EXAMPLE I.

At what time will Arcturus come to the meridian of Greenwich on the 1st of December 1822? The right ascension of Arcturus being $14^h.7'.32''$.†

*'s right ascension (1822) + 24^h - - $=38^h.\ 7'.32''$.
⊙'s right ascension, December 1st, 1822 - $=16^h.28'.16''$.

Time of *'s culminating nearly - - $=21^h.39'.16''$.

⊙'s right ascension, December 1st, 1822, at noon $=16^h.28'.16''$.
⊙'s right ascension, December 2d, 1822, at noon $=16^h.32'.35''$.

Increase of the ⊙'s right ascension in 24 hours $=\ 0^h.\ 4'.19''$.

Then $24^h.\,4'.19'' : 24^h :: 21^h.39'.16'' : 21^h.35'.23''$‡ apparent time of Arcturus's culminating, or $9^h.35'.23''$, December 2d, in the morning.

NOTE. Arcturus will be on the meridian of Philadelphia, longitude 75°.13′ W, on the same day at $21^h.34'.33''$, Philadelphia time; for $15° : 10'' :: 75°.13' : 50''$ and $21^h.\,35'.\,23'' - 50'' = 21^h.34'.33''$. It will also be on the meridian of Pekin, longitude 116°.24′ E, on the same day at $21^h.36'.40''$, Pekin time; for $15° : 10'' :: 116°.24' : 77'' = 1'.17''$, and $21^h.35'.23'' + 1'.17'' = 21^h.36'.40''$. Hence it appears that the same fixed star is on the meridian of every place, in any kingdom of moderate extent, nearly at the same hour reckoned at that place.

PRACTICAL EXAMPLES.

1. At what time will *Aldebaran* culminate at Greenwich on the 20th November 1822?

The right ascension of *Aldebaran* being $4^h.25'.43''$ (Tab. VIII.)

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* For the sun's right ascension varies about four minutes of time every day, but the right ascension of a star remains nearly the same during the whole year, and $24^h : 4' :: 1^h\ (=15°) : 10''$.

† See the table annexed to the Nautical Almanac for the year 1822, or Table VIII. of this work.

‡ The same fixed star is on the meridian of any place at nearly the same hour, on the same day of the month, for several years; the variation in 40 years will seldom exceed two minutes of time.

the sun's right ascension at noon (*Naut. Alm.*) $15_h.41'.29''$ on November 20th, and on the 21st $15^h.45'.40''$.

Answer. $12^h.42'.1''$.[*]

2. At what hour will *Regulus* culminate at Greenwich, on the 6th of February 1822?

The right ascension of *Regulus* being $9^h.58'.53''$ (Tab. VIII.); ⊙'s right ascension Feb. 6th (*Naut. Alm.*) being $21^h.18'.40''$, and on the 7th $21^h.22'.40''$.

Answer. $12^h.38'.6''$.

3. At what time, on the 18th of December 1822, will *Sirius, the Dog Star*, culminate at Greenwich?

The right ascension of *Sirius* being $6^h.37'.18''$ (Tab. VIII.); ⊙'s right ascension 18th December, (*Naut. Alm.*) at noon, $17^h.42'.55''$, and on the 19th, $17^h.47'.21''$.

Answer. $12_h.52'$.

4. At what time, on the 1st of December 1822, will *Castor* culminate at Greenwich?

The right ascension of *Castor* being $7^h.23'.13''$ (Tab. VIII.); ⊙'s right ascension (*Naut. Alm.*) December 1st $= 16^h.28'.16''$, and on the 2d $= 16^h.32'.35''$.

Answer. $14^h.52'.16''$.

PROBLEM VIII.

(E) *To find the time of the moon, or any planet's culminating.*

RULE 1. Subtract the sun's right ascension at noon, from the right ascension of the planet, and the remainder is the time of culminating nearly.

If the sun's right ascension, in time, be greater than that of the planet, add 24 hours to the planet's right ascension before you subtract.

2. Find (from the Nautical Almanac) the daily variation of the sun's right ascension, and the daily variation of the planet's right ascension, and take their *difference* or *sum*. When the daily variation of the planet's[†] right ascension is greater than that of

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[*] The motion of clocks or watches may be examined, and their errors rectified, by the culminating of the stars. For instance, *Aldebaran* is on the meridian of Greenwich, November 20th, at $12^h.42'.1''$ *apparent* time, from which take the equation of time on the same day $14'.12''$, as directed in the IId page in the month in the Nautical Almanac, and the remainder $12^h.27'.49''$ is the true mean time which the clock ought to shew when the star is on the meridian.

[†] The right ascensions of the planets are not given in the Nautical Almanac, but they may be calculated from their geocentric latitudes and longitudes, which are inserted in that work, and the obliquity of the ecliptic. The time at which each of the planets passes the meridian of Greenwich is given in the IVth page of each month in the Nautical Almanac, and that of the moon in the VIth.

the sun's, its motion is *progressive*; and when less, its motion is *retrograde*.

Then, *for the moon*, or the *progressive* motion of a planet.

24 hours diminished by the aforesaid *difference* : 24^h : : the time of culminating nearly : the true time of culminating.

For the *retrograde* motion of a planet.

24 hours increased by the aforesaid *sum* : 24^h : : the time of culminating nearly : the true time of culminating.

NOTE. When the planet is *stationary*, the time of its passage over the meridian will evidently be determined in the same manner as that of a fixed star having the same right ascension with the planet.

EXAMPLE.

Required the time of the moon's culminating at Greenwich on the 13th of August 1822.

☽'s right ascension at noon 13th August 1822 = 94°.59′ = 6^h.19′.56″
☉'s right ascension at noon 13th August 1822 - = 9^h.30′. 4″

Time of culminating nearly - - - - 20^h.49′.52″

☉'s right ascen. Aug. 13th = 9^{h}30′. 4″
☉'s right ascen. Aug. 14th = 9^{h}33′.50″

Variation in 24 hours = 0^h. 3′.46″

☽'s right ascension Aug. 13th = 94°.59′
☽'s right ascension Aug. 14th = 110°.41′

Variation in 24 hours - = 15°.42′

From 15°.42′, in time = 1^h.2′.48″, take 3.′46″ and the remainder = 59′.2″. Then,

24^h − 59′.2″ : 24^h : : 20 .49′52″ : 31^h.43′.17″ true time of the moon's passing the meridian of Greenwich.

PRACTICAL EXAMPLES.

1 Required the time of the moon's culminating at Greenwich on the 14th of April 1822.

The sun's right ascension at noon (*Naut. Alm.*) being 1^h.28′.42″ and the moon's 293°.57′, and on the 15th 1^h.32′.23″ and 307°.0′.

Answer. 18^h.44′.57″.

3. Required the time of the moon's culminating at Greenwich on the 18th of October 1822.

The sun's right ascension at noon (*Naut. Alm.*) being 13^h.30′.56″, and the moon's 239°.49′, and on the 19th 13^h.34′.41″ and 252°.54′.

Answer. 2^h.33′.37″.

PROBLEM IX.

(F) *Given the observed altitude of a fixed star to find its true altitude.*

RULE. From the observed altitude, subtract the refraction (Table IV.) If the star be observed at sea, subtract the dip of the horizon (Table V.) *See* R. 94.

EXAMPLE I.

The observed altitude of *Spica Virginis* was 20°.39′.40″, the error* of the quadrant 19″ subtractive, and the height of the eye 18 feet above the level of the sea; required the correct altitude.

Observed altitude of *Spica Virginis*	20°.39′.40″
Refraction, Table IV. - - =	−2′.32″
	20°.37′. 8″
Dip for 18 feet, Table V. - =	−4′. 3″
	20°.33′.5″
Index error of the quadrant -	−19″
True altitude of the star - =	20°.32′.46″

Or, 20°.39′.40″−(2′.32″+4′.3″+19″)=20°.32′.46″. *Answer.*

2. Suppose the observed altitude of *Regulus* to be 45°.13′.15″, the height of the eye 14 feet above the level of the sea, and the error of the quadrant 5′.6″ additive, required the true altitude.

Answer. 45°.13′.50″.

PROBLEM X.

(G) *Given the observed altitude of the sun's lower or upper limb,*† *to find the true altitude of its centre.*

RULE. To the observed altitude apply the semidiameter (taken from page III. of the month in the Nautical Almanac) by addition or subtraction, according as the lower or upper limb has been observed; from this result subtract the refraction‡ (Table IV.), and then add the parallax in altitude§ (Table VI.). If the altitude be taken at sea, the dip of the horizon (Table V.) must be deducted, the last result will be the true altitude of the sun's centre.

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* Observations taken with a quadrant are liable to errors, arising from the bending and elasticity of the index, and the resistance it meets with in turning round its centre. These errors, though they cannot in all cases be avoided, may be pretty accurately allowed for by a correct observer.

† The sun's upper limb is the upper edge of its face, or the uppermost extremity of the vertical diameter; and the lower limb is the lower edge, or the lower extremity of the vertical diameter.

‡ S. 87. et seq. § S. 95.

EXAMPLE I.

On the 13th of March 1822, if the altitude of the sun's lower limb, observed at sea, be 18°.40′; required the true central altitude, the height of the eye being 22 feet.

Observed altitude of the ⊙'s lower limb = 18°.40′
⊙'s semidiameter, *Naut. Alm.* - - = + 16′. 6″

18°.56′. 6″
Refraction for 19° of altitude (Table IV.) = −2′.44″

18°.53′.22″
⊙'s parallax in altitude (Table VI.) = + 8″

18°.53′.30″
Dip for 22 feet (Table V.) - - = −4′.28″

True altitude of the ⊙'s centre - = 18°.49′. 2″

Or, (16′.6″+8″)−(2′.44″+4′.28″)=9′.2″ and 18°.40′+9′.2″ =18°.49′.2″. *Answer.*

Note. The index error, if any, of the quadrant must be applied to the observed altitude, previous to the other corrections.

2. On the 3d of September 1822, suppose at sea the altitude of the sun's upper limb to be 28°.31′.30″, the index error 40″ additive, height of the eye 12 feet, and the sun's semidiameter 15′.54″; what is the true altitude of the sun's centre?

Answer. 28°.11′.19″.

PROBLEM XI.

(H) *Given the observed altitude of the moon's lower or upper limb, to find the true altitude of its centre.*

Rule. Find the moon's semidiameter and horizontal parallax for the time and place of observation (C. 271.), and increase the semidiameter by the augmentation answering to the moon's altitude (Table VII.).

To the observed altitude* apply the augmented semidiameter by addition or subtraction, according as the lower or upper limb has been observed, and, if the observation has been made at sea, subtract the dip of the horizon (Table V.); the result will be the *apparent altitude* of the moon's centre.

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* Here, as in the preceding problem, the index error of the quadrant, if any, must be applied to the observed altitude, previous to the other corrections.

To the cosine of the moon's apparent altitude, add the logarithm of the horizontal parallax in seconds (found above), the sum, rejecting 10 from the index, will be the logarithm of the moon's parallax in altitude in seconds (T. 96.), from which take the refraction of the moon in altitude (Table IV.), and the remainder will be the *moon's correction*, which added to the apparent altitude will give the true altitude.

EXAMPLE I.

On the 18th of July 1822, in longitude 113°.10′ W. at 20^h.45′.30″, if the observed altitude of the moon's lower limb be 45°.22′.3″, error of the quadrant 58″ subtractive, and the eye 21 feet above the level of the sea; what is the true central altitude?

Time at the given place - -	20^h.45′.30″
Longitude 113°.10′ W. in time -	= 7^h.32′.40″ W.
	28^h.18′.10″
	24
Time at Greenwich on the 19th of July	= 4^h.18′.10″

☽'s semidiameter at noon, 19th = 16′.9″, Horizontal parallax - - 59′.15″
☽'s semidiameter at midnight, 19th = 16′.3″, Horizontal parallax - - 58′.54″

Difference 0′.6″ Difference 0′.21″

	12^h : 6″ :: 4^h.18′.10″ : 0′. 2″		12^h : 21″ :: 4^h.18′.10″ : 7″
☽'s semidiameter at noon -	16′. 9″	Horizontal parallax at noon -	59′.15″
☽'s semidiam. at 4^h.18′.10″	= 16′. 7″	Horizontal paral. at 4^h.18′.10″	= 59′ 8″
☽'s augmentation (Table VII.) =	11″		60
☽'s true semidiameter -	= 16′.18″	In seconds	= 3548
☽'s observed altitude -	= 45°.22′. 3″	Cos ☽'s app. alt. 45°.33′.1″	= 9·84911
Error of the quadrant -	= −58″	Horizontal parallax 3548″ log	= 3·54998
	45°.21′. 5″	Parallax in altitude 2507″ log	= 3.39909
Semidiameter - -	= +16′.18″		
	45°.37′.23″	Then 2507″ - -	= 41′.47″
Dip (Table V.) - -	= −4′.22″	☽'s refraction - -	= −0′.56″
App. altitude ☽'s centre	= 45°.33′.1″	☽'s correction - -	= 40′.51″

Lastly, 45°.33′.1″ + 40′.51″ = 46°.13′.52″, true altitude of the moon's centre.

2. On the 22d of April 1822, in longitude 105° E. of Greenwich, at 13^h.30′, if the observed altitude of the moon's upper

limb be 23°.48′, and the eye 18 feet above the level of the sea; required the moon's true central altitude.

The moon's semidiameter and horizontal parallax at noon (*Naut. Alm.*), being 16′.39″ and 61′.8″; and at midnight, 16′.38″ and 61′.4″.

Answer. The time at Greenwich is $6^{h}.30'$; ☽'s true semidiameter 16′.44″, horizontal parallax 61′.6″; apparent altitude of the ☽'s centre 23°.27′.13″; the ☽'s correction 53′.52″, and the true altitude of her centre 24°.21′.5″.

PROBLEM XII.

(I) *Given the sun's meridian altitude to find the latitude of the place of observation.*

Rule. Reduce the sun's declination to the meridian of the place of observation* (B. 270.)

Subtract the sun's corrected altitude (G. 277.) from 90°, the remainder is the zenith distance.

If the sun be south of the observer when the altitude is taken, call the zenith distance north; but, if north, call it south.

Then, if the zenith distance and declination have the same name, their sum is the latitude; but, if they have contrary names, their difference is the latitude, and it is always of the same name with the greater of the two quantities.

EXAMPLE I.

On the 17th of October 1822, in longitude 52°W., suppose the meridian altitude of the sun's lower limb to be 28°.40′, the observer at sea, the sun to the south of him, and the eye 14 feet above the surface of the water. Required the latitude of the place of observation?

Obser. alt. ⊙'s lower limb	28°.40′
Semidiameter - - =	+ 16′. 6″
	28°.56′. 6″
Refraction (Table IV.) =	− 1′.44″
	28°.54′.22″
⊙'s paral. in alt. (Tab. VI.)	+ 8″
	28°.54′.30″
Dip for 14 feet (Table V.) =	− 3′.44″
True altitude ⊙'s centre =	28°.50′.56″
Zenith distance - -	= 61°. 9′. 4 N.

Time at given place -	0^{h}. 0′
Longitude 52° w. -	= 3^{h}.28′
Time at Greenwich -	= 3^{h}.28′
⊙'s declination at noon	9°. 9′.20″s.
⊙'s declination 18th at noon	9°.31′.20″
Increase in 24 hours -	= 22′. 0″
24^{h}. : 22′ :: 3^{h}.28′ :	3′.10″
⊙'s decline. 17th at noon =	9°.9′.20″s.
⊙'s true declination - =	9°.12′.30″s.
Zenith distance -	= 61°. 9′ 4″N.
True latitude -	= 51°.56′.34″N.

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* The same rule, with a little variation, will answer for a planet; and for the moon, correcting for the declination as at C. 271. See *Dr. Mackay's* Navigation, 1st Edition, page 151 to 155.

2. On the 18th of March 1822, in longitude 106° E, suppose the meridian altitude of the sun's lower limb, when north of the observer, to be 74°.56′, and the height of the eye 16 feet above the level of the sea, required the latitude.

The sun's semidiameter (*Naut. Alm.*) being 16′.5″; the declination at noon on the 17th 1°.26′.59″ S., and on the 18th 1°.3′.17″ S.

Answer. The sun's true altitude = 75°.8′.3″; time at Greenwich $16^h.56'$ on the 17th, sun's corrected declination 1°.10′.19″S. latitude 16°.2′.16″ S.

3. On the 28th of September 1822, suppose the meridian altitude of *Arcturus*[*] to be 36°.18′ north of the observer, its declination 20°.6′.50″ N, and the eye 20 feet above the level of the sea, required the latitude.

Answer. 33°.40′.44″ S.

CHAP. XI.

THE APPLICATION OF RIGHT-ANGLED SPHERICAL TRIANGLES TO ASTRONOMICAL PROBLEMS.

(K) The celestial sphere is represented by Plate III. fig. 1.

1. Let the circular space, *South*, *Zenith*, *North*, *Nadir*, represent the brazen meridian[†] of a celestial, or terrestrial globe, having its north pole elevated above the horizon.

2. Imagine the globe to be cut in halves by the brass meridian, and the semi-globe to be of transparent glass with the circles of the sphere drawn on it. Now if a sheet of paper be put upon the section, and a light be placed in the point aries (the eye being in libra), the shadows of all the most useful circles of the sphere, will form a plane figure similar to Figure I. Plate III.

3. NAS will present the axis of the globe, N the north pole, S the south pole.

4. ÆAQ the equator.

5. HO the horizon.

6. *Zenith* A the quadrant of altitude screwed on the zenith, and passing through aries. Or, *Zenith* A *Nadir*, the prime vertical passing through aries A.

7. ♑ ⊙ A ⊙ ♋ the ecliptic, *n* its north pole, *m* its south pole.

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[*] See the note on D. 273.

[†] It is here presumed that the learner has some knowledge of the globes, and of the circles described thereon. It would be a very good exercise to solve the succeeding problems both by the globes and by calculation.

8. ♋ ♋ the tropic of cancer, or any parallel of the sun's north declination.

9. ♑ ♑ the tropic of capricorn, or any parallel of the sun's south declination.

10. ON the elevation of the north pole above the horizon, equal to Æ *zenith*, the latitude of the place.

11. HÆ the elevation of the equator above the horizon, equal to *zenith* N, the complement of the latitude.

12. T⊙W a parallel circle 18 degrees below the horizon, or the boundary between twilight and dark night.

13. *Zenith* SEC, zenith G*d*, &c. azimuth, or vertical circles.

14. NSDS, NG⊙CWS, &c. meridians or circles of terrestrial longitude.

15. *nvxm*, *ncvm*, &c. circles of celestial longitude.

NOTE. The several triangles in this general figure, and the lines which form them, are explained at the head of each problem, to which the triangle is applied.

(L) PROBLEM I. (*Plate III. Fig.* 1.)

Given the obliquity of the ecliptic and the sun's longitude, to find his right ascension and declination.

In the right-angled spherical triangle AG⊙, or AB⊙.

1. A⊙ = sun's longitude, an arc of the ecliptic from aries.

2. AG, or AB = sun's right ascension, an arc of the equinoctial; the point B always comes to the meridian with the sun.

3. G⊙, or B⊙ = sun's declination, an arc of a meridian passing through the sun's place. This arc likewise comes to the meridian with the sun.

4. GA⊙, or BA⊙ = the obliquity of the ecliptic, or the angle formed between the equinoctial and the ecliptic.

5. A⊙G, or A⊙B = the angle formed by the ecliptic, and the meridian passing through the sun's place.

Any two of these *five* quantities being given, the rest may be found.

EXAMPLE.

On the 17th of May 1822, when the sun's longitude is $1^{s}.25^{\circ}.57'.25''$; required his declination, right ascension, and the angle formed between the ecliptic and the meridian passing through the sun's place. The obliquity of the ecliptic being $23^{\circ}.28'$.

1. *To find the sun's declination, in the triangle* AB⊙.

$$\text{Rad} \times \text{sine } \overset{*}{\text{B}}\odot = \text{sine BA}\odot \times \text{sine A}\odot.$$

$$\because \; (\text{log sine BA}\odot + \text{log sine A}\odot) - 10 = \text{log sine B}\odot.$$

or, Rad : sine $\text{A}\odot = 55°.57'.25''$ :: sine $\text{BA}\odot = 23°.28'$: sine $\text{B}\odot = 19°.16'$.

2. *To find the sun's right ascension, in the triangle* $\text{AB}\odot$.

$$\text{Rad} \times \cos \text{BA}\odot = \cot \text{A}\odot \times \overset{*}{\text{tang}}\ \text{AB}.$$

$$\because (10 + \log \cos \text{BA}\odot) - \log \cot \text{A}\odot = \log \text{tang AB}.$$

or, Cot $\text{A}\odot = 55°.57'.25''$: rad :: cos $\text{BA}\odot = 23°.28'$: tang $\text{AB} = 53°.37'.40'' = 3^h.34'.30''.40'''$.

3. *To find the angle* $\text{A}\odot\text{B}$ *between the ecliptic and meridian*[†].

$$\text{Rad} \times \cos \text{A}\odot = \cot \text{BA}\odot \times \cot \overset{*}{\text{A}\odot\text{B}}.$$

$$\because (10 + \log \cos \text{A}\odot) - \log \cot \text{BA}\odot = \log \cot \text{A}\odot\text{B}.$$

or, Cot $\text{BA}\odot = 23°.28'$: rad :: cos $\text{A}\odot = 55°.57'.25''$: cot $\text{A}\odot\text{B} = 76°.20'.23''$.

(M) Astronomers reckon the sun's longitude and right ascension from aries quite round the globe. Hence, though at equal distances from the equinoctial points, aries and libra, the sun may have the same quantity of declination and of the same name, viz. from aries to libra, *north;* and from libra to aries, *south;* yet the longitudes and right ascensions differ materially at these points where the declinations agree. Hence,

(N) If the sun be in the second quadrant of the ecliptic, subtract the longitude from 180°, use the remainder to find the right ascension; and take the right ascension, when found, from 180°.

(O) If the sun be in the third quadrant, subtract 180° from his longitude, use the remainder to find his right ascension; and add 180° to the right ascension when found.

(P) If the sun be in the fourth quadrant of the ecliptic, subtract his longitude from 360°, use the remainder to find the sun's right ascension; and take the right ascension, when found, from 360°.

(Q) In the first quadrant of the ecliptic between the 20th March and 21st June, the sun's declination is north and increasing; and in the third quadrant between the 22d September and 21st December, the sun's declination is south and increasing.—In the second quadrant of the ecliptic from 21st June to 22d September, the sun's declination is north and decreasing; and in the fourth quadrant from 21st December to the 20th March, the sun's declination is south and decreasing.

PRACTICAL EXAMPLES.

1. On the 17th of May 1822, the obliquity of the ecliptic $= 23°.27'.53''$, the sun's declination $= 19°.15'.57''$ N. and increas-

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[†] The finding of this angle is of no other use than as an exercise of right-angled spherical triangles.

ing; required his longitude, right ascension, and the angle formed between the ecliptic and the meridian passing through the sun's place.

The sun's longitude =55°.57′.37″, or 1^{s}.25°.57′.37″.
Ans. The sun's right ascension = 53°.38′, or 3^{h}.34′.32″.
The angle A⊙B =76°.20′.

The answers here would have been ambiguous, had it not been shewn that the sun was in the first quadrant of the ecliptic.

(R) This problem is useful for calculating tables of the sun's longitude. For the latitude of the place being accurately determined, and a quadrant fixed in the plane of the meridian, the observation of the sun's meridian altitude gives his declination, or distance from the equinoctial; whence his longitude for any day is readily obtained.

2. On the 5th of August 1822, the obliquity of the ecliptic = 23°.27′.53″, the sun's right ascension = 134°.54′; required the sun's longitude, declination, and the angle formed between the ecliptic and the sun's meridian.

The sun's longitude = ♌ 12°.25′.53″, or 4^{s}.12°.25′.53″.
Ans. Declination = 17°.5′.28″.
The angle A⊙B = 73°.40′.34″.

The sun's right ascension being greater than 90°, and less than 180°, he is in the second quadrant of the ecliptic.

3. On the 10th of November 1822, the sun's right ascension = 225°.5′.16″, his declination 17°.5′.18″ S. required his longitude, the obliquity of the ecliptic, and the angle formed by the ecliptic and the sun's meridian.

The sun's longitude = ♏ 47°.33′.26″, or 7^{s}.17°.38′.26″.
Ans. Obliquity of the ecliptic = 23°.27′.53″.
The angle A⊙B = 73°.40′.19″.

The sun's right ascension being greater than 180°, and less than 270°, he is in the third quadrant of the ecliptic.

4. On the 1st of February 1822, the sun's longitude is ♒ 12°. 8′.59″, or 10^{s}.12°.8′.59″, his declination = 17°.10′.15″ S.; required his right ascension, the obliquity of the ecliptic, and the angle formed by the ecliptic and the sun's meridian.

The sun's right ascension = 314°.37′, or 20^{h}.58′.28″.
Ans. Obliquity of the ecliptic = 23°.27′.55″.
The angle A⊙B = 73°.46′.

PROBLEM II. (*Plate III. Fig.* 1.)

(S) *Given the latitude of the place, and the sun's declina-*

tion†, *to find his amplitude, ascensional difference, and the time of his rising and setting.*

In the right-angled spherical triangle ABS or AGC.

1. AB or AG = the sun's, or star's, ascensional difference.
2. BS or GC = the sun's, or star's, declination.
3. AS or AC = the amplitude.
4. BAS, measured by the arc QO; or CAG, measured by the arc HÆ, = the complement of the latitude.
5. ASB or ACB = the angle formed between the horizon and the meridian passing through the sun.

Any two of the above five quantities being given, the rest may be found.

(T) *The same things may be found from the right-angled triangle* SON *or* CHS.

For, in the triangle SON, SO is the complement of AS; SN is the complement of BS; ON is the complement of the arc QO, which measures the angle BAS; and the angle SNO, measured by the arc BQ, is the complement of the ascensional difference. In the same manner the triangle CHS may be compared with the triangle AGC.

EXAMPLE I.

Given the latitude of London 51°.32′ N. on the longest day when the sun has 23°.28′ N. declination; required the sun's amplitude, ascensional difference, time of rising, setting, the length of the day and night, &c.

1. *In the triangle* ABS *to find the amplitude* AS.

Rad × sine BS = sine BAS × sine AS.

∴ (10 + log sine BS) − log sine BAS = log sine AS.

or, Sine BAS = 38°.28′ : rad : : sine BS = 23°.28′ : sine AS = 39°.48′.

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† This problem may be applied to the rising or setting of a planet, or of a star. But where great exactness is required, the declination of the sun or planet must be calculated as near to the time of rising and setting as possible; especially for the moon on account of her swift and irregular motion. Likewise, the declination of the sun near the equinoxes alters considerably in the compass of an hour. Hence what is generally called a parallel of the sun's declination, and drawn as such in the projection of the sphere, is not strictly a parallel circle, but a spiral line.

When the latitude of the place and the declination of the object have the same name, the right ascension diminished by the ascensional difference leaves the oblique ascension, and increased by the same gives the oblique descension.

But when the latitude of the place and declination have contrary names; the right ascension increased by the ascensional difference gives the oblique ascension; and being diminished by the ascensional difference leaves the oblique descension.

viz. cosine of the latitude, is to radius; as the sine of the sun's declination is to the sine of the amplitude.

(U) This part of the problem is useful in navigation, for finding the variation of the compass.—It is evident, by comparing the triangle ABS with the triangle AGC, that the amplitude is always of the same name with the declination, whether north or south.

2. *To find the ascensional difference* AB.

$$\text{Rad} \times \text{sine AB} = \text{cot BAS} \times \text{tang BS}.$$

$$\because (\text{log cot BAS} + \text{log tang BS}) - 10 = \text{log sine AB}.$$

or, Rad : cot BAS $= 38°.28'$: : tang. BS $= 23°.28'$: sine AB $= 33°.7'$.
viz. Radius, is to the tangent of the latitude; as tangent of the sun's declination, is to the sine of the ascensional difference.

(W) By comparing the triangle ABS with the triangle AGC, it is evident that when the sun's declination and the latitude of the place are of the same name, the ascensional difference in time subtracted from six hours gives the time of rising, and added to six hours gives the time of sun-setting: but when the latitude of the place and the sun's declination are of contrary names, the ascensional difference added to six hours gives the time of sun-rising, and subtracted gives the time of setting.

In the present case the sun rises at $3^h.47'.32''$, and sets at $8^h.12'.28''$; double the time of rising gives the length of the night, and double the time of setting gives the length of the day.

(X) When the sun's declination is equal to, or greater than the complement of the latitude; viz. when Q ♋ is equal to or exceeds QO (a circumstance that can never happen in latitudes lower than 66°.32′) the parallel of declination ♋ ♋ will not cut the horizon HO, and consequently the sun will never set at these times. The same observations hold true with stars whose co-declination or polar distance CN, is equal to, or less than, the latitude of the place, or elevation of the pole ON, and in the same hemisphere.

EXAMPLE II.

Required the amplitude, time of rising, culminating, and setting of *Arcturus* at Greenwich, latitude $51°.28'.40''$ N. on the first of December 1822, its right ascension being $14^h.7'.32''$ †, and declination $20°.6'.50''$ N.

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† See Table VIII. at the end of the Book.

The sun's right ascension at noon December 1st (*Naut. Alm.*) being $16^h.28'.16''$, and on December 2d, $16^h.32'.35''$.

1. *To find the amplitude* AS, *in the triangle* ABS.

The star's declination being north, the triangle ABS must be used, wherein BAS = the co-lat. and BS the declination are given, to find AS.

Rad × sine BS = sine $\overset{*}{\text{AS}}$ × sine BAS.

∵ (10 + log sine BS) − log sine BAS = log sine AS.

or, Sine BAS = $38°.31'.20''$: rad :: sine BS = $20°.6'.50''$: sine AS = $33°.30'.50''$.

(Y) The amplitude is always of the same name as the declination; and since the variation of a star's declination is extremely small†, the same star may be considered as having constantly the same amplitude, or to rise and set in the same point of the horizon, during the whole year, in the same latitude.

2. *To find the ascensional difference* AB, *in the triangle* ABS.

Rad × sine $\overset{*}{\text{AB}}$ = cot BAS × tang. BS.

∵ (log cot BAS + log tang BS) − 10 = log sine AB.

or, Rad : cot BAS = $38°.31'.20''$ :: tang BS = $20°.6'.50''$: sine AB = $27°.23'.21''$.

Then $27°.23'.21'' \times 4 = 1^h.49'.33''$, and $6^h. + 1^h.49'.33'' = 7^h.49'.33''$ the arc BADÆ, or half the time of the star's continuance above the horizon‡.

3. *Find the time of the* * *'s culminating*, Prob. VII. (D. 273.)

Then, From the time of the *'s culm. (D. 273.) = $21^h.35'.23''$.
Take the semi-diurnal arc - - - = $7^h.49'.33''$.

Time of the *'s rising, December 1st - = $13^h.45'.50''$.

And, To the time of the *'s culminating (D. 273.) = $21^h.35'.23''$.
Add the semi-diurnal arc - - - - = $7^h.49'.33''$.

Sum $29^h.24'.56''$.

Then $29^h.24'.56''. - 24^h = 5^h.24'.56''$. time of setting, December 2d.

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† The amplitude of this star on the same day, in the year 1796, was $33°.45'.46''$, making a variation of only $14'.56''$ in its amplitude during 26 years. In the first edition of this work (which was published in the year 1801) the places of all the stars were corrected to the year 1796.

‡ When the declination of the star and the latitude of the place are of the same name, the ascensional difference added to 6 hours gives the semidiurnal arc; but when of contrary names, the ascensional difference subtracted from 6 hours, leaves the semidiurnal arc.

(Z) On account of the small change in the declinations of the stars, the same star, in any latitude, may be considered as having the same ascensional difference during the year; consequently the diurnal difference of the same star's rising, culminating, and setting, in the same latitude, is *nearly* equal to the diurnal difference of the sun's right ascension.

The sun's mean apparent daily motion is $59'.8''$ nearly, which is equal to $3'.56''.32'''$, the daily difference between the rising, southing, and setting of any fixed star, in the same latitude.

PRACTICAL EXAMPLES.

1. Given the sun's amplitude $=39°.48'$ N. his declination $=23°.28'$ N.; to find the latitude of the place, time of sun rising and setting, and the length of the day and night.

Answer.
The latitude $=51°.32'$ N.
Sun rises at $3^h.47'.32''$.
Sun sets at $8^h.12'.28''$.
Length of day $=16^h.24'.56''$ length of night $=7^h.35'.4''$.

2. Required at what time *Sirius, the Dog star*, will rise, culminate, and set at Greenwich lat. $51°.28'.40''$ N. on the 18th of December 1822.

The right ascension of *Sirius* being $6^h.37'.18''$, and declination $16°.28'.39''$ S; the sun's right ascension at noon (*Naut. Alm.*) being $17^h.42'.55''$, and on December 19th at noon $17^h.47'.21''$.

Answer.
Ascensional difference $=21°.48'.43''=1^h.27'.13''$.
Semi-diurnal arc $=4^h.32'.47''$.
Sirius will be on the meridian at $12_h.52'$.
——— rise at $8^h.19'.13''$. in the evening.
——— set at $5\ .24'.47''$ next morning.

Star's declination being south, the triangle AGC is used.

3. Required the amplitude, time of rising, setting, and culminating of *Aldebaran* at Greenwich on the 20th of November 1822.

The right ascension of *Aldebaran* being $=4^h.25'.43''$, declination $=16°.8'.36''$ N., the sun's right ascension at noon (*Naut. Alm.*) being $15^h.41'.29''$, and on the 21st November, at noon, $15^h.45'.41''$.

Answer.
Ascensional difference $=21°.19'.19''=1^h.25'.17''$.
Semi-diurnal arc $7^h.25'.17''$.
Aldebaran culminates at $12^h.42'.\ 1''$.
——— rises at $5^h.16'.44''$. evening.
——— sets at $8^h.\ 7'.18''$. next morning.
The amplitude $=26°.30'.53''$. towards the N.

4. Given the latitude $=51°.32'$ N., the sun's amplitude $=$

59°.48′ N. of the east; required the sun's declination, ascensional difference, time of rising, setting, &c.

Answer. {
Sun's declination = 23°.28′. N.
Ascensional diff. = 33°. 7′.
Sun rises at $3^h.47'.32''$.
Sun sets at $8^h.12'.48''$.

PROBLEM III. (*Plate III. Fig.* 1.)

(A) *The latitude of the place, and the sun's (or a star's) declination being given, to find the altitude and azimuth, &c. at six o'clock.*

In the right-angled spherical triangle ADS.

1. AD = the complement of DO the sun's or star's azimuth from the North, or elevated pole.
2. DS = the altitude at six o'clock.
3. AS = the declination.
4. SAD = NO the elevation of the pole, or latitude of the place.

Any two of the above quantities being given, the rest may be found.

The triangle S *Zenith* N is complemental to the triangle ADS; it therefore follows, that the several things concerned in the problem may be determined in this triangle, right-angled at N.

(B) As the meridian of a place, with respect to the sun, is called the 12 o'clock hour circle, so that circle NSAS which cuts it at right angles, being 6 hours from the meridian, is called the 6 o'clock hour circle.

The sun and stars are on the eastern half of this circle 6 hours before they come to the meridian, and on the western half 6 hours after they have passed the meridian.

EXAMPLE I.

In latitude 51°.32′ North, when the sun has 19°.39′ North declination; required his altitude and azimuth at 6 o'clock.

1. *To find the altitude* DS, *in the triangle* ADS.

$$\text{Rad} \times \text{sine } \overset{*}{\text{DS}} = \text{sine A} \times \text{sine AS}.$$

$$\because (\log \text{sine A} + \log \text{sine AS}) - 10 = \log \text{sine DS}.$$

$$\text{or, Rad} : \text{sine AS} = 19^\circ.39' :: \text{sine SAD} = 51^\circ.32' : \text{sine DS} = 15^\circ.16'.$$

2. *To find* AD *the co-azimuth, in the triangle* ADS.

$$\text{Rad} \times \cos \text{A} = \cot \text{AS} \times \overset{*}{\text{tang}}\ \text{AD}.$$

$$\because (10 + \log \cos \text{A}) - \log \cot \text{AS} = \log \text{tang AD}.$$

$$\text{or, Cot AS} = 19^\circ.39' : \text{rad} :: \cos \text{SAD} = 51^\circ.32' : \text{tang AD} =$$

$12^{\circ}.31'.23''$. Hence $90^{\circ}-12^{\circ}.31'.23''=77^{\circ}.28'.37''=$ OD the azimuth from the north.

EXAMPLE II.

At what time will *Arcturus* appear upon the six o'clock hour line at Greenwich, latitude $51^{\circ}.28'.40''$ N., and what will its altitude and azimuth be on the first of April 1822?

Its right ascension being $14^{h}.7'.32''$ and declination $20^{\circ}.6'.50''$ N. (Table VIII.), and the sun's right ascension on the first of April (*Naut. Alm.*) being $0^{h}.41'.11''$, and on the second of April $0^{h}.44'.49''$.

1. *To find the time of the star's culminating.*

By Problem VII. (D. 273.) *Arcturus* will culminate at $13^{h}.24'.19''$.

Consequently (B. 289.) it will be upon the six o'clock hour circle at $(13^{h}.24'.19''-6^{h}=)\ 7^{h}.24'.19''$ in the evening, in the eastern hemisphere; and at $(13^{h}.24'.19''+6^{h}=)\ 19^{h}.24'.19''$, or $7^{h}.24'.19''$ the next morning, in the western hemisphere.

2. *To find the altitude* DS, *in the triangle* ADS.

$$\text{Rad}\times\text{sine DS}=\text{sine SAD}\times\text{sine AS}.$$

$$\because (\text{log sine SAD}+\text{log sine AS})-10=\text{log sine DS},$$

or, Rad : sine $\text{AS}=20^{\circ}.6'.50''$ :: sine $\text{SAD}=51^{\circ}.28'.40''$: sine $\text{DS}=15^{\circ}.36'.25''$.

3. *To find the azimuth* AD *from the east, in the triangle* ADS.

$$\text{Rad}\times\cos\text{SAD}=\cot\text{AS}\times\text{tang AD}.$$

$$\because (10+\text{log cos SAD})-\text{log cot AS}=\text{log tang AD}.$$

or, Cot $\text{AS}=20^{\circ}.6'.50''$: rad :: cos $\text{SAD}=51^{\circ}.28'.40''$: tang $\text{AD}=12^{\circ}.50'.56''$. Hence $\text{OD}=77^{\circ}.9'.4''$ the azimuth from the north.

(C) On account of the small change in the right ascension and declination of a star, it may, without material error, be said to have the same altitude and azimuth every time it arrives at the six o'clock hour circle; and the difference of the times it arrives there, may be considered as equal to the diurnal difference of the sun's right ascension, viz. about four minutes. (Z. 288.)

PRACTICAL EXAMPLES.

1. At what time will *Aldebaran* appear upon the six o'clock hour circle at Greenwich, latitude $51^{\circ}.28'.40''$ N., and what will its altitude and azimuth be on the 20th November 1822?

The right ascension being $4^h.25'.43''$, and declination 16°.8′.36″ N. (Table VIII.); also, the sun's right ascension on the 20th November (*Naut. Alm.*) = $15^h.41'.29''$ and on the 21st $15^h.45'.40''$.

Answer. Aldebaran culminates at $12^h.42'$ (D. 273.), therefore it is on the 6 o'clock hour circle in the eastern hemisphere at $6^h.42'$ in the evening; and in the western hemisphere at $6^h.42'$ the next morning.

Its altitude at each time being 12°.32′.3″, and the azimuth 79°.46′.50″ from the north.

2. Required the sun's altitude and azimuth at 6 o'clock, in latitude 51°.32′ N. when his declination is 15°.28′ North?

Answer. Altitude = 11°.59′.17″, azimuth 80°.17′.14″ from the North.

3. Given the sun's declination 19°.39′ North, his altitude at 6 equal 15°.16′; required the azimuth and the latitude?

Answer. Azimuth = 77°.28′.37″ from the elevated pole, latitude = 51°.32′ North.

4. Given the sun's declination = 19°.39′ North, his azimuth at 6 in the morning N. 77°.28′.37″ E.; required his altitude and the latitude?

Answer. Altitude = 15°.16′, latitude = 51°.32′ North.

5. At what time will *Regulus* appear upon the six o'clock hour circle NSAS at Greenwich, latitude 51°.28′.40″ North, and what will its altitude and azimuth be on the 6th February 1822?

The right ascension being $9^h.58'.53''$ and declination 12°. 50′.1″ N. (Table VIII.), also the sun's right ascension at noon on the same day (*Naut. Alm.*) = $21^h.18'.41''$, and on the 7th February $21^h.22'.41''$?

Answer. *Regulus* culminates at $12^h.38'$ (D. 273.), is on the 6 o'clock hour circle, at $6^h.38'$ in the evening in the eastern hemisphere; and at $6^h.38'$ the next morning in the western hemisphere. Its altitude = 10°.1′.11″, and its azimuth = 81°. 55′.28″ from the north.

6. At what time will *Castor* appear upon the 6 o'clock hour circle at Greenwich, latitude 51°.28′.40″ N., and what will its altitude and azimuth be on the 1st of December 1822?

The right ascension being $7^h.23'.13''$ and declination 32°.16′ 10″ N. (Table VIII.), also the sun's right ascension on the first of December at Noon (*Naut. Alm.*) = $16^h.28'.16''$, and on the second $16^h.32'.35''$.

Answer. *Castor* culminates at $14^h.52'.16''$, (D. 273.) will be on the six o'clock hour circle at $8^h.52'.16''$ in the evening, in the eastern hemisphere, and at $8^h.52'.16''$ next morning in the western hemisphere; altitude = 24°.41′.21″, azimuth = 68°.31′. 55″ from the north.

(D) PROBLEM IV. (*Plate III. Fig. 1.*)

The latitude of a place, and the declination of the sun (or of a star) being given; to find the altitude, and the time when it will be due east and west.

In the right-angled spherical triangle AGS.

1. AG = the time from six o'clock when the sun is due east or west. Or it is the complement of the time from a star's culminating.

2. AS = the altitude of the sun, or a star, above the horizon, when on the prime vertical ZSAN.

3. GS = the sun's or star's declination, of the same name as the latitude.

4. The angle GAS = the latitude of the place; for the elevation HÆ of the equinoctial is always equal to the complement of the latitude.

Any two of these quantities being given, the rest may be found.

The triangle SN *Zenith*, right-angled at *Zenith*, is complemental to the triangle AGS.

EXAMPLE I.

In latitude 51°.32′ North, when the sun's declination is 19°.39′ North, required his altitude, and the time when he will appear due east or west?

1. *To find the altitude* AS *in the triangle* AGS.

$$\overset{*}{\text{Rad}} \times \text{sine GS} = \text{sine GAS} \times \text{sine AS}.$$

$$\because (10 + \log \text{sine GS}) - \log \text{sine GAS} = \log \text{sine AS}.$$

or, Sine GAS = 51°.32′ : sine GS = 19°.39′ : : rad : sine AS = 25°.26′.

2. *To find* AG *the hour from six, in the triangle* AGS.

$$\overset{*}{\text{Rad}} \times \text{sine AG} = \text{tang GS} \times \text{cot GAS}.$$

$$\because (\log \text{tang GS} + \log \text{cot GAS}) - 10 = \log \text{sine AG}.$$

or, Rad : tang GS = 19°.39′ : : cot GAS = 51°.32′ : sine AG = 16°.28′.48″.

Now 16°.28′.48″ = $1^h.5'.55''$ the time from six, hence the sun will be east at $7^h.5'.55''$ in the morning, or west at $4^h.54'.5''$ in the afternoon.

(E) As the declination increases, the altitude and hour from six increase, and when the declination vanishes, the sun appears in the horizon upon the prime vertical. When the latitude is equal to 90°, the sun's altitude upon the prime vertical is equal to his declination.

When the latitude of the place is less than the declination, the sun never appears on the prime vertical.

EXAMPLE II.

At what time will *Arcturus* appear upon the prime vertical, or be due east or west from Greenwich, latitude 51°.28′.40″ N., on the first of April 1822; and what will be its altitude?

Its right ascension being 14h.7′.32″, and declination = 20°.6′.50″ North (Table VIII.), the sun's right ascension on the same day at noon (*Naut. Alm.*) being = 0h.41′.11″, and on April 2d = 0h.44′.49″.

1. *To find the altitude* AS, *in the triangle* AGS.

$$\text{Rad} \times \text{sine GS} = \text{sine GAS} \times \text{sine AS}.$$

$$\because (10 + \log \text{sine GS}) - \log \text{sine GAS} = \log \text{sine AS}.$$

or, Sine GAS = 51°.28′.40″ : rad : : sine GS = 20°.6′.50″ : sine AS = 26°.4′.31″.

2. *To find* AG, *the complement of the time from the star's culminating, in the triangle* AGS.

$$\text{Rad} \times \text{sine AG} = \text{tang GS} \times \text{cot GAS}.$$

$$\because (\log \text{tang GS} + \log \text{cot GAS}) - 10 = \log \text{sine AG}.$$

or, Rad : tang GS = 20°.6′.50″ : : cot GAS = 51°.28′.40″ : sine AG = 16°.57′.

Hence 90° − 16°.57′ = 73°.3′, which reduced to time = 4h.52′.12″, the time from the star's culminating. The star comes to the meridian at 13h.24′.19″ (D. 273.); therefore 13h.24′.19″ − 4h.52′.12″ = 8h.32′.7″, the time in the evening when the star will appear due east; and 13h.24′.19″ + 4h.52′.12″ = 18h.16′.31″; or 6h.16′.31″ next morning, when the star will appear due west.

(F) The height of the same star upon the prime vertical in any place, is always nearly the same, for the reasons already assigned (C. 290.); and the difference of the times of its coming to the prime vertical, will be equal to the difference of the times of its culminating, which is nearly equal to the diurnal difference of the sun's right ascension.

PRACTICAL EXAMPLES.

1. The sun's declination being 19°.39′ North, and having 25°.26′ altitude upon the prime vertical; required the latitude of the place, and the hour of the day?

Answer. Latitude=51°.32′ N. time $7^{h}.5'.55''$ or $4^{h}.54'.5''$, according as the observation was made in the forenoon or afternoon.

2. At what time will *Aldebaran* appear due east or west at Greenwich, latitude 51°.28′.40″ N. on the 20th of November 1822, and what will its altitude be at that time?

Its right ascension being=$4^{h}.25'.48''$ and declination 16°.8′.36″ N. (Tab. VIII.), the sun's right ascension at noon 20th November (*Naut. Alm.*)=$15^{h}.41'.29''$, and on the 21st $15^{h}.45'.40''$.

Answer. Aldebaran culminates at $12^{h}.42'$ (D. 273.), is due east at $7^{h}.35'.17''$; due west at $17^{h}.48'.48''$, and its altitude= 20°.46′.3″. The arc AG=13°.19′.19″.

3. The sun's declination being 19°.39′ N. and he was observed to be due west at $4^{h}.54'$ in the afternoon; required his altitude, and the latitude of the place?

Answer. Altitude=25°.26′, latitude=51°.32′ N.

4. At what time will *Regulus* appear due east or west at Greenwich, latitude 51°.28′.40″ N., on the 6th of February 1822, and what will be its altitude at that time?

Its right ascension being = $9^{h}.58'.53''$, and declination 12°.50′.1″ N. (Tab. VIII.) the right ascension of the sun at noon on the 6th of February (*Naut. Alm.*) being=$21^{h}.18'.41''$, and on the 7th=$21^{h}.22'.41''$.

Answer. Regulus culminates at $12^{h}.38'.5''$, is due east at $7^{h}.19'.56''$, due west at $17^{h}.56'.14''$, and its altitude=16°.29′.37″. The arc AG=10°.27′.37″.

5. In latitude 51°.32′ North, the sun's altitude when on the prime vertical was 25°.26′; required his declination, and the hour of the day?

Answer. { Declination=19°.39′ North.
Time $7^{h}.5'.55''$, or $4^{h}.54'.5''$, according as the observation was made in the forenoon or afternoon.

PROBLEM V. (*Plate III. Fig.* 1.)

(G) *Given the latitude of the place, and the sun's altitude, when on the equinoctial, to find his azimuth and the hour of the day.*

In the right-angled spherical triangle A*d*G.

1. AG=the complement of ÆG the hour from noon, being an arc of the equinoctial, on which G represents the sun's place between noon and 6 o'clock.

2. A*d*=the complement of H*d* the sun's azimuth from the meridian, or south point of the horizon.

3. *d*G=the sun's altitude.

4. GAd = the complement of the latitude, being measured by the arc HÆ the elevation of the equator above the horizon.

Any two of these quantities being given, the rest may be found.

This triangle is complemental to the triangle GÆ *zenith*; for GZ is the complement of dG the altitude; ÆG is the hour from noon; ÆZ the latitude; and the angle ÆZG, measured by the arc Hd, is the sun's azimuth from the meridian.

EXAMPLE.

In latitude 51°.32′ North, when the sun was on the equinoctial, his altitude was 22°.15′; required his azimuth, and the hour of the day?

1. *To find* Ad *the complement of the azimuth, in the triangle* AdG.

$$\text{Rad} \times \text{sine } \overset{*}{\text{A}d} = \text{tang } d\text{G} \times \text{cot GA}d.$$

∵ (log tang dG + log cot GAd) − 10 = log sine Ad.
or, Rad : tang dG = 22°.15′ :: cot GAd = 38°.28′ : sine Ad = 30°.59′.

Hence the azimuth = 59°.1′ from the south towards the east or west, according as the time is before or after noon.

2. *To find* AG, *the hour from* 6, *in the triangle* AdG.

$$\text{Rad} \times \text{sine } d\text{G} = \text{sine } \overset{*}{\text{AG}} \times \text{sine GA}d.$$

∵ (10 + log sine dG) − log sine GAd = log sine AG.
or, Sine GAd = 38°.28′ : rad :: sine dG = 22°.15′ : sine AG = 37°.29′.45″, which reduced into time = 2^h.29′.59″; this added to, and subtracted from, 6 hours, gives 8^h.29′.59″, or 3^h.30′.1″, according as the sun is in the eastern or western semi-circle of the globe.

PRACTICAL EXAMPLES.

1. In latitude 51°.32′ N. when the sun has no declination, what is his altitude, and azimuth at three hours and a half from noon?

Answer. The altitude = 22°.15′. Azimuth = S. 59°.1′ E. or W.

2. At the time of the equinox the sun's altitude was found to be 22°.15′, and his azimuth S. 59° E.; required the hour of the day, and the latitude of the place?

Answer. Time 8^h.30′. Latitude 51°.32′ North.

3. In latitude 51°.32′ North, the sun being in the equinox, there is given the angle (AGd) which the vertical circle passing

through the sun forms with the equinoctial $=57^\circ.46'$; to find the hour of the day and the altitude and azimuth of the sun?

Answer. Time $8^h.29'.56''$, or $3^h.30'.4''$. Altitude $22^\circ.15'$; azimuth S. 59°. E. or W.

(H) PROBLEM VI. (*Plate III. Fig. 2.*)

The difference of longitudes between two places, both in one parallel of latitude, being given, to find the distance between them.

Before we give any examples to this problem, it will be necessary to remark, that all the meridians on the terrestrial globe are great circles meeting each other in the poles of the equator. Therefore the meridional distances of places vary as the latitudes; that is, the distance between the meridians PE and PW is greater on the equator WOQE (supposing WOQE to be a part of the equator) than it is on the parallel of latitude MN, and greater at MN than at *l*L, &c.

Longitude is always counted on the equator, where a degree is reckoned 60 geographical miles, hence a degree of longitude in the parallel MN must be less than 60 miles. This shows that what is called meridional distance in navigation, is always less than the longitude, except the ship sail on the equator.

Since all the meridians PW, PO, PQ, &c. cut the equator at right angles (I. 134.), they must necessarily cut all the parallels of latitude at right angles.

The arcs of the equator WO, OQ, &c. are the measures of the spherical angles WPO, OPQ, &c. at the pole (D. 183).

If W*m* or O*n* represent the latitude of two places on the same parallel, then *m*P or *n*P will be their co-latitudes.

The arcs WO and *mn* are similar (P. and R. 135.); hence *mr* : *mn* :: CW : WO, viz. CW : *mr* :: WO : *mn*. But *m*V is the sine of the latitude W*m*, *mr*$=$VC is its cosine, and CW is the radius of the sphere. Therefore,

Radius : cosine of *any latitude* :: the length of any arc on the equator : the length of a similar arc in *that latitude.*

PRACTICAL EXAMPLES.

1. Suppose a ship, in latitude 40° North, sail due east until her difference of longitude be 60°, required the distance sailed in that parallel?

Rad : cos 40° :: diff. long $=3600$ miles : the distance 2757·7 miles.

EXAMPLE II.

2. Suppose a ship, in latitude 40° north, sail due east 2757·7 miles, what is her difference of longitude?

Answer. 3600 miles.

(I) By the first of these examples, geographers have calculated tables showing how many miles answer to a degree of longitude for each degree of latitude; thus in latitude 45°, a degree of longitude is 42·43 miles; in the latitude of London 51°.32′, the length of a degree is 37·32 miles, &c.

3. If a ship sail 2757·7 miles directly eastward or westward, so as to alter her longitude 60°, what latitude is she in?

Answer. 40° north or south.

4. Suppose a ship to sail from latitude 40° North to latitude 10° North, till her difference of longitude be 60°, and it be required to find her departure, course, and distance sailed.

Here *l*L the parallel of 40° is the latitude left, A*a* the parallel of 10' is the latitude arrived at, and MN the parallel of 25° is the middle latitude between them; hence MP is the complement of the middle latitude, and MN is the meridional distance, or the true departure nearly*; BW is a quadrant or 90°, and WE is the difference of longitude.

(K) Hence the following CONSTRUCTION (*Plate III. Fig.* 3.):

1. With the chord of 60°, which is equal to the sine of 90°, and centre P, describe the arc WE, on which set off the difference of longitude (60°=3600 miles); draw PW and PE.

2. With the sine of (65°) the complement of the middle latitude, taken from the line of sines on Gunter's scale, describe the arc MN, and draw the straight line MN.

3. Set off the difference of latitude (30°=1800 miles) from P to *m*; draw *mn* perpendicular to P*m*, and equal to MN; lastly, draw P*n*, and the construction is finished. The several lines are named as in the figure.

(L) OR THUS.

Make the angle MPN (Plate III. fig. 4.)=the complement of the middle latitude, and PN the difference of longitude, then MN will be the departure or meridional distance; for, in the plane triangle MPN, *per* Plane Trigonometry, radius : PN :: sine∠MPN : MN, the same proportion as is used in the first example. The triangle *mpn* is the same as before.

(M) OR THUS. (*Plate III. Fig.* 5.)

Draw the meridian AP, in which assume any point P; make the angle at P equal to the complement of the middle latitude, in the opposite quadrant to which the ship sails; set off PN equal to the difference of longitude, and draw MN perpendicular to

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* See the Theory of Navigation, Book IV. Chapter II.

AP; lastly, set off the difference of latitude from M to A, and draw AN; then the names of the several lines are as expressed in the figure.

BY CALCULATION. (*Plate III. Fig.* 3, 4, *or* 5.)

Rad : diff. long : : sine comp. midd. lat. : merid. dist. or departure = 3262·7.

Diff. lat : rad : : departure : tang course = 61°.7′.

Sine course : dep. : : rad : distance = 37126·2.

CHAP. XII.

THE APPLICATION OF OBLIQUE ANGLED SPHERICAL TRIANGLES TO ASTRONOMICAL PROBLEMS.

PROBLEM VII. (*Plate III. Fig.* 1.)

(N) *Given the sun's declination, and the latitude of the place, to find the apparent time of day-break in the morning, and the end of twilight in the evening.*

1. In the oblique-angled spherical triangles ⊙ *zenith* N.

1. ⊙N = the sun's distance from the north pole.
2. ⊙*zenith* = the sun's distance from the zenith.
3. *Zenith* N = the complement of the latitude.
4. ⊙N *zenith*, measured by the arc *a*Æ = the time from noon.

OR, 2. In the oblique-angled spherical triangle ⊙ N*adir* S.

1. ⊙ S = the sun's distance from the south pole.
2. ⊙ N*adir* = the sun's distance from the nadir.
3. S N*adir* = the complement of the latitude.
4. ⊙ S N*adir*, measured by the arc *a*Q, = the time from midnight.

EXAMPLE.

Given the latitude of the place 51°.32′ N., and the sun's declination 10° N., to find the time of day-break in the morning, and the end of twilight in the evening.

In the triangle ⊙ *zenith* N.

⊙N = 80° the sun's distance from the north pole = 90° − 10°.

⊙ *zenith* = 108° the sun's distance from the zenith = 18° + 90°.

Zenith N = 38°.28′ the complement of the latitude.

To find the angle *zenith* N⊙, measured by the arc *ax*, = the time from noon.

⊙N = - 80°	Cosec ZN = 38°.28′ - ·20617	} reject
⊙*zenith* = 108°	Cosec⊙N = 80°. 0′ - ·00665	} indices.
Zenith N = 38°.28′	Sine - - 113°.14′ 9·96327	
———	Sine - - 5°.14′ 8·96005	
2 | 226. 28	———	
———	2 | 19·13614	
Half sum - 113. 14	———	
⊙*Zenith* = 108. 0	Cosine 68°.17′.29″ 9·56807	
———	2	
Remainder - 5. 14	———	
———		

∠ ⊙NZ = $136°.34'.58'' = 9^{h}.6'.20''$, time from noon* when the sun is 18° below the horizon. Consequently the day breaks at $2^{h}.53'.40''$ in the morning, and twilight ends at $9^{h}.6'.20''$ in the evening, supposing the sun's declination to undergo no change between the beginning of twilight in the morning, and the ending thereof at night, being about 18 hours.

The same things might have been found from the triangle ⊙ S *Nadir*, for S⊙ = 90° + 10° = 100°, *Nadir*⊙ = 180° − 108° = 72°, and *Nadir* S = comp. lat. = 38°.28′. Then by the method above find the angle ⊙ S *Nadir* (measured by the arc *aQ*) = $43°.25'.12'' = 2^{h}.53'.40''$ as before, the time from midnight, when the sun is 18° below the horizon.

(O) When the declination of the sun, the latitude and declination being of the same name, is greater than the difference between the complement of latitude and 18°, the parallel of declination (♋ SSS ♋) will not cut the parallel of 18° (T⊙W) below the horizon: consequently there will be no real night at these times, but constant day or twilight, as is the case at London from the 22d of May to the 21st of July.

(P) Since the sun sets more obliquely at some times of the year than at others, it necessarily follows that he will be longer in descending 18° below the horizon at one season than at another.

When the sun is on the same side of the equinoctial as the visible pole, the duration of twilight will constantly increase as he approaches that pole, till he enters the tropic, at which time the duration of twilight will be the longest. It will then decrease till some time after the sun passes the equinox, but will increase again before he arrives at the other tropic; therefore, there must be a point between the tropics, where the duration of twilight is the shortest, which point may be found by the following proportion; viz.

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* The ∠ ⊙ NZ represents the time from *apparent noon*, that is, from the time the ⊙ comes to the meridian, and not the time from 12 o'clock, as shown by a well regulated clock or time-piece. If the equation of time, given in the second page of each month in the Nautical Almanac, be applied to this *apparent time* by addition or subtraction, as there directed, you will obtain the *true* time, or that shown by a clock.

Rad : tang 9° :: sine of the latitude : sine of the sun's declination.*

The declination must always be of a contrary name with the latitude.

PRACTICAL EXAMPLES.

1. Given the sun's declination 10° south, and the latitude of the place $51^\circ.32'$ N. to find the time of day-break in the morning, and the end of twilight in the evening?

Answer. The $\angle\odot$ s *nadir* $=73^\circ.35'.34''=4^h.54'.22''$ the time from midnight. Hence, day breaks at $4^h.54'.22''$ in the morning, and twilight ends at $7^h.5'.38''$ in the evening; admitting the sun's declination to be constant for one day.

2. Given the sun's declination $23^\circ.28'$ S. and the latitude of the place $51^\circ.32'$ N. to find the time of day-break in the morning, and the end of twilight in the evening?

Answer. The $\angle\odot$ s *nadir* $=90^\circ.16'=6^h.1'.4''$ the time from midnight when the $\odot$ is 18° below the horizon. Consequently day breaks at $5^h.58'.56''$ and twilight ends at $6^h.1'.4''$, on the shortest day at London.

3. In the latitude of London, $51^\circ.32'$ N., what time does the day break when the sun has $11^\circ.30'$ N. declination?

Answer. $2^h.41'$, rejecting seconds.

4. Required the beginning and ending of twilight at London, lat. $51^\circ.32'$ N., when the sun's declination is $15^\circ.12'$ N.?

Answer. $2^h.4'.12''$ and $9^h.55'.48''$.

5. Required the time the twilight begins and ends at London, latitude $51^\circ.32'$, when the sun has $11^\circ.30'$ South declination?

Answer. $5^h.2'$ and $6^h.58'$, rejecting seconds.

6. At London, latitude $51^\circ.32'$ N., required the sun's declination, day of the month, and duration of twilight when it is the shortest?

Answer. Sun's declination $=7^\circ.7'.25''$ south, answering to

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* Vide L'Hospit. Infinim. Petit. sect. iii. prop. 13. Dr. Gregory's Astron. page 323. Heath's Royal Astron. page 225. Emerson's Miscellanies, page 492. Vince's Astronomy, vol. i. page 18, &c. Also, Leybourn's edition of the Ladies' Diary, vol. iv. from page 314 to 339, wherein the subject is amply discussed.

March 2d and October 11th; between which days, the twilight increases, and from the latter to the former it decreases. The duration of twilight is 1h.56'.32"; this is found by taking the difference between the time of sun-rise and day-break.

PROBLEM VIII. (*Plate III. Fig.* 6.)

(Q) *Given the day of the month, the latitude of the place, the horizontal refraction, and the sun's horizontal parallax, to find the apparent time of his centre appearing in the Eastern and Western part of the horizon.*

Example. Given the sun's declination at noon on the 21st of June 1822=23°.28'; the latitude of the place=51°.32' N.; the horizontal refraction=33'; the sun's horizontal parallax =9"; to find the apparent time of his rising and setting.

This example being given for the longest day, the sun's declination may be considered the same at his rising as at noon; for the declination does not vary above 5" in 24 hours at this time, though near the equinoxes it varies about 1' in an hour.

Let s be the true point of the sun's rising, and *b* that of his apparent[*] rising, then *b*⊙=33'−9"=32'.51" the distance of the sun's centre below the horizon; hence the true zenith distance zs will be increased, by refraction, to z⊙, but the declination will remain the same, for NS=N⊙.

In the oblique-angled triangle ZN⊙.

z⊙=90°.32'.51" app. dist. of ⊙'s centre from the zenith.
N⊙=66°.32'. 0" app. dist. of ⊙'s centre from the north pole.
ZN = 38°.28'. 0" the complement of the latitude.

2)195.32.51	ZN=38°.28' cosec - -	·20617
	N⊙=66°.32' cosec - -	·03749
½ Sum 97.46.25½	Half sum=97°.46'.25½" sine	9·99599
z⊙=90.32.51	Remainder=7°.13'.34½" sine	9·09964
Rem. 7.13.34½		2)19·33929
	Cosine - 62°.8'.15" -	9·66964
	2	

∠ ZN⊙=124.16.30=8h.17'.6" the apparent time from noon when the sun's centre appears in the western part of the horizon. Hence the apparent time of

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[*] The apparent time of rising and setting of the heavenly bodies always differs from the true time, for they are elevated by refraction (S. 87, &c.), and depressed by parallax (S. 95, &c.).

All the heavenly bodies, when in the horizon, appear 33' above their true plac the[*] sun's rising is $3^h.42'.54''$, and the apparent time of setting is $8^h.17'.6''$. The true time of the sun's rising and setting, on this day, has been determined (Prob. II. W. 286.) to be $3^h.47'.32''$ and $8^h.12'.28''$; hence the apparent day is $9'.16''$ longer than the astronomical day.

(R) As the refraction causes an error in the rising and setting of the celestial objects, so it will cause an error in the amplitudes, as may be seen by comparing the triangle ABS with the triangle AD*b*. If the sun's amplitude be taken when his centre is in the visible horizon, an allowance depending on the horizontal refraction, parallax, height of the eye, and latitude of the place, should be applied to the observed amplitude, in order to obtain the true amplitude. This inconvenience may be avoided, by observing the amplitude when the altitude of the sun's lower limb is equal to $16'+$ the dip of the horizon.

PRACTICAL EXAMPLES.

1. Required the apparent time of the sun's rising and setting at Glasgow, latitude 55°.32' N., longitude 4°.15' W., on the 12th of October 1822?

The sun's declination at noon, for the meridian of Greenwich (*Naut. Alm.*), being 7°.17'.34" S. and on the 13th 7°.40'.8" S.

Answer. The sun's declination, when it is noon at Glasgow, is 7°.17'.50" S. (B. 270.). The ☉'s declination being south, the triangle ZN☉ will fall on the left hand of the prime vertical ZA. Then

$$\left.\begin{array}{l} \mathrm{Z}\odot = 90^\circ.32'.51'' \\ \mathrm{N}\odot = 97^\circ.17'.50'' \\ \mathrm{ZN} = 34^\circ.28' \end{array}\right\}$$

With these three sides find the hour $\angle \mathrm{ZN}\odot = 80^\circ.14'.3'' = 5^h.20'.57''$. the time from noon when the sun sets, hence the sun rises at $6^h.39'.3''$.

Call these the *approximating* times of rising and setting; calculate the sun's declination to the approximating time of rising and setting, which you will find to be 7°.12'.49" S., and

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by the effect of refraction (see Table IV.); but they are depressed by parallax. Hence, by these causes combined, a body becomes visible when its distance from the zenith is equal to $90^\circ + 33' -$ horizontal parallax, and as the sun's horizontal parallax is about 9" (see Table VI.) his centre will appear in the horizon when it is 32'.51" below it, or 90°.32'.51" from the zenith. If to this zenith distance, the sun's semi-diameter 15'.47" be added, the zenith distance of the centre for the setting of the upper limb will be had, if subtracted, that for the lower limb will be obtained; the former being 90°.48'.38" and the latter 90°.17'.4", and the respective times $8^h.19'.20''$ and $8^h.14'.52''$.

A star has no sensible parallax, and, therefore, will appear in the horizon when it is 33' below it, or 90°.33' from the zenith.

[*] The equation of time on this day at noon (see page II. of the Nautical Almanac) is 1'.15" at Greenwich; this applied as directed in the 2d page, and page 149 of the Nautical Almanac, will give the true time of the sun's rising and setting, as shown by a well-regulated clock or time-piece.

7°.22′.51″ South.; then, with the polar distance $N\odot = 97^\circ.12'.49''$ repeat the operation, and the *true apparent* time of rising will be $6^h.38'.33''$; with the polar distance $N\odot = 97^\circ.22'.51''$ repeat the operation a second time, and the *true apparent* time of the sun's setting will be $5^h.20'.27''$. To obtain the *true*, or *mean*, time of rising and setting, as shown by a well regulated clock, or time-piece, the equation of time must be applied as directed in page 2, and page 165 of the Nautical Almanac for 1822.

PRACTICAL EXAMPLES.

1. Required the apparent time of the rising and setting of the sun, in latitude 51°.32′ N. supposing his declination to be 23°.29′ S?

Answer. $8^h.8'.2''$ and $3^h.51'.58''$.

2. Required the apparent time of the rising and setting of the sun, in latitude 51°.32′ N. when his declination is 17°.32′ N. supposing it to undergo no change from sun-rise to sun-set?

Answer. $4^h.22'.16''$ and $7^h.37'.44''$.

3. Required the apparent time of the rising and setting of the sun in latitude 56° N., supposing the sun's declination at his rising to be 22°.34′.35″, and at his setting 22°.30′.7″?

Answer. $3^h.22'.20''$ and $8^h.37'$.

PROBLEM IX. (*Plate III. Fig.* 7.)

(S) *Given the latitude of the place, the day of the month, the moon's horizontal parallax and refraction, to find the time of her rising.*

Required the time of the moon's rising at Greenwich, latitude 51°.28′.40″ N., on the 4th of August 1822.

The horizontal parallax of the moon being from 53′ to 62′, always exceeds the horizontal refraction; therefore, when the moon's centre appears in the horizon it is really above it, by a quantity equal to the horizontal parallax minus the refraction.

The moon's declination must be found as near to the time of her rising* as possible, because it is subject to a considerable variation in the course of a few hours.

Let *n* be the place of the moon, when in the horizon, *m* the point where she becomes visible, and *e* her place on the meridian; then *de* will represent her change of declination from the time of her rising to the time of her transit.

Find the time of the moon's passage over the meridian (E. 275.), and calculate the declination to that time (B. 270.).

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* The moon's declination at any hour may be found by interpolation or second differences, see Naut. Alm. 1822, page 174; but in general it may be found sufficiently exact by common proportion.

With this declination, the latitude, &c. calculate the hour angle, as in the preceding problem, and call this the *estimated* time of the moon's rising; correct the moon's declination, and her horizontal parallax, to the estimated time of rising; repeat the operation; and correct the result for the daily retardation of the moon's transit.

SOLUTION.

1. *Find the time of the moon's passage over the meridian.*

The moon passes over the meridian at $13^h.32'.51''$ (by Prob. VIII. page 276.)

2. *To find the ☽'s declination when she is on the meridian.*

☽'s declin. at midnight (*Naut. Alm.*) 4th Aug. 1822 $= 7°.44'$ S.
☽'s declin. at noon ditto 5th Aug. 1822 $= 4°.44'$ S.

Decrease of declination in 12 hours - - - $= 3°.\ 0'$

$12^h. : 3°.0' :: 1^h.32'.51'' : 23'.12''$ the change of the ☽'s declination from midnight on the 4th of August, to the time of her transit; hence $7°.44' - 23'.12'' = 7°.20'.48''$ S. the moon's declination when on the meridian.

3. *In the triangle* ZN*m*, *find the* ∠ZN*m*.

The horizontal refraction $= 33'$ (Table IV.). The ☽'s horizontal parallax at midnight (*Naut. Alm.*) $= 57'.4''$, at noon on the 5th $= 57'.19''$, and by calculating the proportional part for $1^h.32'.51''$ (C. 271.), the time past midnight when the ☽ is on the meridian, the horizontal parallax at that time will be $57'.6''$. Hence we have

$$\text{Given}\left\{\begin{array}{l} \mathrm{Z}m = 90°.+33'-57'.\ 6'' = 89°.35'.54'' \\ \mathrm{N}m = 90°.+\ \ 7°.20'.48'' = 97°.20'.48'' \\ \mathrm{ZN} = 90°.-\ \ 51°.28'.40'' = 38°.31'.20'' \end{array}\right\}\begin{array}{c}\text{To find the}\\ \angle\,\mathrm{ZN}m.\end{array}$$

The ∠ ZN*m* will be found $= 80°.1'.12'' = 5^h.20'.5''$ = nearly the time the moon rises before she comes to the meridian. Whence $13^h.32'.51'' - 5^h.20'.5'' = 8^h.12'.46''$ P.M. the *estimated* time of the moon's rising; and the time of setting will be nearly $6^h.52'.56''$ A.M. on the 5th of August.

4. *To find the ☽'s declination at the estimated time of rising.*

☽'s decl. at noon 4th Aug. 1822. $= 10°.37'$
☽'s decl. at midnight 4th Aug. 1822 $= 7°.44'$

Decrease of declination in 12 hours $= 2°.53'$

$12^h. : 2°.53' :: 8^h.12'.46'' : 1°.58'.24''$
☽'s decl. at ris. $= 10°.37' - 1°.58'.24'' = 8°.58'.36''$S.

Horizont. parallax $= 56'.49''$
Horizont. parallax $= 57'.\ 4''$

Incr. in 12 hours $= 0'.15''$

$12^h : 15'' :: 8^h.12'.46'' : 10''$
☽'s hor. parallax $= 56'.59''$

5. *Again, in the triangle* zN*m*, *we have*

$$\text{Given}\left\{\begin{array}{l} zm = 90^\circ + 33' - 56'.59'' = 89^\circ.36'.1'' \\ \text{N}m = 90^\circ + 8^\circ.38'.36'' = 98^\circ.38'.36'' \\ z\text{N} = 90^\circ - 51^\circ.28'.40'' = 38^\circ.31'.20'' \end{array}\right\}\begin{array}{c}\text{To find the} \\ \angle z\text{N}m.\end{array}$$

The ∠ zN*m* will be found $= 78^\circ.20'.4'' = 5^h.13'.20''$, the time the moon rises before she comes to the meridian; whence $13^h.32'.51'' - 5^h.13'.20'' = 8^h.19'.31''$ P.M. the time of the moon's rising, nearly.

To find the correct time. The ☽ passes the meridian on the 5th of October (Prob. VIII. page 276.) at $14^h.18'.12''$, then $14^h.18'.12'' - 13^h.32'.51'' = 45'.21''$ daily variation of transit; hence, $24^h : 45'.21'' :: 8^h.19'.31'' : 15'.44''$, and $8^h.19'.31'' - 15'.43'' = 8^h.3'.47''$ P.M. *true* time of the moon's rising.

(T) Should more examples be thought necessary, they may be formed and solved in a similar manner; likewise the rising of the different planets may be found by the same method. The times of the rising and setting of the moon and of the planets, are given in *White's Ephemeris*, and in the *Connaissance des Tems*. These works will serve as a check upon the calculation. The rising and setting of the moon, and of the planets, are not inserted in the Nautical Almanac.

PROBLEM X. *(Plate III. Fig. 1.)*

(U) *The latitude and longitude of a fixed star, or planet, being given, to find its right ascension and declination, et contra.**

In the oblique triangle n*N*C.

N*n* = obliquity of the ecliptic, or distance between the poles of the equator and ecliptic.

*n*C = the complement of the star's latitude VC.

NC = the complement of the star's declination RC.

The angle N*n*C, measured by the arc V♋, is the complement of the star's longitude from aries.

The supplement of the angle *n*NC, measured by the arc RQ, is the complement of the star's right ascension from aries.

Or, *In the oblique triangle* S*m*X.

S*m* = the obliquity of the ecliptic.

*m*X = the complement of the star's latitude VX.

SX = the complement of the star's declination RX.

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* A neat method of solving this problem may be seen at page 44 and 45, Vol. I. third edition, of Dr. Mackay's useful treatise on the theory and practice of finding the longitude, or in Dr. Maskelyne's introduction to Taylor's logarithms. The problem may be varied so as to admit of several cases, but the two above specified are the only useful ones.

The supplement of the angle x*ms*, measured by the arc vv ♋, is the complement of the star's longitude from aries.

The angle xs*m*, measured by the arc RQ, is the complement of the right asceusion from aries.

EXAMPLE.

The latitude of Aldebaran in Taurus being = 5°.28'.27" S.; its longitude 2^s.6°.56'.24"[*]; required its right ascension and declination; the obliquity of the ecliptic being 23°.28'?

In the triangle s*m*x.

s*m* = 23°.28'. 0" = the obliquity of the ecliptic.
x*ms* = 156°.56'.24" = star's longitude + 90°.
*m*x = 84°.31'.33" = the complement of the star's lat.

1. To find the co-declination sx.

s*m* = 23°.28'.00" log sine =	9·60012	Nat. sine 22°.22' = - -	38053
*m*x = 84°.31'.33" log sine =	9·99802		2
½x*m*s = 78°.28'.12" log sine =	9·99114 9·99114		76106
		*m*x − s*m* = 61°.3'.33" Nat. cos =	48391
22°.22' log sine - =	9·58042	Nat. cos 73°.54'.37" = - -	27715
		sx = 106°.5'.23" obtuse.	

Hence 106°.5'.23" − 90° = 16°.5'.23" = RX the declination N.

2. *To find the co-right ascension* xs*m*.

Sine sx = 106°.5'.23" : sine x*m*s = 156°.56'.24" : :
Sine *m*x = 84°.31'.33" : sine xs*m* = 23°.56'.32".

Hence AR = 66°.3'.28", the right ascension.

OR THUS:

Draw the perpendicular XL.

Rad : cos L*m*x : : tang *m*x : tang L*m*, and L*m* + *m*s = LS.
Cos *m*L : cos *m*x : : cos SL : cos SX.
Sine *m*L : sine SL : : cot L*m*x : cot xs*m*.

PRACTICAL EXAMPLES.

1. Required the right ascension and declination of Regulus in the Lion's heart; latitude being 0°.27'.30" N., and longitude 4^s.26°.57'.5", obliquity of the ecliptic being 23°.28'?

Answer. Right ascension = 149°.19'.54" and declination = 12°.58'.21" N.

2. Required the right ascension and declination of Spica Virginis, in the sheaf of Virgo; latitude 2°.2'.23" South, and longitude 6^s.20°.57'.10"; obliquity of the ecliptic 23°.28'?

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[*] The latitudes and longitudes of the stars, in this, and many of the following examples, in the first edition of this work (published in the year 1801) were adapted to the year 1796; any change of data is at present unnecessary, as the method of solution remains the same, being independent of an ephemeris, and the answers, when obtained, would only differ a few seconds from those now given.

Answer. Right ascension = 198°.34'.32", and declination = 10°.4'.31" South.

3. Required the right ascension and declination of Betelguese in the shoulder of Orion; latitude 16°.3'.3" South, longitude 2^s.25°.51'.46"; obliquity of the ecliptic 23°.28'?

Answer. Right ascension = 85°.59'.28", declination = 7°.21'. 17" North.

4. Required the right ascension and declination of Antares in Scorpio; latitude 4°.32'.35" S., longitude 8^s.6°.51'.51"; obliquity of the ecliptic 23°.28'?

Answer. Right ascen. = 244°.11'.3", declination = 25°.57'.22" South.

5. Required the right ascension and declination of the moon, when her latitude is 4°.0'.34" N., longitude 7^s.14°.26'.21"; and the obliquity of the ecliptic 23°.27'.48"?

Answer. Right ascension = 223°.11'.11", declin. = 12°.21'.14" South.

6. Given the right ascension of Aldebaran = 4^h.25'.43", and its declination 16°.8'.36" N. in the year 1822, required its latitude and longitude; the obliquity of the ecliptic being 23°.28'?

In the triangle nNC.

nNC = 156°.25'.45" = AR + 90°, AR being = 4^h.25'.43".
NC = 73°.51'.24" = complement of RC.
Nn = 23°.28'. 0" = obliquity of the ecliptic.

1. *To find the co-latitude* nC.

Nn = 23°.28' log sine - =	9·60012	Nat. sine 21°.30'.12" - -	= 36655
NC = 73°.51'.24" log sine =	9·98252		2
½nNC = 78°.12'.52" log sine =	9·99075 9·99075		73310
		NC − Nn = 50°.23'.24" Nat. cos	= 63755
21°.30'.12" log sine =	9·56414		
		Nat. cos 84°.31'.1" - -	= 09555
		Co-lat. nC = 95°.28'.59" obtuse.	

Hence the latitude = 5°.28'.59"* South.

OR THUS,

Draw the perpendicular CL, Then, in the right-angled triangle CLN,		In the oblique-angled triangle nNC (B. 178.)	
rad. sine of 90° - -	10·00000	cos NL = 72°.28'.25" - -	9·47877
: cos CNL = RQ = 23°.34'.15"	9·96216	: cos nL = 95°.56'.25" - -	9·01490
:: tang NC = 73°.51'.24" -	10·53842	:: cos NC = 73°.51'.24" -	9·44417
: tang NL = 72°.28'.25". -	9·50058	: cos nC = 84°.30'.59" -	8·98030
Then NL + Nn = nL = 95°.56'.25"		Hence the lat = 5°.29'.1" South.	

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* The latitude of this star, on the same day, in the year 1796, was 5°.28',27" S making a difference of only a few seconds in 26 years. (See the note page 30

2. *To find the co-longitude* cnN.

Sine *n*c = 95°.28′.59″ : sine *n*NC = 156°.25′.45″ :: sine NC = 73°.51′.24″ : sine c*n*N = 22°.41′.53″. Hence the longitude = 67°.18′.7″ *, or 2^s.7°.18′.7″.

7. Required the latitude and longitude of Sirius in the Great Dog, right ascension being 99°.0′.21″, and declination = 16°.26′.35″ South; the obliquity of the ecliptic being 23°.28′?

Answer. Latitude 39°.33′.30″ S. longitude 3^s.11°.13′.44″.

8. Required the latitude and longitude of Procyon in the Little Dog; right ascension being 112°.6′.47″, declination 5°.45′.3″ N.; obliquity of the ecliptic 23°.28′?

Answer. Latitude 15°.58′.14″ S., longitude 3^s.22 .55′.42″.

9. Required the latitude and longitude of Arcturus in Boötes: right ascension 211°.33′.17″, declination 20°.15′.58″ N.; the obliquity of the ecliptic being 23°.28′?

Answer. Lat. 30°.52′.43″ N., long. 6^s.21°.20′.41″.

10. Required the latitude and longitude of Fomalhaut, in the Southern Fish; right ascension being 341°.32′.34″, declination 30°.42′.51″ S.; obliquity of the ecliptic 23°.28′?

Answer. Lat. 21°.6′.29″ S., long. 11^s.0°.56′.42″.

11. Required the latitude and longitude of the moon, on the 16th of June 1822, when her right ascension is 41°.24′, declination 21°.21′ N., and the obliquity of the ecliptic 23°.27′.53″?

Answer. Lat. 5°.6′ N., longitude 45°.28′.

PROBLEM XI. *(Plate III. Fig. 8.)*

(W) *The right ascensions and declinations of two stars, or the latitudes and longitudes of two stars given, to find their distance.*

Required the distance between Sirius in the Great Dog, right ascension = 99°.0′.21″, and declination = 16°.26′.35″ S., and Procyon in the Little Dog, right ascension = 112°.6′.47″, and declination 5°.45′.3″ N.?

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* The longitude of this star in the year 1796 was found, by the same process, to be 66°.56′.24″, making an increase of 21′.43″ in 26 years.

By comparing together the places of the fixed stars deduced from observation, astronomers have found that their longitudes increase about 50¼″ annually. This increase of longitude must necessarily cause an irregular motion in the same star with regard to the equinoctial. Hence the right ascensions and declinations of stars are continually varying, so that stars which formerly had north declination have now south declination, *et contra.* Their latitudes are also subject to a small variation. See the Tables, page 153, &c. *Naut. Alm.* 1822.

In the triangle XST let the star at T represent Procyon, because its right ascension is the greatest, and the star at X represent Sirius.

The ∠ XST = 13°.6′.26″ diff. of their right ascensions = RQ.

SX = 73°.33′.25″ the complement of Sirius's declination RX.

ST = 95°.45′.3″ Procyon's declination + 90° = SQ + QT.

Hence, here are two sides and the contained angle given, to find the third side XT, by any of the methods in the preceding problems.

Answer. XT = 25°.42′.10″.

EXAMPLE II.

The latitude and longitude of Sirius being = 39°.33′.30″ S., and 3^{s}.11°.13′.44″; and the latitude and longitude of Procyon = 15°.58′.14″ South, and 3^{s}.22°.55′.42″. Required their distance?

In the triangle XmT as before, we have

Xm = 50°.26′.30″ the complement of Sirius's lat. VX.

Tm = 74°.1′.46″ the complement of Procyon's lat. ♋T.

The ∠ XmT = 11°.41′.58″ the difference of their longitudes V♋.

Hence, as before, here are two sides and the contained angle given, to find the third side, XT = 25°.42′ *Answer.*

EXAMPLE III.

Required the distance between Capella in the Goat, right ascension = 75°.21′.19″, declination = 45°.46′.15″ North, and Procyon in the Little Dog, right ascension = 112°.6′.47″, and declination = 5°.45′.3″ North?

In the triangle TNC, let the star at T represent Procyon, and C Capella.

NC = 44°.13′.45″ the co-declination of Capella.

TN = 84°.14′.57″ the co-declination of Procyon.

∠ TNC = 36°.45′.28″ the diff. of their right ascensions = RQ.

Here we have two sides and the contained angle given, to find the third side, as in the preceding example.

Answer. CT = 51°.6′.56″.

EXAMPLE IV.

Required the distance between Capella, whose latitude is 22°.51′.57″ North, longitude 2^{s}.18°.57′.57″, and Procyon, latitude 15°.58′.14″ South, longitude 3^{s}.22°.55′.42″?

In the triangle T*n*C. Let the star at T represent Procyon, that at C, Capella.

*n*C = 67°. 8′. 3″ = comp. Capella's lat. OC.
*n*T = 105°.58′.14″ = 90°.+Procyon's lat. = NT.
∠T*n*C = 33°.57′.45″ = diff. of their longitudes = σ ♋.

Here we have two sides and the contained angle given, to find the third side CT = 51°.6′.39″ *Answer*.

PROBLEM XII. *(Plate III. Fig. 8.)*

(X) *The places of two stars being given, and their distances from a third star, to find the place of this third star.*

EXAMPLE I.

Suppose the distance of a comet or new star C, was measured by a sextant to be 65°.47′.42″ from Sirius the dog star, lat. 39°.33′.30″ South, longitude 3s.11°.13′.44″; and 51°.6′.56″, from Procyon the Little Dog, lat. 15°.58′.14″ South, longitude 3s.22°.55′.42″; it is required to find the latitude and longitude of this comet or star.

Let X represent the place of Sirius, or one of the known stars, T that of Procyon the other known star, having the greatest longitude, and C the place of the comet or unknown star.

First. In the triangle X*n*T, we have given

X*n* = 129°.33′.30″ Sirius's distance from the pole of the ecliptic nearest the comet or star;
T*n* = 105°.58′.14″ Procyon's distance from the same pole.
∠X*n*T = 11°.41′.58″ the diff. of longitude between the given stars.

To find XT the distance between the two known stars = 25°.42′.10″, and the angle *n*XT = 26°.42′.39″: To find the former of these quantities, there are two sides and the included angle given, and to find the latter the three sides and one angle are given.

Secondly. In the triangle XTC, we have given

XT = 25°.42′.10″ } To find the ∠CXT at the star X. By
TC = 51°. 6′.56″ } any of the methods (W. 231. or P. 248,)
and XC = 65°.47′.42″ } you will find CXT = 49°.14′.

This angle being greater than *n*XT shews the longitude of the star at C to be less than at X. Hence CXT ∼ *n*XT = 22°.31′.21″ = *n*XC.

Thirdly. In the triangle X*n*C, we have given

X*n* = 129°.33′.30″ } To find the side *n*C the comet or star's
XC = 65°.47′.42″ } co-lat. and X*n*C the diff. between the
and *n*XC = 22°.31′.21″ } star's long. at C and Sirius's long.

nc=67°.8′.26″ the star's co-latitude, hence its latitude oc=22°.51′.34″, and x*nc*=22°.16′.42″, and since the star's longitude has been shewn to be greater than the comet's; from Sirius's longitude=101°.13′.44″ take 22°.16′.42″, the remainder 78°.57′.2″ is the unknown star's longitude.

EXAMPLE II.

Suppose the distance of an unknown star at x was measured to be 65°.47′.42″, from a known star (Capella) at c having 22°.51′.57″ North latitude, and 2^s.18°.57′.57″ of longitude; and its distance from another known star at T (Procyon)=25°.42′.10″, this star having 15°.58′.14″ South latitude, and 3^s.22°.55′.42″ of longitude; required the latitude and longitude of the unknown star?

First. In the triangle c*m*T, we shall have

c*m*=*m*o+oc=112°.51′.57″=Capella's distance from the pole of the ecliptic nearest the comet, or unknown star.

c*m*T=33°.57′.45″ = the diff. long. between Capella and Procyon.

T*m* = 74°.1′.46″=Procyon's distance from the pole of the ecliptic.

To find TC=51°.6′.56″, the distance between the known stars; and the angle *m*CT=43°.37′.41″.

Secondly. In the triangle XCT, we have given

XC=65°.47′.42″
XT=25°.42′.10″ } To find the angle XCT at the star c.
TC=51°. 6′.56″

By any of the methods (W. 231. or P. 248.) XCT will be found=24°.58′; this angle being less than *m*CT shews the longitude of the star at x to be greater than that at c.

Hence *m*CT∼XCT=18°.39′.41″=XC*m*.

Thirdly. In the triangle XC*m*, we have

c*m*=112°.51′.57″
cx= 65°.47′.42″
∠XC*m*= 18°.39′.41″ } To find x*m* the unknown star's co-latitude, and x*m*c the difference between the star's longitude at x, and Capella's longitude at c.

x*m*=50°.26′ the star's co-lat., hence its latitude vx=39°.34′, and x*m*c=22°.15′.6″, and since the given star's longitude has been shewn to be less than the required star's longitude, this difference must be added to Capella's longitude, viz. to 2^s.18°.57′.57″ add 22°.15′.6″, the sum is 3^s.11°.13′.3″ the longitude required.

EXAMPLE III.

Suppose the distance from an unknown star at T was measured to be 25°.42′.10″, from a star at X, whose latitude is 39°.33′.30″ South, longitude 3ˢ.11°.13′.44″; and 51°.6′.56″ from another known star at C, whose latitude is 22°.51′.57″ North, and 2ˢ.18°.57′.57″ longitude; required the latitude and longitude of the unknown star.

First. In the triangle X*m*C, you will have given

X*m* = the complement of X's latitude,

C*m* = the distance of C from the same pole of the ecliptic, and the angle X*m*C = the difference between their longitudes = VO.

Hence XC will be found = 65°.47′.42″, and the angle *m*XC = 153°.17′.21″.

Secondly. In the triangle XTC, are given

XC
XT
and TC } To find the angle CXT = 49°.14′
Now *m*XC ∼ CXT = 104°.3′.21″ = *m*XT.

Thirdly. In the triangle *m*XT, are given

X*m*
XT
∠ *m*XT } To find T*m* = 74°.1′.46″, the distance of the star T from the pole of the ecliptic, and the angle X*m*T = 11°.41′.58″, the difference between the longitude of X and T.

(Y) By the assistance of the foregoing problem, and a knowledge of any one fixed star's true situation, the places of all the rest may be determined. Indeed, it is by this problem principally that astronomers have rectified the places of all the fixed stars; and thence, by a similar mode of proceeding, found the true places of the planets. *Leadbetter* tells us in his Astronomy, page 230, that he made use of this method for constructing his astronomical tables. The preceding examples contain every variation that can happen with respect to the stars' latitudes and longitudes, but the same things may be determined from their right ascensions and declinations, as in the following example:

EXAMPLE IV. *(Plate III. Fig. 9.)*

Suppose the distance of a star or planet at Y, was observed to be 65°.47′.42″, from Sirius the Dog-star at X, whose right ascension is 99°.0′.21″, and declination 16°.26′.35″ South; and to be 51°.6′.56″ distant from Procyon at T in the Little Dog, whose right ascension is 112°.6′.47″, and declination 5°.45′.3″ North; it is required to find the right ascension and declination of this unknown star or planet

First. In the triangle XNT, we have

The angle XNT=13°.6'.26"=the difference between the right ascensions of X and T, measured by the arc RQ.

XN=106°.26'.35"=X's distance from the pole N.

TN= 84°.14'.57"=T's distance from the same pole.

Hence two sides and the included angle are given, to find the third side XT=25°.42'.10", and the angle NXT=31°.22'.

Secondly. In the triangle YXT, we have given

YX=65°.47'.42"
YT=51. 6.56
XT=25. 42.10

To find the angle YXT=49°.14'.
Then YXT ~ NXT=17. 52.
=YXN.

The ∠ YXT being greater than the ∠ NXT shews the right ascension of the unknown star at Y to be less than the right ascension of that at X.

Thirdly. In the triangle XYN, are given

XY= 65°.47'.42"
XN=106. 26.35
∠ YXN= 17. 52.

To find YN the complement of the declination BY, and the angle YNX the difference between the right ascensions of Y and X, =BR.

You will find
YN=44°.14', consequently BY=45°.46' N.
YNX=23°.38'.55", hence the right ascension of Y=AB=75°.21'.26".

(Z) In the same manner we may find the distance between any two places on the earth; suppose X and T to be two places whose latitudes and longitudes are known, then in the triangle XNT, XN is the distance of X from the pole, TN the distance of T from the same pole, and the angle XNT, measured by the arc RQ, the difference of longitude between the two places, hence the distance XT is easily found. In a similar manner the distance between T and Y, or Y and X may be obtained.

EXAMPLE V.

What is the extent of Africa, from Cape Verd, latitude 14°. 46' N., longitude 17°.47' W. to Cape Guardafui, latitude 11°. 47' N., longitude 51°.35' E.?

Answer. 67°.19' the arc of a great circle contained between the places, or 4678 *English* miles, of 69½ to a degree, or 4039 geographical miles, of 60 to a degree.

PROBLEM XIII. *(Plate III. Fig. 1.)*

(A) *Given the latitude of the place, the sun's declination and altitude, to find the azimuth.*

EXAMPLE. I.

In latitude 51°.32′ North the sun's altitude was observed to be 50°, and his declination 23°.28′ North; required his azimuth from the North.

In the triangle SZN, we have given

SN = 66°.32′. the co-declination, or sun's polar distance; for DS is the sun's declination North.

SZ = 40. 0. the co-altitude of the sun; for CS is his altitude above the horizon HO.

ZN = 38. 28. the co-latitude of the place, or what ON the elevation of the pole wants of 90°.

To find the angle SZN, measured by the arc of the horizon CO, the sun's distance from the North.

Call the co-latitude ZN the base, and draw the perpendicular *sf.* Then (W. 231.)

tang $\frac{1}{2}$ ZN = *zt* = 19°.14′ - - - - - - - - 9·54269

: tang $\frac{1}{2}$ (SN + SZ) = 53°.16′ - - - - - - 10.12710

:: tang $\frac{1}{2}$ (SN ∼ SZ) = 13°.16′ - - - - - - 9·37250

: tang *ft* - - - = 42°.10′ - - - - - - 9·95691

Take the difference between *ft* and half the complement of latitude. Then

Radius, sine of 90°. - - - - - - - - - 10·00000

: tang *ft* ∼ *zt* = 22°.56′ - - - - - - - - 9·62645

:: tang of the sun's altitude = CS = 50° - - - 10·07619

: sine of azimuth from the east or west = 30°.17′ 9·70264

Hence 90° + 30°.17′ = 120°.17′ the sun's azimuth from the North.

(B) A slight review of *Figure I. Plate III.* will naturally point out to us the following observations:

If the latitude of the place and the declination of the sun be of the same name, the sun is less than 90° from the elevated pole; therefore the co-declination is the polar distance.

If the sun or star be in the equinoctial, it is 90° from the pole, and forms a quadrantal triangle. If the declination be of a contrary name with the latitude, the distance of the object from the elevated pole will be equal to the declination increased by 90°.

(C) The preceding solution is on a supposition that the sun's declination, as taken from the Nautical Almanac, remains the same in all places, and at all times of the day. But this cannot be strictly true, for the declination is subject to a daily variation, and is adapted to the meridian of Greenwich. (B. 270.) An azimuth will not be materially affected by this change, but it will make a considerable difference when we want to find the time of the day.

EXAMPLE II.

Suppose the sun's altitude at London, latitude 51°.32′North, to be 1°.14′ when the declination is 20°.11′ South, what is the azimuth from the North?

Answer. 125°.34′.

EXAMPLE III.

Given the latitude of the place 51°.32′ North, the sun's altitude 25°, his declination 4°.47′ South; required the azimuth from the North?

Answer. 137°.16′.

EXAMPLE IV.

Suppose in latitude 51°.32′ North, the altitude of the star Arcturus in Boötes was 44°.30′, when its declination was 20°.16′ N.; required its azimuth from the North at that time.

Answer. 117°.8′.

EXAMPLE V.

In latitude 48°.51′ North, when the sun's declination is 18°.30′ North, and his altitude 52°.35′; what is his azimuth from the North?

Answer. 134°.36′.8″.

(D) This problem is not only useful for finding the variation of the compass, but it will be found very convenient for determining the true meridian, and the four cardinal points. For, if with a line representing the vertical on which the sun is, an angle be made equal to the supplement of the azimuth from the North, the meridian will be obtained: provided care has been taken to correct the sun's altitude by refraction.

(E) An azimuth may be constructed by a scale of chords, sufficiently accurate for Nautical purposes; Thus,

With the chord of 60° describe a circle (*Plate IV. Fig.* 2.), set off the complement of the latitude from Z to P, and the complement of the altitude from Z to *a*, draw ZON through the centre C and *aoa* at right angles to it. Set off the polar distance from P to *p*, draw *pp* through the centre C, and from the point I where it intersects *aoa* draw IR parallel to ZON, and through C the centre draw CS parallel to *aoa*.

With C as a centre, and the distance *oa*, cross the line IR in *m*; through *m* draw C*n*, then S*n*, measured on a scale of chords will be the azimuth from the south (in North latitude) if the point I fall on the left-hand of ZN, and from the north if it fall on the right-hand.

See the Mathematical Repository, vol. i. page 283, second edition, 1799, and vol. ii. page 26 and 27. See also Dr. Mackay's Navigation, first edition, 1804, page 200.

PROBLEM XIV. *(Plate III. Fig. 1.)*

(F) *Given the latitude of the place, the sun's declination and altitude, to find the hour of the day.*

EXAMPLE I.

At London, latitude 51°.32′ North, the altitude of the sun's centre was 38°.19′, and his declination 19°.39′ North, required the hour of the day?

In the triangle SZN, we have given

SN = 70°.21′ the sun's distance from the elevated pole.

SZ = 51. 41 the sun's co-altitude.

ZN = 38. 28 the co-latitude.

To find the angle ZNS, measured by the arc ÆD, the distance between the meridian which the sun is on at the time of observation, and the meridian of the place of observation.

Since ÆZ is equal to the latitude ON; it follows, that if the latitude and declination be both north or both south, the difference between them is equal to the meridional zenith distance; if the one be north and the other south, their sum is the meridional zenith distance. Hence the following general method of solution:

Nat cos of merid zenith dist
(51°.32′ − 19°.39′) - - - 31°.53 = 84913
Nat sine of the sun's altitude - - 38. 19 = 62001

Remainder - - - 22912

Common logarithm of - - 22912 = 4·36007
Secant of the latitude - - 51°.32′ = ·20617
Secant of the declination - 19°.39′ = ·02606

Num. in col. ris., Requisite Tables, 3^{h}.30′ 4·59230
Or, Increase the index by 5 +

Log. versed sine - - 52°.29′.20″ - 9·59230
Or add the logarithm of 5000000000 = ½ radius. - - - - + 9·69897

Divide by 2 | 19·29127

Log sine of half the hour angle 26°.14′.40″ = 9·64563
2

The hour angle - - 52.29.20 = 3^{h}.29′.57″.20‴.

(G) This solution is analogous to Problem 8th in the *Requisite Tables* published by order of the Commissioners of longitude; or to Problem VII. Chap. II. Book III. of Dr. Mackay's Navigation. If you have the *Requisite Tables* * by you, the sum of the three logarithms above may be looked for in the column of rising, and the hours and minutes answering thereto will be the time from noon. Or, increase the index by 5, and the sum is the log. versed sine of the same angle. Or, the log. versed sine of an arc increased by the logarithm of 5 (with 9 for its index), will give double the logarithmical sine of half the required angle. (P. 131.)

EXAMPLE II.

In latitude 39°.54′ North, longitude 80°.39′.45″ West of Greenwich, suppose the altitude of the sun's lower limb to be 15°.40′.57″ on the 9th of May 1822, at 5h.34′.21″ P.M. per watch. Required the error of the watch.

1. *To find the time at Greenwich, nearly.*

$80^{\circ}.39'.45'' \times 4 = 5^{h}.22'.39''$ the difference of time between the clocks at Greenwich and the place of observation. Hence, $5^{h}.34'.21'' + 5^{h}.22'.39'' = 10^{h}.57'$ P.M. the time at Greenwich.

2. *To correct the ⊙'s declin.*

⊙'s decl. May 9th 1822	$= 17^{\circ}.16'.51''$
Ditto May 10th -	$= 17^{\circ}.32'.47''$
Increase in 24 hours -	$= 15'.56''$

$24^{h} : 15'.56'' :: 10^{h}.57' : 7'.16''$.

Hence the ⊙'s true declin. at the time and place of observation $= 17^{\circ}.24'.7''$N.

3. *To correct the ⊙'s altitude.*

⊙'s observed alt. -	$= 15^{\circ}.40'.57$
Refraction † - -	$= -\ 3'.17''$
	$15^{\circ}\ 37'.40''$
⊙'s semi-diameter ‡ -	$= +\ .\ 5'\ 52''$
⊙'s parallax ‖ - -	$= +\ 8''$
True alt. ⊙'s centre -	$= 15^{\circ}.53'.40''$

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* The XVIth of the *Requisite Tables*, or the XXXV, XXXVI, and XXXVII Tables in Dr. Mackay's Navigation.

The XVIth of the Requisite Tables is divided into three columns. The FIRST column, called *half elapsed time*, is only a table of secants with 10 rejected from the index; the time being reduced to degrees.

The SECOND, called *middle time*, is the logarithmical sine of half the elapsed time, increased by the logarithm of 2, and the index of the sum diminished by 5.

The third, called *column of rising*, is a table of logarithmical versed sines with their indices diminished by 5.

Thus for 1 hour of time, or 15 degrees.

co-sec of 15° = 10·58700,	sine of 15° = 9·41300,	ver. sine of 15° = 8·53243
−10	log. of 2 = 0·30103,	−5
Log. ½ elap. time 0·58700	sum 9·71403	logarith. rising. 3·53243
	− 5	
	Log. mid. time 4·71403	

Proceed in the same manner for any other time.

† Table IV. ‡ Nautical Almanac. ‖ Table VI.

4. *To find the true time. In the triangle* SZN,

reject indices.
Complement of latitude NZ=50°. 6′. 0″ cosecant - ·11511
Complement of altitude SZ=74. 6. 20
Polar distance SN=72. 35. 53 cosecant - ·02035

2 | 196.48.13

½ sum - - 98. 24. 6½ sine - - 9·99531
Complement of altitude 74. 6.20

Remainder - 24. 17.46½ sine - 9·61432

2 | 19·74509

Cosine of half ZNS=41°.47′ - - =9·87254
8

True time 5h.34′.16″.
Time *per* watch 5. 34. 21.

Watch too fast 5″.

(H) In the practical application of this problem, it will be proper to take several altitudes of the sun within a minute or two of each other, and to note the corresponding times per watch. The mean of these altitudes, and of the times, will be more correct than a single altitude and time, and will in a great measure counteract the errors arising from the imperfection of the instrument.

EXAMPLE III.

On the 4th of March 1822, in latitude 45°.37′ North, and longitude 169°.59′.30″ West, suppose the following altitudes of the sun's lower limb to be observed:

Times per watch P.M.		Altitudes of the sun.
2h.53′.32″	- - - -	24°.55′
2. 54. 30	- - - -	24. 48
2. 55. 36	- - - -	24. 40
2. 56. 46	- - - -	24. 31
4 | 11. 40. 24		4 | 98. 54
Mean 2. 55. 6. P.M.		Mean 24. 43. 30

Required the apparent time of observation, and the error of the watch.

The sun's declination at Greenwich at noon being 6°.31′.59″ S. *(Naut. Alm.)* and on the 5th 6°.8′.53″ S. also the ⊙'s semidiameter 16′.9″.

Answer. The time at Greenwich is 14^{h}.15′.4″, the ⊙'s declination at that time = 6°.18′.16″ S. true altitude of the ⊙'s centre=24°.57′.45″; hence the apparent time of observation =2^{h}.55′.50″, and the watch is 44″ too slow.

EXAMPLE IV.

On the 15th of September 1822, in latitude 33°.56′ South, and longitude 84°.6′.15″ West, suppose the mean time per watch of several altitudes of the sun's lower limb to be 8^{h}.11′.5″ A.M. and the mean of the altitudes themselves 24°.43′; required the error of the watch?

The ⊙'s declination at noon *(Naut. Alm.)* on the 16th being 2°.46′.24″ N. and on the 17th 2°.23′.11″, also the ⊙'s semidiameter 15′.56″.

Answer. The time at Greenwich on the 16th is 1^{h}.47′.30″ P.M. the ⊙'s declination at that time=2°.44′.40″ N. true alt. of the ⊙'s centre=24°.57′.2″; hence the apparent time is 3^{h}.48′.54″ from noon, or 8^{h}.11′.6″ A.M. and the watch is right.

PROBLEM XV. *(Plate III. Fig. 1.)*

(I) *Given the latitude of the place, the declination and altitude of a known fixed star, to find the hour of the night when the observation was made.*

EXAMPLE I.

In latitude 7°.45′ South, and longitude 30°.18′ East of Greenwich, on the 7th of September 1822, suppose the altitude of the star Procyon, when east of the meridian, to be 28°.12′; required the true time of the day? The declination of Procyon being 5°.40′.29″ N.

Procyon's alt =(28°.12′−refraction 1′.47″=)28°.10′.13″
Co-lat NZ =82°.15′ - - - Cosec - ·00399
Co-alt SZ =61°.49′.47″
Polar dist SN =95°.40′.29″ - - Cosec - ·00213

2 | 239. 46. 2

Half sum =119. 53. 1 - - sine - - 9·93807
Co-alt SZ = 61. 49.47

Remainder= 58. 3.14 - - sine - 9·92864

2 | 19·87283

Cos of half the hour ∠ 30°.15′.13″ - - 9·93641
8

The hour ∠ ZNS in time= 4^{h}. 2′.1″.44‴

(K) The hour ∠ found by calculation, as above, is the distance of the star from the meridian of the place of observation; consequently when the star is eastward of the meridian at the time of observation, this hour angle must be subtracted from the time of the star's culminating, and when westward it must be added.

*'s right asc. (1822, Tab. VIII.) $+24^{h}$. - $=31^{h}.29'.59''$
⊙'s right ascen. Sept. 7th, 1822, *(Naut. Alm.)* $=11.\ 2.\ 0$

Time of *'s culminating nearly - - $=20.\ 27.\ 59$

⊙'s right ascension, Sept. 7th, 1822, at noon $=11^{h}\ 2'\ 0''$
⊙'s right ascension, Sept. 8th, 1817, at noon $=11.\ 5.\ 36$

Increase of the ⊙'s right ascension in 24 hours $=\ 0.\ 3.\ 36$

Then $24^{h}.3'.36''$: 24^{h} :: $20^{h}.27'.59''$: $20^{h}.24'.55''$ time of Procyon's culminating at Greenwich. Then,
15° : 10″ :: 30°.18′ : 20″, hence $20^{h}.24'.55''+20''=20^{h}.25'.15''$ the time of the *'s culminating in longitude 30°.18′. E.
Lastly. $20^{\circ}.25'.15''-4^{h}.2'.1''=16^{h}.23'.14''$ the correct time.

(L) In order to attain the greatest accuracy several altitudes of a star should be observed in succession, and if two or more separate stars be observed, and the error of the watch be deduced from each star, the medium result will be more correct than that derived from the observation of a single star.

If an equal number of stars can be observed on each side of the meridian, and nearly equidistant therefrom, the errors arising from the imperfection of the instrument, &c. will be rendered almost insensible.

EXAMPLE II.

On the 29th of January 1822, in latitude 53°.24′ North, and longitude 25°.18′ West; suppose the following altitudes of Procyon to the west, and Alphacca to the east of the meridian, were observed by two separate persons at the same instant of time; required the error of the watch.

The right ascension of Alphacca being $15^{h}.27'.9''$ declination 27°.19′.12″ N., and of Procyon $7^{h}.29'.59''$ and 5°.40′.29″ N. The sun's right ascension at noon on the 29th $=20^{h}.46'.11''$ *(Naut. Alm.)* and on the 30th $=20^{h}.50'.17''$.

Time per watch.		Procyon's altitude.		Alphacca's altitude.
14^h.53'. 0"	- -	20°.41'.40"	- -	41°.17'.40"
14. 57. 0	- -	20. 7. 40	- -	41. 57. 40
15. 0.32	- -	19. 37. 40	- -	42. 24. 40
15. 4. 0	- -	19. 8. 40	- -	42. 55. 40
4 | 59. 54.32	-	4 | 79. 35. 40	-	4 | 168. 35. 40
Mean 14. 58.38		Mean 19. 53. 55		Mean 42. 8. 55

Answer. The hour ∠ by Alphacca's corrected altitude is 3^h.40'.47", and the * culminates at 18^h.37'.31", hence the correct time = 14^h.56'.54".

The hour ∠ by Procyon's corrected altitude is 4^h.15'.55", and the * culminates at 10^h.41'.42", hence the correct time = 14^h.57'.37". The mean time between the two is 14^h.56'.55", consequently the watch is 1'.43" too fast.

EXAMPLE III.

In latitude 48°.56' North, longitude 66° West, suppose the altitude of Aldebaran, when west of the meridian, to be 22°.20'.15" on the 15th of April 1822; required the apparent time of observation?

The right ascension of Aldebaran being 4^h.25'.43" and declination 16°.8'.36" N. Also, the ⊙'s right ascension at noon on the 15th *(Naut. Alm.)* = 1^h.32'.23", and on the 16th = 1^h.36'.5".

Answer. The hour ∠ by the *'s corrected altitude = 4^h.57'.33", and the * culminates at 2^h.52'.9", hence the correct apparent time = 7^h.49'.42".

(M) SCHOLIUM.

The three preceding problems (XIII, XIV, and XV.) are all included in one triangle (*Zenith* NS· *Plate III. Fig.* 1.), but they have been considered separately, on account of their importance, and to prevent confusion.

Any three of the five quantities, viz. the sun's or star's declination, the latitude of the place, the altitude of the object, the azimuth, and the hour of the day being given, the rest may be found.

A sixth quantity might be taken into consideration, were it materially useful, viz. the ∠ opposite to the complement of the latitude, being formed by the intersection of the vertical circle ZSC, and the azimuth circle NSD.

PRACTICAL EXAMPLES.

1. Given the sun's declination 19°.39' North, in North lati-

tude, his corrected altitude 38°.19′, and his azimuth = S. 72°.13′ E.; required the hour of the day and the latitude of the place?

Answer. The hour angle is 52°.30′ = 3h.30′, the hour from noon, hence the time is 8h.30′ A. M. and the latitude 51°.32′ North.

When the altitude of the sun and his declination are equal, the triangle is isosceles, and the hour angle is equal to the azimuth. But when the altitude of the sun is less than the declination, the case is ambiguous.

2. Given the sun's declination = 19°.39′ North, his altitude corrected = 38°.19′ at 3h.30′ in the afternoon; required the latitude of the place, supposed to be north, and the sun's azimuth?

Answer. This problem is ambiguous, the azimuth may be N. 72°.13′.30″ W. and the latitude 9°.16′ N. or the azimuth may be N. 107°.46′.30″ W. and the latitude 51°.30′.54″ N. But when the sun's altitude is less than the declination, the problem will be limited.

3. Given the latitude of the place = 51°.32′ N. the sun's declination = 19°.39′ North, the sun's azimuth = S. 72°.13′ W.; required the sun's altitude and the hour of the day?

Answer. The sun's altitude is 38°.18′.30″, the hour angle 52°.30′.20″, and the time from noon 3h.30′. If the latitude of the place be less than the sun's declination, when both have the same name, the case will be ambiguous; see the following example.

4. Given the latitude of the place = 13°.30′ North, the sun's declination = 23°.28′ N. the sun's azimuth = N. 67°.30′ East; required the sun's altitude and the hour of the day?

Answer. The angle opposite to the complement of latitude may be either acute = 78°.20′, or obtuse = 101°.40′. In the former case the sun's altitude is 7°. 7′ nearly, and the time from noon 5h.52′.12″; and in the latter case the sun's altitude is 57°. 4′.34″, and the time from noon 2h.12′.45″. Hence it follows that in this latitude on the longest day, the sun appears upon the same point of the compass twice in the forenoon and twice in the afternoon.

5. Given the latitude 13°.30′ North, the sun's corrected altitude = 60°.30′ at two o'clock in the afternoon; required the sun's azimuth and its declination?

Answer. The angle opposite to the complement of latitude may be either acute = 80°.52′, or obtuse = 99°.8′; in the former case the declination is 10°.21′.42″; and the azimuth 87°.12′, and in the latter case the declination is 20°.37′.34″, and azimuth 71°.51′.30″.

6. In latitude 51°.32′ North at 3h.30′ in the afternoon, the

sun's azimuth was S. 72°.13′ West; required the sun's altitude, his declination, and the angle formed by the intersection of the vertical circle, and the azimuth circle passing through the sun?

Answer. The sun's altitude $= 38°.18'.53''$, the declination $= 19°.39'.19''$, and the angle *zenith* SN formed by the vertical and azimuth circle $= 38°.58'.28''$.

7. The sun's corrected altitude at $8^{h}.30'$ in the forenoon was 38°.19′, the angle of position, viz. formed by the vertical and azimuth circles, was 38°.58′; required the sun's azimuth, declination, and the latitude of the place?

Answer. The latitude is 51°,32′.34″: but it cannot be determined by the *data* whether it be north or south, the sun's declination $= 19°.39'.19''$, and the azimuth $= 72°.13'$ from the east or west.

PROBLEM XVI. (*Plate III. Fig.* 10 and 11.)

(N) *Given two altitudes of the sun and the time between the observations, to find the latitude of the place.*

EXAMPLE I. (*Plate III. Fig.* 10.)

Given the sun's declination 19°.39′ North, his altitude in the forenoon 38°. 19′, and at the end of one hour and a half the same morning the altitude was 50°.25′; required the latitude of the place, supposing it North?

On a supposition that the sun's declination is the same at both observations, (which will not materially affect the result of the operation, when the observations are taken at no great distance from each other, and the time be not near the equinoxes;) the triangle AND is isosceles, A being the place of the sun at the first observation, and D its place at the second, and $AN = ND$. Draw NG perpendicular to AD, then will $AG = GD$ and $GND = \frac{1}{2}AND$ half the elapsed time. (K. 143.)

1. In the right-angled triangle NGD,

There is given $ND = 70°.\ 21'$ the sun's distance from the pole at each observation, and the angle $GND = 11°.15' = \frac{1}{2}$ elapsed time, to find GD and AD.

$$\text{Rad} \times \text{sine } \overset{*}{GD} = \text{sine } GND \times \text{sine } ND.$$

$$\therefore (\log \text{sine } GND + \log \text{sine } ND) - 10 = \log \text{sine } GD.$$

or, rad : sine $ND = 70°.21'$ :: sine $GND = 11°15'$: sine $GD = 10°.\ 35'.\ 13''$, the double of which is $21°.10'.26'' = AD$.

2. *To find the* ∠ GDN.

$$\text{Rad} \times \cos ND = \cot GND \times \cot \overset{*}{GDN}$$

$$\therefore (10 + \log \cos ND) - \log \cot GND = \log \cot GDN.$$

or, cot $GND = 11°.15'$: rad :: cos $ND = 70°.21'$: cot $GDN = 86°.10'.24''$.

3. *To find the* ∠ ADZ.

In the oblique-angled triangle ADZ, there is

$$\text{Given}\left\{\begin{array}{l}\text{First co-alt AZ}=51^\circ.41'\\ \text{Sec. co-alt DZ}=39^\circ.35'\\ \text{And}\qquad\text{AD}=21^\circ.10'.26''\end{array}\right\}\begin{array}{l}\text{Hence }\tfrac{1}{2}\text{ADZ}=57^\circ.41'.33''\\ \text{And }\angle\text{ ADZ}=115^\circ.23'.6''\end{array}$$

The ∠ ADZ being greater than the ∠ GDN, this example belongs to Figure 10th.

And $\angle\text{ ADZ} - \angle\text{ GDN} = 115^\circ.23'.6'' - 86^\circ.10'.24'' = 29^\circ.12'.42'' = \angle\text{ NDZ}$.

4. *To find* NZ *the co-latitude.*

In the oblique triangle DZN, there is

$$\text{Given}\left\{\begin{array}{l}\text{The co-decl ND}=70^\circ.21'\\ \text{Second co-alt DZ}=39^\circ.35'\\ \text{And the }\angle\text{ NDZ}=29^\circ.12'.42''\end{array}\right\}\begin{array}{l}\text{To find NZ the co-lat}\\ =38^\circ.28'.11''.\end{array}$$

This may be done by the rules Case III. (N.245.) Or by Formula I. Case VIII. page 203. viz.

$$\text{Tang}\,\varphi=\frac{\cos\angle\text{ NDZ . tang ND}}{\text{rad}};\ \text{and}\cos\text{NZ}=\frac{\cos\text{ ND}}{\cos\varphi}.\cos(\text{DZ}\sim\varphi).$$

Cos ∠ NDZ = 29° 12'.42" -	9·94092	Cos ND = 70°.21' - -	9·52669
Tang ND = 70°.21' -	10·44725	Cos (DZ ∼ φ) = 28°.10'.2" -	9 94526
Sum	20·38817	Sum	19·47195
Subtract rad =	10·	Cos φ = 67°.45' 2" sub.	9·57823
Tang φ = 67°.45'.2" -	10·38817	Cos NZ = 38°.28'.11" -	9·89372

Hence the latitude is 51°.31'.49" North.

(O) The principal use to which this problem is applied, is in questions of a similar nature with the preceding; but several other things may be determined from the same *data;* such as the hour from noon when each altitude was taken, the azimuth at each observation, the difference of azimuths, &c. It will be sufficient in this place to point out the method of solution of these less useful cases.

The hour from noon when the second altitude was taken, is measured by the angle DNZ, for ⊙ is the place of the sun when on the meridian. Now in the triangle DNZ, we have ND = 70°.21', DZ = 39°.35', ZN = 38°.28'.10", and the angle NDZ = 29°.12'.42", to find the angle DNZ the hour from noon = 30° or 2 hours; consequently the second altitude was taken at 10 o'clock, at the first at $8^h.30'$ in the forenoon. Again, in the triangle ANZ we have AN = 70°.21', the sun's polar distance at the first observation, AZ = 51°.41' his zenith distance, and the angle ANZ = 3h.30' or 52°.30' the hour from noon when the first altitude was taken, to find the angle AZN = 107°.47' the azimuth from the North at the first observation. Also, in the triangle DNZ we have DN = 70°.21′ the sun's polar distance at the second observation, DZ = 39°.35′ the zenith distance, and the angle DNZ = 30° or 2 hours, the hour from noon when the second altitude was taken, to find the angle NZD = 132°.21′.12″ the azimuth from the north at the second observation. The difference of the azimuth is therefore = 24°.34′.12″. The azimuths might have been found independent of the time, for in each of the above triangles, the co-declination, the co-altitude, and the co-latitude are given.

EXAMPLE II. (*Plate III. Fig.* 11.)

When the sun's declination was 22°.40′ North, his altitude at $10^{h}.54'$ in the forenoon was 53°.29′, and at $1^{h}.17'$ in the afternoon it was 52°.48′; required the latitude of the place of observation, supposing it to be North.

$12^{h}.\ 0'$
Time of the first alt. 10. 54

The time from noon 1. 6
The time from noon 1. 17 when the second alt. was taken.

Elapsed time 2. 23 = 35°.45′ the angle AND.

Half elapsed time $1^{h}.11'.30''$ = 17°.52′.30″ the angle GND.

Supposing the sun's declination constant during the elapsed time, AND is an isosceles triangle, and the method of solution is precisely the same as in the former example.

I. For ND = AN = 67°.20′, and the angle GND = 17°.52′.30″, whence the angle GDN is found = 82°.54′.56″, and AD = 32°.54′.22″.

II. In the triangle ADZ are given AZ the first co-alt. = 36°.31′, DZ the second co-alt. = 37°.12′, and the side AD = 32°.54′.22″, to find the angle ADZ = 65°.44′.40″; hence the difference between GDN and ADZ gives NDZ = 17°.10′.16″.

The ∠ GDN being greater than the ∠ ADZ, this example belongs to Figure 11.

III. In the triangle DZN there are given ND the polar distance = 67°.20′, DZ the second co-altitude = 37°.12′, and the angle NDZ = 17°.10′.16″, to find the co-lat. NZ = 32°.51′.36″; hence the latitude is 57°.8′.24″.

The azimuth at the time of each observation may be found as in the preceding example.

(P) If the times as given in Example II. and such like, were exactly true, one altitude *only* would at all times determine the latitude. But we must not depend upon these times; they only serve to point out the elapsed time between the observations. If the watch go true during the elapsed time, or measure that time truly, it is of no consequence to the solution of the problem, whether it be too fast or too slow.

The methods of solving this important problem directly by trigonometry, as is done in the foregoing examples, have been considered by mathematicians as too troublesome for general practice. They have, therefore, contrived various other rules dependent upon an assumed latitude, in such a manner that when the latitude assumed is true, the result of the operation will exactly agree with it. But when the latitude obtained by calculation differs materially from the assumed latitude, the solution must be repeated.

The rule now chiefly used, and which is printed in almost every Treatise on Navigation, the Requisite Tables, &c., was first given by Mr. Douwes, examiner of the marine cadets at Amsterdam; this rule, and tables adapted to it, were first published in England, by Mr. R. Harrison, in the year 1759, but without any account of the construction of the tables. In the year 1760 the learned Dr. Pemberton gave a very circumstantial account of the construction of the tables, and the reason of the process; showed the nature and limits of the approximation, and estimated the errors which may arise from the several circumstances that occur in it. Vide Phil. Trans. 1760.

The following rule depending only on the elapsed time, the two altitudes and declination, is deduced from Examples I. and II. It is in all cases accurate, and in general less troublesome in the process than the common method by assumed latitudes.*

(Q) GENERAL RULE.

I. Radius, is to the sine of the polar distance; as the sine of half the elapsed time, is to the sine of half the first arc, the double of which gives *arc the first.*

II. The co-tangent of half the elapsed time, is to radius; as the cosine of the polar distance, is to the co-tangent of *arc the second.* This arc is acute or obtuse like the polar distance.

III. Add the two zenith distances and arc the first into one sum. From half this sum subtract the greatest zenith distance.

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* Dr. Brinkley, Professor of Astronomy in the University of Dublin, has published a set of rules for solving this problem, which are annexed to the Nautical Almanac for the year 1822.

Add together the co-secant of arc the first, the co-secant of the least zenith distance, (rejecting the indices,) the sine of the half sum, and the sine of the remainder; half this sum is the logarithmical cosine of half the third arc, double of which gives *arc the third.* The *difference** between arc the third and arc the second gives *arc the fourth.*

IV. Add the sine of the polar distance, the sine of the least zenith distance, and double the sine of half the fourth arc into one sum, and reject 30 from the index. Consider the remainder as a common logarithm, and find the natural number answering to it, to as many places of figures as the tables extend to, if the index be 9; if 8, to one less; if 7, to two less; and if 6, to three less (for this index can never exceed 9, nor fall short of 6). Subtract double this natural number from the natural cosine of the difference between the polar distance and the least zenith distance, the remainder will be the natural sine of the true latitude.

EXAMPLE III.

In a supposed latitude of 50°.41′ N. at $10^{h}.17'.30''$ *per* watch, the true altitude of the sun's centre was 17°.13′, and at $11^{h}.17'.30''$ it was 19°.41′, the sun's declination on that day was 20° South; required the true latitude independent of the supposed one?

Here $11^{h}.17'.30''-10^{h}.17'.30''=1^{h}$ the elapsed time, hence ½ the elapsed time $=30^{m}.=7°.30'$.

I.			
Rad, sine of	90°	-	10·00000
: sine pol. dist.	110°	-	9·97299
: : sine ½ elap. time	7°.30′		9·11570
: sine ½ arc 1st	7°.2′.43″		9·08869
Arc 1st = 14°.5′.26″			

II.			
Cot ½ elap. time	7°.30′	-	10·88057
: rad, sine of	90°.	-	10·00000
: : cos. pol. dist.	110°.	-	9·53405
: cot arc 2d	92°.34′.41″		8·65348

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* 1. When the supposed latitude, or latitude by account, is less than, or nearly equal to, the declination, and both the same way, the magnetic azimuths of the sun ought to be attended to at the time of taking the altitudes. For when the magnetic azimuth at taking the least altitude is greater than the magnetic azimuth at taking the greatest altitude, counting from the elevated pole; the *supplement* of arc the second must be *added* to the *supplement* of arc the third to obtain *arc the fourth.*

2. Should the double natural number used near the conclusion of the problem, at any time exceed the natural cosine of the difference between the least zenith distance and polar distance, the latitude will be of a different name with the supposed latitude.

3. Hence it appears that in places near the equator where the azimuths increase and decrease slowly, the above method of solution requires a little more attention. However, should any doubt remain whether the latitude found by the general rule be true or not: find the fourth arc as directed in the first note, and use it instead of that found by the rule; one of the two results will most certainly be the true latitude, and some circumstances will generally occur that will seldom fail to point out which is the proper one.

III. Arc first - - = 14°. 5′.26″ co-secant ·61359

Greatest zen. dist. 172. 47

Least zen. dist. 70. 19 - co-secant ·02615

2 | 157. 11. 26

Half sum - - 78. 35. 43 sine - 9·99134

Greatest zen. dist. 72. 47

Remainder - - 5. 48. 43 sine - 9·00544

2 | 19·63652

Cos ½ third arc 48°.50′.56″ - - 9·81826

2

Third arc - = 97. 41. 52

Second arc = 92. 34. 41

Fourth arc - = 5. 7. 11

½ fourth arc - = 2. 33. 35

IV. Sine polar distance 110° - - - 9·97299

Sine least zen. dist. 70°.19′ - - 9·97385

½ fourth arc - - 2°.33′.35″ sine × 2 =17·29986

Nat. number of this com. log. - - 176=7·24668

2

352

Nat. cos. diff. polar dist.
and least zen. dist.=39°.41′ - =76959

Sine true lat - 50° - =76607

Note. This example, by the approximating rules, requires a repetition of the process, vide page 18. Requisite Tables, Example 2.

EXAMPLE IV.

In a supposed latitude of 51°.0′ North, when the sun's declination was 22°.23′ South, some time past noon the sun's corrected altitude was found by observation to be 14°.46′ and 1^h.22′ afterwards, his altitude was 8°.27′; required the true latitude independent of the supposed one?

Answer. The true latitude is 50°.33′.44″ N.

EXAMPLE V.

In a supposed latitude of 47°.19′ North, when the sun's declination was 12°.16′ North, some time in the afternoon the true altitude of the sun's centre was observed to be 51°.59′, and $2^{h}.50'$ afterward it was 49°.9′; required the true latitude independent of the supposed one?

Answer. The true latitude is 49°.19′.30″ North.

EXAMPLE VI.

In a supposed latitude of 30°.0′ North, when the sun's declination was 10°.24′ North, at $8^{h}.54'$ in the forenoon the sun's true altitude was 41°.30′; and at $1^{h}.24'$ in the afternoon his altitude was 61°.47′; required the true latitude independent of the supposed one?

Answer. The latitude is 31°.33′.26″ North.

EXAMPLE VII.

In a supposed latitude of 40°.0′ North, when the sun's declination was 2°.46′ South, at $9^{h}.20'$ in the forenoon the sun's altitude corrected was 33°.11′, and at $1^{h}.20'$ in the afternoon his altitude was 42°.44′; required the true latitude independent of the supposed one?

Answer. The latitude is 40°.50′.10″ North.

EXAMPLE VIII.

In a supposed latitude of 50°.40′ North, when the sun's declination was 20° South, the true altitude of the sun's centre was 19°.41′, and one hour afterward his altitude was 17°.13′; required the true latitude independent of the supposed one?

Answer. The true latitude is 50° North.

EXAMPLE IX.

In a supposed latitude of 60° North, when the sun was on the equinoctial, or had no declination, the true altitude of his centre was 28°.53′, and two hours afterwards his corrected altitude was 20°.42′; required the true latitude independent of the supposed one?

Answer. The true latitude is 59°.59′.32″.

In this example the polar distance is equal to 90°; and in all such examples, the elapsed time will be arc the first, and arc the second will be 90°.

EXAMPLE X.

In North latitude when the sun's declination was 23°.29′ North, at $8^{h}.54'$ in the forenoon the sun's corrected altitude was 48°.42′, and at $9^{h}.46'$ the altitude was 55°.48′; required the true latitude? *British Palladium* 1773, *page* 72.

Answer. The true latitude is 49°.49′.28″ North.

EXAMPLE XI.

In a supposed latitude of 6°.50′ North, at $7^{h}.30'$ in the forenoon the true altitude of the sun's centre was 22°.30′, and at $10^{h}.36'.40''$ his altitude was 63°.40′, the sun's declination being 22°.48′ North; required the true latitude, without using the supposed one?

Here $10^{h}.36'.40'' - 7^{h}.30' = 3^{h}.6'.40''$ the elapsed time, hence half the elapsed time $= 1^{h}.33'.20'' = 23°.20'$.

I.			II.		
Rad, sine of	90° -	10·00000	Cot ½ elap. time	23°.20′ -	10·36516
: sine pol. dist.	67°.12′	9·96467	: rad, sine of	90° - -	10·00000
: : sine ½ elap. time	23°.30′	9·59778	: : cos pol. dist.	67°.12′ -	9·58829
: sine ½ arc 1st	21°.25′	9·56245	: cot arc 2d	80°.30′.37″	9·22313

Arc 1st = 42°.50′.

III.	Arc first - - -	42°.50′	co-secant	·16758
	Greatest zenith dist.	67°.30′		
	Least zenith dist.	26°.20′	co-secant	·35302
	2	136°.40′		
	Half sum - -	68°.20′	sine -	9·96818
	Greatest zenith dist.	67°.30′		
	Remainder -	0°.50′	sine -	8·16268
			2	18·65146
	Cos ½ arc third	77°.46′.40″ 2	- -	9·32573
	Third arc - -	155°.33′.20″		
	Second arc - -	80°.30′.37″		
	Fourth arc - -	75°. 2′.43″		
	½ fourth arc -	37°.31′.21″		

IV. Sine polar distance 67°.12′ - - - 9·96467
Sine least zenith dist. 26°.20′ - - - 9·64698
½ fourth arc - - 37°.31′.21″ sine × 2 = 19·56933

Nat. Numb. - - - - - - = 15170 = 9·18098
2

Nat. cos. diff. polar dist. 30340
and least zenith distance - - = 75623

Nat. sine lat. - - 26°.55′ - 45283

This latitude being so very different from the latitude by account, it will be necessary to examine it by the notes to the rule.

There can be no error, except in the fourth arc, and this is examined with very little additional trouble, there being only three numbers to take out of the tables:

Arc the second = 80°.30′.37″ its supplement = 99°.29′.23″
Arc the third = 155°.33′.20″ its supplement = 24°.26′.40″

Sum = arc the fourth 123°.56′. 3″

½ arc fourth 61°.58′
Sine polar distance 9·96467
Sine least altitude 9·64698
½ fourth arc 61°.58′; sine × 2 = 19·89160

Natural numb. of this common log. 31860 = 9·50325
2

63720
Natural cosine of the diff. polar dist.
and the least zenith distance - 75623

Sine latitude - - - 6°.50′ N. 11903

One of the two latitudes found above is certainly the true one, the latter, being the nearest to the latitude by account, may be taken as the proper one.

EXAMPLE XII.

In latitude 1°.50′ North, by account at 10h.24′ in the forenoon the true altitude of the sun's centre was 64°.59′, and at 11. 20′ it was 78°.57′, the sun's declination being 0°.30′ North;

required the true latitude, without making use of the supposed one?

Answer. The latitude by the rule is 1°.36′.24″ North, but the place being so near to the equator, it will be proper to examine arc the fourth as directed in the notes to the rule. By this method the latitude is 0°.39′ South. The former is nearest the latitude by account.

(R) In all the preceding examples the two observations of the sun's altitude are supposed to have been made at the same place, and the latitude determined by the solution agrees to that place. Although the two altitudes are generally taken at the same place at land, yet at sea that is seldom the case. An allowance, therefore, ought to be made for the run of the ship during the elapsed time; thus, find the angle contained between the ship's course and the sun, if it be eight points no correction is necessary, but if less or more than eight points the correction must be applied to the first altitude, by addition or subtraction. Consider the angle contained between the ship's course and the bearing of the sun as a course, the distance made good during the elapsed time as a distance; with these find a difference of latitude and apply it as above.* The result will reduce the first altitude to what it would have been if taken at the same place where the second was taken. The latitude must be found with these altitudes, thus corrected, in the same manner as before: this will be the latitude of the place where the second altitude was taken.

The difference of longitude during the elapsed time may likewise be taken into consideration, though in general it is of little or no consequence. The change of the sun's declination during the elapsed time might likewise be considered, but this, like the longitude, will cause no sensible error; particularly, if the declination answering to the middle time between the observations be used.

(S) PROBLEM XVII. (*Plate III. Fig.* 12.)

Given the apparent distance of the moon from the sun, or a star, and their apparent zenith distances, to find their true distance, as seen from the earth's centre.

Let ZM be the observed zenith distance of the moon, Z*m* the true zenith distance; M*m* being the difference between the moon's refraction and her parallax in altitude. And let Z ⊙ represent the observed zenith distance of the sun or of a star; ZS the true zenith distance, ⊙ S being the difference between the sun's refraction and his parallax, or the refraction of a star.

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* Vide Mr. Cotes's De estimatione errorum in mixtâ Mathesi.

GENERAL PRINCIPLES.

I. Find the segments of the base ⊙V and VM, (by the rule W. 231.). With ⊙Z and ⊙V find the angle Z⊙V, which will be equal to the angle T⊙S, (N. 135.) With ⊙S, and the angle T⊙S, find ⊙T, which will be equal to SP. The triangle T⊙S, being indefinitely small, may be considered as a plane triangle.

Again, with VM and ZM find the angle ZMV, with M*m* and the angle ZMV find the base RM, considering the right-angled triangle *m*RM as a plane triangle.

Lastly, M⊙ + SP − RM = S*m* the true distance in all cases except where the angle at the zenith is acute, and the angle at the moon obtuse, then M⊙ + SP + RM = S*m*.

II. Or, with Z⊙, ZM, and ⊙M find the vertical angle ⊙ZM, (P. 248.), and with ZS, Z*m*, and the angle ⊙ZM, find the true distance S*m*. (N. 245.)

The various methods which have hitherto been made use of for determining the distance between the moon and the sun, or a star, are derived from one or other of the above principles. Those methods which are derived from the latter, are generally preferable to those derived from the former, as being more correct and simple. When the observed distance is small, or the moon's parallax great, and the star's refraction considerable, two other corrections are necessary in order to render the first of these principles generally correct. We have considered the little triangles as right-angled, but the fact is NP⊙ and NRM are each of them isosceles triangles, and, therefore, ⊙P and R*m* are not strictly perpendicular to S*m* and ⊙M; these corrections, therefore, consist in determining how much SP and RM deviate from the bases of right-angled triangles. A true method of determining these *four* corrections may be seen in the Edinburgh Transactions, Article VII., *Physical Class.*

(T) INVESTIGATION OF A GENERAL RULE FOR DETERMINING THE TRUE DISTANCE OF THE MOON FROM THE SUN, OR FROM A FIXED STAR. (*Plate III. Fig.* 13.)

Let ⊙M be the observed distance, and S*m* the true distance. Also, let ZM be the observed zenith distance of the moon and Z*m* the true zenith distance; Z⊙ the observed zenith distance of the sun or of a star, and ZS the true zenith distance, as above. Then, by one of the formulæ F. 184, we have, in the triangle ⊙Z*m*,

$$\frac{2\ \text{sine}^2\ \tfrac{1}{2}\ \text{Z}}{\text{rad}^2} = \frac{\text{rad} \ .\ \cos\left(\text{Z}\odot \sim \text{ZM}\right) - \text{rad} \ .\ \cos \odot\text{M}}{\text{sine Z}\odot \times \text{sine ZM}}$$; and, likewise

in the triangle SZ*m*, by the same formula,

$$\frac{2\ \text{sine}^2\ \tfrac{1}{2}\text{Z}}{\text{rad}^2} = \frac{\text{rad} \ .\ \cos\left(\text{ZS} \sim \text{Z}m\right) - \text{rad} \ .\ \cos \text{S}m}{\text{sine ZS} \times \text{sine Z}m}.$$

By putting these values equal to each other, and dividing by the radius, we obtain

$$\frac{\cos(z\odot \sim z\mathrm{M}) - \cos \odot\mathrm{M}}{\text{sine } z\odot \times \text{sine } z\mathrm{M}} = \frac{\cos(\mathrm{zs} \sim zm) - \cos sm}{\text{sine } \mathrm{zs} \times \text{sine } zm}.$$

Hence, $\cos sm - \cos(\mathrm{zs} \sim zm) = \cos \odot\mathrm{M} - \cos(z\odot - \sim z\mathrm{M}) \times \frac{\text{sine } \mathrm{zs} \times \text{sine } zm}{\text{sine} z\odot \times \text{sine} z\mathrm{M}}$.

Now $\mathrm{zs} \sim zm$ and likewise $z\odot \sim z\mathrm{M}$ will always be acute, being each less than 1°; but $\odot\mathrm{M}$ and sm, being each of the same species, may be either acute or obtuse, therefore when the observed distance is more than 90°,

$$\operatorname{Cos} sm = \cos\odot\mathrm{M} + \cos(z\odot \sim z\mathrm{M}) \times \frac{\text{sine } \mathrm{zs} \times \text{sine } zm}{\text{sine } z\odot \times \text{sine } z\mathrm{M}} \sim \cos(\mathrm{zs} \sim zm).$$

And when the observed distance is less than 90°,

$$\operatorname{Cos} sm = \cos(\mathrm{zs} \sim zm) - \cos\odot\mathrm{M} \times \frac{\text{sine } \mathrm{zs} \times \text{sine } zm}{\text{sine } z\odot \times \text{sine } z\mathrm{M}} \sim \cos(\mathrm{zs} \sim zm.)$$

Hence the following

GENERAL RULE.*

To the natural cosine of the difference between the apparent altitudes†, add the natural cosine of the apparent distance if more than 90°, or subtract it if less than 90°, and find the common logarithm of the remainder; to which add the logarithm secants of the apparent altitudes, the logarithm cosines of the true altitudes, and reject the tens from the index. The difference between the natural number answering to this sum in the table of logarithms, and the natural cosine of the difference between the true altitudes, will give the natural cosine of the true distance.

NOTE. The moon's correction, which is the difference between the refraction and the parallax in altitudes, must always be added to the apparent altitude to obtain the true altitude;

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* Several other rules, differing in the form of expression, may be deduced from the foregoing demonstration, but this has been preferred on account of its shortness, and the ease with which it may be applied: requiring no other tables in its application than those which are common and well known.

A collection of short rules, without demonstration, may be seen in Nicholson's Philosophical Journal, for November 1806, vol. XV. page 254.

Dr. Mackay's 1st Method (page 150, vol. I. 3d edition) of his valuable treatise on the longitude, is the simplest I have ever met with, when his tables are used.

Mendoza's method is likewise very short and easy, but his tables are large and expensive.

† The apparent altitudes, are the observed altitudes corrected for dip and semidiameter, and the apparent distance, is the observed distance corrected for semidiameters.

but the sun or star's correction must always be subtracted. The sun's correction is the difference between the refraction (Table IV.), and the parallax in altitude (Table VI.), and a star's correction is the refraction (Table IV.).

EXAMPLE I.

The apparent distance of the moon's centre from the star Regulus was 63°.35′.13″, when the apparent altitude of the moon's centre was 24°.29′.44″, the apparent altitude of the star 45°.9′.12″; and the moon's horizontal parallax 55′.2″; required the true distance?

☽'s horizontal parallax = 55′.2″, the parallax in altitude = 50′.5″ (T.96.), refraction = 2′.4″ (Table IV.) hence the ☽'s correction = (50′.5″ − 2′.4″ =) 48′.1″, and the ✱'s correction = 57″. (Table IV.)

Diff. app. altitudes 20°.32′.28″ Nat. cos. 93570
Apparent distance 63°.35′.13″ Nat. cos. 44485

Common log. - - 49085 = 4·69095

Moon's apparent altitude 24°.29′.44″ log. sec. 10·04097
Star's apparent altitude[*] 45°. 9′.12″ log. sec. 10·15169
Moon's corrected altitude 25°.17′.45″ log. cos. 9·95622 [†]
Star's corrected altitude 45°. 8′.15″ log. cos. 9·84843

Natural number - - - - - - 48782 4·68826

Diff. true altitudes 19°.50′.30″ Nat. cos. 94063

True distance - 63°. 4′.33″ Nat. cos. 45281

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[*] Instead of the logarithmical secants of the apparent altitudes, you may take the logarithmical cosines, add them together and take the sum from 20, the remainder will be the same as the sum of the logarithmical secants without the indices.

[†] The sum of these four logarithms, rejecting the tens, is 9·99731. This number may be found at once from the XLIId Table of Dr. Mackay's treatise on the Longitude, 3d edition, using the ☽'s horizontal parallax 55′.2″, and the apparent altitude 24°.29′.44″. The operation would be rendered shorter by such a table, but it occupies 45 pages of *Royal Octavo*, and, therefore, could not be inserted in this work. The IXth of the Requisite Tables is not so extensive, but then it is proportionably imperfect. The author of the article *Longitude* in Dr. *Rees's* new Cyclopedia has been pleased to remark, that the rule given by me may be simplified by using the IXth of the Requisite Tables, or Dr. Mackay's XLIId Table; this I was fully aware of: my object was not to give a *short* rule for solving the problem by auxiliary tables not in common use, but to render the subject plain and easy to a learner without the help of any other tables than those at the end of this treatise.

EXAMPLE II.

The apparent distance of the moon's centre from the sun's was 106°.46′.44″, when the apparent altitude of the sun's centre was 45°.32′.30″, and the moon's 19°.43′.22″; the moon's correction 50′.3″, and the sun's 50″; required the true distance of their centres?

Diff. of app. alts. - 25°.49′. 8″ Nat. cos. 90018
Apparent distance 106°.46′.44″ Nat. cos. 28868

Sum, common log. -	118886=		5·07513
Moon's app. altitude - -	19°.43′.22″	log. sec.	10·02625
Sun's app. altitude - -	45°.32′.30″	log. sec.	10·15466
Moon's corrected alt. -	20°.33′.25″	log. cos.	9·97143
Sun's corrected alt. - -	45°.31′.40″	log. cos.	9·84544
Natural number - - - - - -	118281	- -	5·07291

Diff. of true alts. 24°.58′.15″ Nat. cos. 90652

True distance 106°. 2′.19″ Nat. cos. 27629

EXAMPLE III.

The apparent distance of the sun and moon's centres was 68°.42′.11″, when the apparent altitude of the sun's centre was 32°.0′.1″, and that of the moon's 24°.0′.10″; the sun's correction was 1′.23″, and the moon's 51′.1″; required the true distance of their centres?

Answer. 68°.19′.46″.

EXAMPLE IV.

The apparent distance of the moon's centre and a star was 2°.20′, when the apparent altitude of the star's centre was 11°.14′, and that of the moon's 9°.39′; the moon's correction was 51′.30″, and the star's 4′.40″; required the true distance of their centres?

Answer. 1°.49′.

EXAMPLE V.

The apparent distance of the moon from the star Spica was observed to be 31°.17′.53″, when the apparent altitude of the moon's centre was 18°.56′.45″, and the apparent altitude of the star 20°.10′.56″; the moon's correction was 50′.46″, and the star's 2′.35″; required the true distance of their centres?

Answer. The true distance is 31°.11′.44″.

EXAMPLE VI.

The apparent distance of the moon's centre from that of the sun was 70°.8′.23″ at a time when the apparent altitude of the sun's centre was 22°.14′.55″, and the apparent altitude of the moon's centre 80°.52′.22″; the sun's correction was 2′.11″, and the moon's correction 8′.47″; required the true distance of their centres?

Answer. 70°.7′.56″.

EXAMPLE VII.

The apparent distance of the sun and moon's centre was 63°.5′.46″, the apparent altitude of the sun 45°.9′.12″, that of the moon 24°.29′.40″; the sun's correction 57″, and the moon's 47′.47″; required the true distance of their centres?

Answer. 62°.44′.

EXAMPLE VIII.

The apparent distance of the sun and moon's centres was 72°.21′.40″, the apparent altitude of the moon 19°.19′, that of the sun 25°.16′; the moon's correction being 50′.40″, and the sun's 1′.53″; required the true distance of their centres?

Answer. 72°.3′.51″

PROBLEM XVIII.

(U) *The latitude of a place and its longitude by account, the distance between the sun and the moon, or the moon and a star* in the* NAUTICAL ALMANAC, *being given, to find the correct longitude.*

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* The principal stars used for finding the longitude at sea are the following: α *Arietis* in the head of Aries. This is a small star, without the zodiac, and cannot be readily found and applied by the generality of persons; it appears about 22° to the right hand of the Pleiades.

2. *Aldebaran* in the Bull's eye is easily distinguished by its largeness, colour, and position to the other stars, being half way between the Pleiades and the star which forms the western shoulder of Orion.

3. α *Pegasi* or *Markab*, about 44° to the right hand of Arietis, a line drawn in imagination from the Pleiades through Arietis will pass through α Pegasi.—The constallation Pegasus is very remarkable, the three principal stars in it, with the head of Andromeda, form a large square, of which the four corner stars are all of the second magnitude.

4. *Pollux*, a little northward of Aldebaran, and about 45° towards the left hand, there are two stars nearly together: the right hand one is Castor, the left hand one Pollux.

5. *Regulus*, about 38° S. E. of Pollux, is easily distinguished by being the southern-most of four bright stars resembling the letter Z inverted, it lies to the north-east of Aldebaran.

I. Turn the longitude, by account, into time (X. 265 *and* Z. 266.) and add it to the astronomical time* where you are if west longitude, but subtract it from that time if east, and you will have the astronomical time at Greenwich nearly, which call *reduced time.*

II. In page VII. of the Nautical Almanac, for the given month and day, take out the moon's semi-diameter, and horizontal parallax, for the noon and midnight between which the reduced time falls, subtract the less semi-diameter from the greater, and the less parallax from the greater.

Then, as 12 hours are to the first difference, so is the reduced time to a fourth number, which must be added to the moon's semi-diameter if the tables be increasing, but subtracted if they be decreasing; to the sum or difference add the augmentation (Table VII.), and you will have the moon's *true semi-diameter* at *reduced time.*

Again, As 12 hours are to the second difference, so is the reduced time to a fourth number, which must be added to or subtracted from the horizontal parallax at the nearest noon, or midnight, preceding the reduced time, according as the tables are increasing or decreasing, and it will give the *horizontal parallax* at *reduced time.* (*See Example* II. *page* 272.)

III. Clear the observed altitude of the moon of dip (Table V.) and semi-diameter †, and you have the *apparent altitude of her centre:* to the cosine of the moon's apparent altitude, add the logarithm of the horizontal parallax at reduced time in seconds; the sum, rejecting 10 from the index, will be the logarithm of

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6. *Spica,* or *α Virginis,* a white sparkling star, about 54° south-east of Regulus.

7. *Antares.* The arc of a great circle passing through Regulus and *Spica Virginis* east-south-east, or to the left hand of Regulus southward, will pass through Antares, which is about 45° from *Spica Virginis.*

8. *Fomalhaut,* about 45° south of *α* Pegasi.

9. *α Aquila,* 47° westward, or to the right hand of *α* Pegasi.

The distance between the moon and the sun, and between the moon and the nine stars above described (being near her path), are given in the VIIIth, IXth, Xth, and XIth pages of the Nautical Almanac, to every three hours apparent time, by the meridian of Greenwich. The observer who uses the Nautical Almanac, and certainly no observer ought to be without it, is under the necessity of taking the distance of one or other of the above stars from the moon. The distances calculated in the Nautical Almanac afford, perhaps, the readiest method of knowing the star from which the moon's distance ought to be observed. For the sextant being fixed to that distance, and the moon found upon the horizon glass, there is nothing more to do than to look to the east or west of the moon, according as the distance corresponds to the VIIIth and IXth, or Xth and XIth pages of the Nautical Almanac, guiding the sextant in a line with the moon's shortest axis.

* By a late order from the admiralty, the Nautical Day begins at midnight, in consequence of which the Navy Logs correspond with the *civil* reckoning; but the tables in the Nautical Almanac are adapted to *Astronomical* time, see the note, page 267.

† *Viz.* take their difference and add it to the observed altitude of the moon's lower limb; or take their sum and subtract it from the observed altitude of the moon's upper limb, according as the lower or upper limb has been observed.

the moon's parallax in altitude in seconds (T.96.), from which take the refraction of the moon in altitude (Table IV.), the remainder will be the *moon's correction.*

The moon's correction must always be added to the *apparent altitude* to obtain the *true altitude.*

IV. *Additional Preparation for the Sun and Moon.*

Clear the observed altitude of the sun of dip and semi-diameter *, and you have the *apparent altitude of his centre.*

From the refraction of the sun's altitude (Table IV.) take his parallax in altitude (Table VI.), and you have the *correction of the sun's altitude;* which must always be subtracted from the apparent altitude to obtain the *true altitude.*

V. To the observed distance of the sun and moon's nearest limbs, add their semi-diameters at reduced time, and the sum will be the *apparent distance* of their centres.

IV. *Additional Preparation for the Moon and a Star.*

From the star's observed altitude take the dip of the horizon (Table V.), the remainder will be its *apparent altitude.* The refraction of a star (Table IV.) is the *correction of its altitude,* and must always be subtracted from the apparent altitude to obtain the *true altitude.*

V. To the observed distance of the moon from a star, add the moon's semi-diameter at reduced time, the sum will be the *apparent distance*; if the farthest limb was observed, subtract the semi-diameter.

VI. *To find the true distance.*

With the apparent altitudes, their corrections, and the apparent distance, find the true distance by the general rule. (T. 333 *and* 334.)

Note. If the watch has not been previously regulated; the true time must now be found with the mean altitude of the sun or star, and the latitude of the ship, as in Prob. XIV. *and* XV. taking care to calculate the sun's declination to the reduced time. (B. 270.)

VII. *To find the longitude.*

In page VIII, IX, X, or XI, of the Nautical Almanac for the given month and day, look for the computed distance between the moon and the other object; if you find it there exactly, the time at Greenwich stands at the top of the column; but if you do not find it exactly, take the nearest distance to it both less and greater; take their difference, and likewise the difference between the computed distance and the *earliest* Ephemeris distance.

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* *Viz.* take their difference, and add it to the observed altitude of the ⊙'s lower limb; or take their sum, and subtract it from the observed altitude of the ⊙'s upper limb, according as the lower or upper limb has been observed.

Then, as the first difference is to 3 hours, so is the second difference to a fourth number, which being added to the time standing over the *earliest* Ephemeris distance, will give the true time at Greenwich.

The difference between the ship's time and the time at Greenwich (turned into degrees) will be the longitude of the place, where the altitude of the object was taken for determining the true time. If the ship's time be greater, the longitude is east; if less, west.

EXAMPLE I.

Suppose on the 6th of January 1822, in longitude 7° west by account, at $1^h.33'$ A. M. per watch well regulated, the distance of the moon's farthest limb from the star Regulus to be 64°.2′; the altitude of the moon's lower limb 24°.18′.30″; the star's altitude 45°.13′.15″; and the eye 18 feet above the surface of the sea, required the true longitude?

I.

Time per watch at ship 5th of January = $13^h.33'$ P.M.
7° longitude W. - - = $0^h.28'$

Reduced time, or time at Greenwich nearly - = $14^h.1'$

II.

☽'s semi-dia. at midnight, 5th Jan. 1822 = 16′.20″ Hor. paral. . . - = 59′.50″
☽'s semi-dia. at noon, 6th Jan. 1822 = 16′.16″ Hor. paral. - - = 59′.41″

* First diff. 0′. 4″ Second diff. 0′. 9″

12^h : 4″ :: $2^h.1'$:	0′. 0″	12^h : 9″ :: $2^h.1'$:	0′ 1″
☽'s semi-diam. at midnight	= 16′.20″	Hor. parallax at midnight -	= 59′.50″
☽'s semi-diam. at red. time	= 16′.20″	Hor. parallax at red. time -	= 59′.49″
☽'s augmentation (Tab. VII.)	= + 7″		60
☽'s true semi-dia. at red. time	= 16′.27″	In seconds	3589

III.

☽'s observed altitude	= 24°.18′.30″	Cos ☽'s apparent altitude	= 9·95897
☽'s semi-dia. = 16′.27″ Dip. (Tab. V.) = 4′. 3″	= + 12′.24″	Horizontal parallax 3589 log	= 3·55497
		Parallax in altitude 3266 log	= 3·51394
App. altitude ☽'s centre	= 24°.30′.54″		
☽'s correction -	= 52′.21″	3266″	= 54′.26″
		☽'s refr. (Tab. IV.)	= 2′. 5″
☽'s corrected altitude -	= 25°.23′.15″		
		☽'s correction -	= 52′.21″

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* The reduced time being so near to midnight, the proportions here made are superfluous, excepting to show the method of proceeding in other cases.

IV.

* 's observed altitude	=45°.13′.15″
Dip (Table V.) -	= − 4′. 3″
* 's apparent altitude -	=45°. 9.12″
* 's refraction (Table IV.) =	− 57″
* 's corrected altitude	=45°. 8′.15″

V.

Observed distance -	=64°. 2′. 0″
☽ 's semi-diameter -	= − 16′.27″
Apparent distance -	=63°.45′.33″

VI. *To find the true distance.*

* 's app. alt.	45°. 9′.12″	Log. sec. -	10·15168
☽ 's app. alt.	24°.30′.54″	Log. sec. -	10·04103
Diff. app. alts.	20°.38′.18″	Nat. cos. 93582	
App. dist.	63°.45′.33″	Nat. cos. 44215	
	Common log. of	49367 =	4·69344
* 's cor. alt.	45°. 8′.15″	Log. cos. -	9·84844
☽ 's cor. alt.	24°.23′.15″	Log. cos. -	9·95589
	Natural number	49032	4·69048
Diff. of true alts.	19°.45′.0″	Nat. cos. 94118	
True dist. -	63°.13′.0″	Nat. cos. 45086	

VII. *To find the longitude.*

Dist. Jan. 5th 1822 at 12^h	=64°.26′.42″
Dist. Jan. 5th 1822 at 15^h	=62°.37′.38″
First difference -	= 1°.49′. 4″

Earliest ephem. distance	=64°.26′.42″
Computed distance -	=63°.13′. 0″
Second difference -	= 1°.13′.42″

Then 1°.49′.4″ : 3^h :: 1°.13′.42″ : 2^h. 1′.38″
Time above the earliest distance = 12^h.

True time at Greenwich = 14^h. 1′.38″
Time at ship = 13^h.33′. 0″

Difference = 0^h.28′.38″ = 7°. 9′.30″ the longitude west of Greenwich, because Greenwich time exceeds the ship's time.

EXAMPLE II.

Suppose on the 26th of April 1822, in longitude 105° E. of Greenwich, by account, at 9^h.16″ P.M. the distance between the sun and moon's nearest limbs to be 68°.9′.30″; the altitude of the sun's lower limb 31°.48′.16″; the altitude of the moon' lower limb 23°.44′.16″; and the eye 18 feet above the level c the sea; required the true longitude.

☽'s semi-diameter at noon, 26th April (*Naut. Alm.*) = 16'.5", and at midnight = 15'.58"; horizontal parallax at noon = 59'.1", and at midnight = 58'.36"; also the ☉'s semi-diameter = 15'.55".

Answer. The reduced time is 2^{h}.16'; ☽'s true semi-diameter = 16'.11"; horizontal parallax = 58'.57"; apparent altitude of the ☽'s centre 23°.56'.24"; and the correction of her altitude = 51'.46".

The apparent altitude of the ☉'s centre = 32°.0'.8"; correction 1'.23"; apparent distance of the ☉ and ☽'s centres 68°.41'.36"; true distance 68°.18'.47", and the true longitude = 105°.37' East.

(*Naut. Alm.* distance at noon being = 67°.6'.15", and at III^{h} = 68°.44'.1".)

EXAMPLE III.

Suppose on the 10th of July 1822, in longitude 116° W., by account, the following observations to be taken:

Time A.M		Alt. ☉'s lower limb.		Alt. ☽'s upper limb.		Distance of nearest limbs.
9^{h}.30'. 0"	-	44°.48'.30"	-	20°.32'. 0"	-	106°.19'.15"
9. 31. 0	-	45. 4. 0	-	20. 18. 30	-	106. 18. 45
9. 31. 50	-	45. 22. 45	-	20. 10. 0	-	106. 18. 30
9. 32. 10	-	45. 45. 0	-	19. 50. 0	-	106. 17. 45
9. 32. 45	-	45. 50. 0	-	19. 30. 0	-	106. 16. 45
5 | 47. 37. 45		5 | 226. 50. 15		5 | 100. 20. 30		5 | 531. 31. 0
Mean 9. 31. 33	-	45. 22. 3	-	20. 4. 6	-	106. 18. 12
Errors of the quadrant		−58	-	−1. 0	-	−2. 37
True mean		45. 21. 5	-	20. 3. 6	-	106. 15. 35

Required the true longitude, the eye being 21 feet above the level of the sea.

☽'s semi-diameter at noon, July 10th (*Naut. Alm.*) = 15'.44", and at midnight 15'.49"; horizontal parallax at noon = 57'.46", at midnight 58'.4"; also the ☉'s semi-diameter = 15'.46".

Answer. The reduced time is 5^{h}.15'.33" July 10th; ☽'s true semi-diameter 15'.52"; horizontal parallax 57'.53"; apparent altitude of her centre 19°.42'.52"; and correction of her altitude 51'.51".

The apparent altitude of the sun's centre is 45°.32'.29"; correction 50"; apparent distance of centres 106°.47'.13"; true distance 105°.51'.29", and the true longitude 116°.0'.2" W. (*Naut. Alm.* distance on the 10th at III^{h} = 107°.2'.33", and at VI^{h} = 105°.28'.18".)

EXAMPLE IV.

Suppose on the 6th of May 1822, in longitude 72°.10′ E. by account, the following observations to be taken:

Times P.M.	Alt. ☽'s lower limb.	Alt. * Spica ♍	Dist. of * from ☽'s far limb.
12^{h}.37′.10″	19°.10′.20″	20°.39′.40″	31°.35′.45″
12. 39. 20	18. 50. 10	20. 29. 20	31. 34. 10
12. 41. 30	18. 40. 30	20. 15. 50	31. 32. 50
12. 43. 20	18. 20. 10	20. 4. 10	31. 31. 35
12. 45. 18	18. 9. 20	19. 47. 30	31. 30. 40
5 | 63. 26. 38	5 | 93. 10. 30	5 | 101. 16. 30	5 | 157. 45.
Mean 12. 41. 19	18. 38. 6	20. 15. 18	31. 33.
Error of the quad.	+ 7. 39	Error — 19	
	18. 45. 45	20. 14. 59	

Required the true longitude? the eye being 18 feet above the level of the sea.

☽'s semi-diameter at noon, May 6th (*Naut. Alm.*) = 14′.42″, and at midnight the same; horizontal parallax at noon 53′.58″, and at midnight 53′.57″.

Answer. The reduced time is 7^{h}.52′.39″, ☽'s true semi-diameter = 14′.48″; horizontal parallax = 53′.58″, apparent altitude of the ☽'s centre = 18°.56′.30″, and her correction = 48′.19″.

App. altitude of the star 20°.10′,56″, correction 2′.35″;
App. distance of centres 31°.18′.12″; true distance 31°.12′.20″.
The true longitude 72°.9′ East.

(*Naut. Alm.* distance at VI^{h} = 30°.16′.54″, and at IX^{h} = 31°.45′.25″).

(W) In all the preceding examples the watch is supposed to have been previously regulated; when that is not the case, the error of the watch must be found from observations of the altitudes of the sun or of a star, taken either before or after that of the distance, as directed in Problems XIV. or XV., pages 316 and 319. Or if the sun or star be sufficiently distant from the meridian, the mean of the sun's or star's altitudes, taken at the same time as the distance is taken, together with the latitude of the place and the sun's declination, &c. may be used to correct the watch; with this corrected time proceed as before.

EXAMPLE V.

At sea, April 26th 1822, in latitude 47°.39′ N. and longitude 178°.20′.30″ West from Greenwich, by account, at 4^{h}.23′ P.M.

per watch not previously regulated; suppose the observed altitude of the sun's lower limb to be 25°.57′.10″, and that of the moon's lower limb 46°.22′.32″, and at the same time the distance of the sun's and moon's nearest limbs to be 75°.33′.37″; required the longitude? The eye being 18 feet above the level of the sea.

☽'s semi-diameter at midnight, April 26th (*Naut. Alm.*) = 15′.58″, and at noon April 27th = 15′.51″; horizontal parallax at midnight = 58′.36″; and at noon April 27th = 58′.11″; also the ☽'s semi-diameter = 15′.55″.

Answer. The reduced time is 16^{h}.16′.22″; ☽'s true semi-diameter 16′.7″; horizontal parallax 58′.28″; apparent altitude of her centre 46°.34′.36″, and correction of her altitude 39′.16″.

The apparent altitude of the sun's centre is 26°.9′.2″; correction 1′.48″; apparent distance of centres 76°.5′.39″; true distance 75°.51′.20″.

The sun's declination (*Naut. Alm.*) on the 26th = 13°.25′.19″ N. on the 27th = 13°.44′.36″ N. which reduced to the time and place of observation (B. 270.) = 13° 38′.23″ N.; with this declination, the ☉'s true altitude 26°.7′.14″; and the co-latitude 42°.21′, find the correct apparent time at ship (F. 316.) = 4^{h}.24′.6″; and hence the true longitude is 177°.51′.45″ west.

(*Naut. Alm.* distance at XV^{h} = 75°.11′.16″, and at XVIII^{h} = 76°.47′.7″.)

CHAP. XIII.

OF THE FLUXIONAL ANALOGIES OF SPHERICAL TRIANGLES.*

PROPOSITION I. (*Plate IV. Fig.* 3.)

(X) *A preparatory Proposition.*

CONSTRUCT a general figure as at Prop. XIX. Book III. Chap. I. (L. 151.) Thus, let A be the pole of the circle HGFE; F the pole of ABH; E the pole of CGI; and C the pole of EDI.

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* See Cote's "De Æstimatione Errorum in mixtâ Mathesi." Cambridge, 1722. Simpson's Fluxions, vol. ii. page 278, et seq.—Crakelt's translation of Mauduit's Trigonometry, Chap. v. page 164, &c.—Traité de Trigonométrie, par M. Cagnoli, Chap. xix. page 310, &c.—La Lande's Astronomy, Paris, 1792, vol. iii. page 588, et seq.; or art. 3997 to 4051, &c. &c.

Suppose these circles to be invariable whilst another great circle DFCB revolves about the pole F, and let C*n* be at right angles to the great circle *mno*FS*d*; then the three triangles ABC, CGF, and EDF will be variable, viz.

I. In the right-angled triangle ABC, the ∠ A will be a fixed quantity, and the other parts will be variable; viz. B*m* will be the increment of AB; *no* the increment of BC; C*o* the increment of AC; and D*s* the increment of the arc ID which measures the ∠ C.

II. In the right-angled triangle CGF, the side FG will be a fixed quantity, and the other parts will be variable; viz. C*o* will be the decrement of CG; *no* the decrement of FC; B*m*, the decrement of the ∠ CFG; and D*s* the increment of the ∠ C.

III. In the right-angled triangle FDE, the hypothenuse EF will be a fixed quantity, and the other parts will be variable; viz. *sd* $=$ *no*, will be the increment of FD $=$ BC; *s*D the decrement of ED; and B*m*, the decrement of the ∠ EFD $=$ ∠ CFG (N. 135.)

PROPOSITION II. (*Plate IV. Fig.* 3.)

(Y) *In any right-angled spherical triangle* ABC, *right-angled at* B, *suppose one of the angles as* A *to remain constant, it is required to find the ratios of the fluxions of the other parts.*

1. In the triangles FB*m*, FC*n*, having the same acute ∠ at F,

sine FB : sine FC :: tang B*m* : tang C*n* (M. 167.)

But FB $=90^\circ$, FC is the complement of BC, and B*m* and C*n*, being very small arcs, have the same ratio to each other as their tangents.

$$\because \text{rad} : \cos \text{BC} :: \text{B}m : \text{CN} = \frac{\cos \text{BC}}{\text{rad}} . \text{B}m.$$

2. In the triangles DCI, C*on*, where the ∠ DCI may be supposed equal to the ∠ C*on*.

Tang DI : sine CI :: tang C*n* : sine *no* (M.167.)

But DI is the measure of the ∠ C, CI $=90^\circ$, and the tangent of C*n* and the sine of *no*, have the same ratio to each other as the arcs,

$$\because \text{tang} \angle \text{C} : \text{rad} :: \text{C}n \left(= \frac{\cos \text{BC}}{\text{rad}} . \text{B}m\right) : no = \frac{\cos \text{BC}}{\text{tang} \angle \text{C}} . \text{B}m.$$

Again, sine DI : sine DC :: sine C*n* : sine C*o* (M. 167.)

But DI $=$ ∠ C, DC $=90^\circ$, and the very small arcs C*n* and C*o*, have the same ratio to each other as their sines.

$$\because \text{sine} \angle \text{C} : \text{rad} :: \text{C}n \left(= \frac{\cos \text{BC}}{\text{rad}} . \text{B}m\right) : \text{C}o = \frac{\cos \text{BC}}{\text{sine} \angle \text{C}} . \text{B}m.$$

3. In the triangles BFm, DFs, where Fs = FD extremely near.

$$\text{sine FB} : \text{sine B}m :: \text{sine F}s : \text{sine D}s. \text{ (M. 167.)}$$

But FB = 90°, Fs = FD = BC, and the small arcs are as their sines.

$$\therefore \text{rad} : \text{B}m :: \text{sine BC} : \text{D}s = \frac{\text{sine BC}}{\text{rad}} . \text{B}m.$$

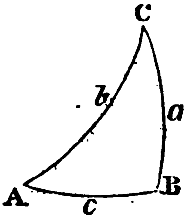
$$\text{Hence } no = \frac{\cos \text{BC}}{\text{tang} \angle \text{C}} . \text{B}m;\ co = \frac{\cos \text{BC}}{\text{sine} \angle \text{C}} . \text{B}m;$$

$$\text{and D}s = \frac{\text{sine BC}}{\text{rad}} , \text{B}m.$$

But no will represent the fluxion of BC; co the fluxion of AC; Ds the fluxion of DI = ∠C; and Bm the fluxion of AB. (B. 126.) Therefore,

$$\dot{\text{BC}} = \frac{\cos \text{BC}}{\text{tang} \angle \text{C}} \times \dot{\text{AB}};\ \dot{\text{AC}} = \frac{\cos \text{BC}}{\text{sine} \angle \text{C}} \times \dot{\text{AB}};\ \angle \dot{\text{C}} = \frac{\text{sine BC}}{\text{rad}} \times \dot{\text{AB}}.$$

(Z) Hence, in any right-angled spherical triangle ABC, right-angled at B; by denoting the sides opposite to the angles A, B, C, by a, b, c, respectively, we derive by substitution and reduction, the following general equations, *which include all the varieties that can possibly happen wherein the* ∠B *is* 90° *and one of the other angles* (*viz.* A) *constant.*



$$\text{*I. } \dot{a} = \frac{\cos a}{\text{tang C}} \times \dot{c} = \frac{\cos \text{C}}{\text{rad}} \times \dot{b} = \frac{\cot a}{\text{tang C}} \times \dot{\text{C}}.$$

$$\text{II. } \dot{b} = \frac{\cos a}{\text{sine C}} \times \dot{c} = \frac{\text{rad}}{\cos \text{C}} \times \dot{a} = \frac{\cot a}{\text{sine C}} \times \dot{\text{C}}.$$

$$\text{III. } \dot{\text{C}} = \frac{\text{sine } a}{\text{rad}} \times \dot{c} = \frac{\text{tang C}}{\cot a} \times \dot{a} = \frac{\text{sine C}}{\cot a} \times \dot{b}.$$

$$\text{IV. } \dot{c} = \frac{\text{tang C}}{\cos a} \times \dot{a} = \frac{\text{sine C}}{\cos a} \times \dot{b} = \frac{\text{rad}}{\text{sine } a} \times \dot{\text{C}}.$$

(A) Any of the foregoing equations may be turned into proportions, or varied in the expression, by reference to pages 104, 105, &c.

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* Simpson's Fluxions, page 280.

Thus, from the fourth set of equations $\dot{c} \times \cos a = \dot{b} \times \sin \angle \text{C}$.

$$\text{that is, } \dot{b} : \dot{c} :: \cos a : \sin \text{c}.$$
$$:: \frac{\cos b}{\cos c} : \frac{\sin c}{\sin b} \text{(O. 169.)}$$
$$:: \cos b \times \sin b : \cos c \times \sin c.$$
$$:: \sin 2b : \sin 2c^{*} \text{ (O. 108.)}$$

PROPOSITION III. (*Plate IV. Fig.* 3.)

(B) *In any right-angled spherical triangle* CGF, *right-angled at* G, *suppose one of the sides as* FG *to remain constant, it is required to find the fluxions of the other parts.*

By the foregoing proposition $no = \frac{\cos \text{BC}}{\text{tang} \angle \text{C}} . \text{B}m$; $co = \frac{\cos \text{BC}}{\sin \angle \text{C}} . \text{B}m$; and $\text{D}s = \frac{\sin \text{BC}}{\text{rad}} . \text{B}m$. But *no* is the decrement of FC; B*m* the decrement of the ∠ F; *co* the decrement of CG; and D*s* the increment of the ∠ C, also FC is the complement of BC. Therefore,

$-\dot{\text{CF}} = \frac{\sin \text{CF}}{\text{tang} \angle \text{C}} \times - \angle \dot{\text{F}}$; $-\dot{\text{CG}} = \frac{\sin \text{CF}}{\sin \angle \text{e}} \times - \angle \dot{\text{F}}$; and $\angle \dot{\text{C}} = \frac{\cos \text{CF}}{\text{rad}} \times - \angle \dot{\text{F}}$; also $\angle \dot{\text{F}} = \frac{\text{tang} \angle \text{C}}{\sin \text{CF}} \times \dot{\text{CF}} = \frac{\sin \angle \text{C}}{\sin \text{CF}} \times \dot{\text{CG}} = - \frac{\text{rad}}{\cos \text{CF}} \times \angle \dot{\text{C}}$.

(C) Hence, if the fixed side FG be represented by *c*, the hypothenuse FC by *b*, the side CG by *a*, and the angles opposite to these sides by C, B, and A respectively (as at Z. 346.) we derive the following general equations, or formulæ, *which comprehend all the different cases that can possibly happen wherein the angle* B *is* 90° *and one of its adjacent sides* (viz. *c.*) *constant.*

$$\dagger \text{ I. } \dot{\text{A}} = \frac{\text{tang C}}{\text{sine } b} \times \dot{b} = \frac{\text{sine C}}{\text{sine } b} \times \dot{a} = - \frac{\text{rad}}{\cos b} \times \dot{\text{C}}.$$

$$\text{II. } \dot{\text{C}} = - \frac{\cos b}{\text{rad}} \times \dot{\text{A}} = - \frac{\text{tang C}}{\text{tang } b} \times \dot{b} = - \frac{\text{sine C}}{\text{tang } b} \times \dot{a}.$$

$$\text{III. } \dot{a} = \frac{\text{sine } b}{\text{sine C}} \times \dot{\text{A}} = \frac{\text{rad}}{\cos \text{C}} \times \dot{b} = - \frac{\text{tang } b}{\text{sine C}} \times \dot{\text{C}}.$$

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* Vince's Trigonometry, 2d edition, page 139; Traité de Trigonométrie, n— M. Cagnoli, art. 677, page 329.

† Simpson's Fluxions, page 280.

IV. $\dot{b} = \frac{\text{sine } b}{\text{tang C}} \times \dot{\text{A}} = \frac{\text{cos C}}{\text{rad}} \times \dot{a} = -\frac{\text{tang } b}{\text{tang C}} \times \dot{a}$.

(D) The preceding formulæ may be varied in the expression by reference to pages 104, 105, &c. or they may be turned into proportions thus, from the first set of equations $\dot{\text{A}} \times \text{sine } b = \text{sine C} \times \dot{a}$, that is,

$$\dot{a} : \dot{\text{A}} :: \text{sine } b : \text{sine C}.^*$$

PROPOSITION IV. (*Plate IV. Fig.* 3.)

(E) *In any right-angled spherical triangle* FDE, *right-angled at* D, *suppose the hypothenuse* EF *to remain constant, it is required to find the fluxions of the other parts.*

By Prop. 2d, $no = \frac{\text{cos BC}}{\text{tang} \angle \text{C}} . \text{B}m$; $co = \frac{\text{cos BC}}{\text{sine} \angle \text{C}} . \text{B}m$; and $\text{D}s = \frac{\text{sine BC}}{\text{rad}} . \text{B}m$.

But *no* is the increment of BC, or of its = FD; B*m* is the measure of the ∠BF*m*, or of its equal *s*FD, and consequently it is the decrement of the ∠ EFD; also BC = FD, and the ∠C is measured by the arc DI which is the complement of ED.

Hence $\dot{\text{FD}} = \frac{\text{cos FD}}{\text{cot ED}} \times -\angle \dot{\text{F}}$.

Again, *co* is the decrement of CG, and CG is the complement of GI the measure of the ∠E, therefore *co* is the increment of the ∠E.

Hence $\angle \dot{\text{E}} = \frac{\text{cos FD}}{\text{cos ED}} \times -\angle \dot{\text{F}}$; also D*s* is the decrement of ED,

Consequently $-\dot{\text{ED}} = \frac{\text{sine FD}}{\text{rad}} \times -\angle \dot{\text{F}}$, and by reduction

$$\angle \dot{\text{F}} = -\frac{\text{cot ED}}{\text{cos FD}} \times \dot{\text{FD}} = -\frac{\text{cos ED}}{\text{cot FD}} \times \angle \dot{\text{E}} = \frac{\text{rad}}{\text{sine F}d} \times \dot{\text{ED}}.$$

(F) Hence, if the hypothenuse EF be represented by *b*, the side FD by *c*, ED by *a*, and their opposite angles by B, C, and A respectively, (as at Z. 346.) we derive the following general formulæ, *which include all the varieties that can possibly happen, wherein the angle* B *is* 90°, *and the hypothenuse* (*viz. b*) *constant.*

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* Vince's Trigonometry, 2d edition, page 140.

$$\text{I. } \dot{\text{A}} = -\frac{\cot a}{\cos c} \times \dot{c} = -\frac{\cos a}{\cos c} \times \dot{\text{C}} = \frac{\text{rad}}{\text{sine } c} \times \dot{a}.$$

$$\text{II. } \dot{\text{C}} = -\frac{\cos c}{\cos a} \times \dot{\text{A}} = -\frac{\text{tang } c}{\cos a} \times \dot{a} = \frac{\text{rad}}{\text{sine } a} \times \dot{c}.$$

$$\text{III. } \dot{c} = -\frac{\cos c}{\cot a} \times \dot{\text{A}} = -\frac{\cot c}{\cot a} \times \dot{a} = \frac{\text{sine } a}{\text{rad}} \times \dot{\text{C}}.$$

$$\text{IV. } \dot{a} = \frac{\text{sine } c}{\text{rad}} \times \dot{\text{A}} = -\frac{\cot a}{\text{tang } c} \times \dot{\text{C}} = -\frac{\cos a}{\cot c} \times \dot{\text{C}}.$$

(G) These formulæ may be varied in the expression or turned into proportions, &c. as before observed. Thus from the third set of equations

$$\dot{\text{C}} = -\frac{\cot c}{\cot a} \times \dot{\text{A}}; \text{ but } \cot c = \frac{\text{rad}^2}{\text{tang } c}, \text{ and } \cot a = \frac{\text{rad}^2}{\text{tang } a},$$

$$\because \dot{\text{C}} = -\frac{\text{tang } a}{\text{tang } c} \times \dot{a}. \text{ Hence } \dot{a} : \dot{c} :: \text{tang } c : \text{tang } a.^*$$

PROPOSITION V. (*Plate IV. Fig.* 4.)

(H) *In any oblique-angled spherical triangle* ABC, *suppose the angle* A *and its adjacent side* AB *to remain constant* †, *it is required to find the fluxions of the other parts.*

Let BC, by its revolution about B, be changed to B*m*, then AC will become A*m*, and the ∠ ABC will be reduced to the ∠ AB*m*. Now, *m*C will be the decrement of AC, the ∠ CB*m* will be the decrement of the ∠ B, and if C*n* be drawn perpendicular to B*n*, *mn* will be the decrement of BC; produce B*n* and BC so that B*p* and B*o* may be quadrants, then *op* will be the measure of the ∠ CB*m*. Hence,

I. C*n* : *op* :: sine BC : sine BO,

and in the small straight-lined triangle C*nm* right-angled at *n*

mn : C*n* :: sine ∠ *m*C*n* : cos ∠ *m*C*n*. (Y. 34.)

∵ *mn* : *op* :: sine BC × sine ∠ *m*C*n* : sine BO × cos ∠ *m*C*n*.

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[*] Vince's Trigonometry, Prop. 49th, page 140, 2d edition.

[†] Traité de Trigonométrie, par Cagnoli, page 311. La Lande's Ast art. 3998 to 4003.

But ∠*mcn* will be the complement of the angle C, because BE*n* is a right angle, and BO = 90° by the construction, hence

$\dot{\mathrm{BC}} : \angle\dot{\mathrm{B}} :: \text{sine BC} \times \cos\angle\mathrm{C} : \text{rad} \times \text{sine}\angle\mathrm{C}.$

$$:: \text{sine BC} : \frac{\text{rad} \times \text{sine}\angle\mathrm{C}}{\cos\angle\mathrm{C}}.$$

$:: \text{sine BC} : \text{tang}\angle\mathrm{C}.$ (N. 104.)

II. Again, rad : *mc* :: sine ∠*mcn* : *mn*. (Y. 34.)

that is, rad : $\dot{\mathrm{AC}} :: \cos\angle\mathrm{C} : \dot{\mathrm{BC}}$.

III. Also, *mc* : *nc* :: rad : cos ∠*mcn*. (Y. 34.)

and *nc* : *op* :: sine BC : sine BO = rad.

∵ *mc* : *op* : sine BC :: sine ∠C,

that is, $\dot{\mathrm{AC}} : \dot{\mathrm{B}}$ :: sine BC : sine ∠C.

IV. By taking the supplemental* triangle to ABC, viz. DFE, (S. 136.) the ∠E and the side FE will be constant quantities, and the rest will be variable; now it is shown in the first case of this proposition, that flux. side opp. a constant ∠ : flux. ∠ adj. to the constant side :: sine of the side opp. to the constant ∠ : tang ∠ opp. to the constant side.†

Hence, $\dot{\mathrm{DF}} : \dot{\mathrm{F}}$ :: sine DF : tang ∠D, that is, (S. 136.)

$\dot{\mathrm{C}} : \dot{\mathrm{AC}}$ :: sine ∠C : tang BC,

and $\dot{\mathrm{B}} : \dot{\mathrm{AC}}$ :: sine ∠C : sine BC. (III. case above.)

∵ $\dot{\mathrm{B}} : \dot{\mathrm{C}}$ :: tang BC : sine BC.

:: rad : cos BC. (U. 103.)

OTHERWISE. (*Plate IV. Fig.* 4.)

(I) Let a perpendicular BD be drawn from the end of the fixed side AB and opposite to the fixed ∠A, then the several parts of the right-angled triangle BDA will be invariable; and in the right-angled triangle BDC, the perpendicular BD will be a fixed quantity, and all the other parts will be variable; hence, by the assistance of the formulæ already given, (C. 347.) all the variations of which this proposition is susceptible may easily be derived; some of the principal will here be inserted, denoting the sides of the triangle by *a*, *b*, *c*, and their opposite angles by A, B, C, as in the figure, where A is the constant angle, and *c* the constant side.

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* Called by the French writers the polar triangle, from the manner in which it is described.

† Cagnoli, page 314, art. 555.

$\dot{b} : \dot{\mathrm{B}} ::$ sine a : sine C. 1st equation C. 347.
$\dot{b} : \dot{a} ::$ rad : cos C. 4th equation C. 347.
$\dot{b} : \dot{c} ::$ tang a : sine C. 3d equation C. 347.
$\dot{a} : \dot{\mathrm{B}} ::$ sine a : tang C. 1st equation C. 347.
$\dot{\mathrm{B}} : \dot{\mathrm{C}} ::$ rad : cos a. 1st equation C. 347.
$\dot{\mathrm{C}} : \dot{a} ::$ tang C : tang a. 2d equation C. 347.

PROPOSITION VI. (*Plate IV. Fig. 5.*)

(K) *In any oblique-angled spherical triangle* ABC, *suppose the angle* A *and its opposite side* BC *to remain constant**, *it is required to find the fluxions of the other parts.*

Let BC change its position to mo, and let these circles intersect each other in an indefinitely small angle at r; make $rn = r\mathrm{B}$, and $r\mathrm{C} = rp$; then, nm will be the decrement of $r\mathrm{B}$, and po will be the increment of $r\mathrm{C}$; and because $np = \mathrm{BC}$, by construction, and $mo = \mathrm{BC}$, by hypothesis; if the common part mp be taken from each, there will remain $nm = op$; then,

I. Considering the indefinitely small triangles $\mathrm{B}nm$, $\mathrm{C}po$ as rectilinear, and right-angled at n and p, we have
In the right-angled triangle $\mathrm{C}po$, rad : $\mathrm{C}o$ :: cos $\angle po\mathrm{C}$: op, and
In the right-angled triangle $\mathrm{B}nm$, rad : $\mathrm{B}m$ :: sine $\angle n\mathrm{B}m$: $nm = op$. (Z. 34.)

$$\because \mathrm{C}o : \mathrm{B}m :: \text{sine} \angle n\mathrm{B}m : \cos \angle po\mathrm{C}.$$

$$\text{that is, } \dot{\mathrm{AC}} : \dot{\mathrm{AB}} :: \cos \angle \mathrm{B} : \cos \angle \mathrm{C}.$$

For, $\mathrm{C}o$ is the increment of AC, $\mathrm{B}m$ is the decrement of AB; and the three angles $p\mathrm{C}o$, $p\mathrm{C}r$, and $\mathrm{AC}r$ are together equal to two right angles (M. 134.), of which $p\mathrm{C}r$ is a right angle, therefore $\angle p\mathrm{C}o + \angle \mathrm{AC}r =$ one right angle $= \angle p\mathrm{C}o + \angle po\mathrm{C}$, hence $\angle po\mathrm{C} = \angle \mathrm{ACB}$.

II. Take the supplemental triangle to ABC, viz. DFE (S. 136.), then, since $\angle$A and BC are constant quantities in the triangle ABC, EF and the $\angle$D will be constant quantities in the triangle DFE; and it has been shown in the first part of this proposition, that the fluxions of the variable sides are as the cosines of their opposite angles, we have
$\dot{\mathrm{DE}} : \dot{\mathrm{DF}} :: \cos \angle \mathrm{F} : \cos \angle \mathrm{E}$, but DE is the measure of the $\angle$B, and DF is the measure of the $\angle$C, also $\cos \angle \mathrm{F} = \cos \mathrm{AC}$, and $\cos \angle \mathrm{E} = \cos \mathrm{AB} \because \angle \dot{\mathrm{B}} : \angle \dot{\mathrm{C}} :: \cos \mathrm{AC} : \cos \mathrm{AB}$.

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* La Lande's Astronomy, vol. iii art. 4008 to 4010. Cagnoli, page 316, &c

III. Again, sine $\angle \mathrm{A}$: sine BC :: sine $\angle \mathrm{C}$: sine AB. (G. 176.)
:: sine $\angle \mathrm{B}$: sine AC.

That is, the sines of the variable angles have a constant ratio to the sines of their opposite sides, consequently their fluxions will be in the same ratio; but the rectangle of radius and the fluxion of the sine of any arc = the rectangle of the fluxion of that arc and its cosine (1st equation page 127.) Hence, flux. sine of $\angle \mathrm{C} = \frac{\cos \angle \mathrm{C}}{\mathrm{rad}} \times \angle \dot{\mathrm{C}}$, flux. sine of $\angle \mathrm{B} = \frac{\cos \angle \mathrm{B}}{\mathrm{rad}} \times \angle \dot{\mathrm{B}}$, flux. sine of $\mathrm{AB} = \frac{\cos \mathrm{AB}}{\mathrm{rad}} \times \dot{\overline{\mathrm{AB}}}$ and flux. sine $\mathrm{AC} = \frac{\cos \mathrm{AC}}{\mathrm{rad}} \times \dot{\overline{\mathrm{AC}}}$, therefore,

$$\text{sine } \angle \mathrm{A} : \text{sine BC} :: \cos \angle \mathrm{C} \times \angle \dot{\mathrm{C}} : \cos \mathrm{AB} \times \dot{\overline{\mathrm{AB}}},$$
$$:: \cos \angle \mathrm{B} \times \angle \dot{\mathrm{B}} : \cos \mathrm{AC} \times \dot{\overline{\mathrm{AC}}},$$

that is, $\angle \dot{\mathrm{C}} : \dot{\overline{\mathrm{AB}}} :: \text{sine} \angle \mathrm{A} \times \cos \mathrm{AB} : \text{sine BC} \times \cos \angle \mathrm{C}$,

and $\angle \dot{\mathrm{B}} : \dot{\overline{\mathrm{AC}}} :: \text{sine} \angle \mathrm{A} \times \cos \mathrm{AC} : \text{sine BC} \times \cos \angle \mathrm{B}$.

But it has been shown in the first part of this proposition, that

$$\dot{\overline{\mathrm{AB}}} : \cos \angle \mathrm{C} :: \dot{\overline{\mathrm{AC}}} : \cos \angle \mathrm{B}.$$

$$\because \angle \dot{\mathrm{C}} : \dot{\overline{\mathrm{AC}}} :: \text{sine} \angle \mathrm{A} \times \cos \mathrm{AB} : \text{sine BC} \times \cos \angle \mathrm{B}.$$

(L) Hence, as in the former propositions, if we denote the sides of the triangle ABC, by a, b, c, and their opposite angles by A, B, C, we derive the following proportions, wherein A and a are invariable quantities.

1. $\dot{b} : \dot{c} :: \cos \angle \mathrm{B} : \cos \angle \mathrm{C}$.
2. $\dot{\mathrm{B}} : \dot{\mathrm{C}} :: \cos b : \cos c$.
3. $\dot{\mathrm{C}} : \dot{c} :: \text{sine } \mathrm{A} \times \cos c : \text{sine } a \times \cos \mathrm{C}$.
4. $\dot{\mathrm{B}} : \dot{b} :: \text{sine } \mathrm{A} \times \cos b : \text{sine } a \times \cos \mathrm{B}$.
5. $\dot{\mathrm{C}} : \dot{b} :: \text{sine } \mathrm{A} \times \cos c : \text{sine } a \times \cos \mathrm{B}$.

These proportions may be varied in their form, thus from the third.

$$\dot{\mathrm{C}} : \dot{c} :: \frac{\text{sine } \mathrm{A}}{\cos \mathrm{C}} : \frac{\text{sine } a}{\cos c}.$$

But sine A : sine a :: sine C : sine c. (G. 176.)

$$\because \dot{\mathrm{C}} : \dot{c} :: \frac{\text{sine } \mathrm{C}}{\cos \mathrm{C}} : \frac{\text{sine } c}{\cos c},$$

viz. $\dot{\mathrm{C}} : \dot{o} :: \text{tang } \mathrm{C} : \text{tang } c$. (N. 104.)

And exactly in the same manner, from the fourth proportion, we derive $\dot{\mathrm{B}} : \dot{b} :: \text{tang } \mathrm{B} : \text{tang } b$.

PROPOSITION VII. (*Plate IV. Fig.* 6.)

(M) *In any oblique-angled spherical triangle* ABC, *suppose the two sides* AB *and* AC *to remain constant**, *it is required to find the fluxions of the other parts.*

Let BC and AC change their positions to B*n* and A*n*, A*n* being equal to AC; produce AC and A*n* to the points *o* and *p*, so that A*o* and A*p* may be quadrants, and join c*n*.

Also, produce BC and B*n* to the points *q* and *r*, making B*q* and B*r* quadrants, and through C draw c*m* parallel to *qr*.

Then *op* will be the measure of the increment of the ∠ A, *qr* the measure of the decrement of the ∠ B, and *mn* will be the increment of the side BC.

I. Sine A*p* : sine A*n* :: sine *op* : sine c*n*,

$$\text{viz. rad} : \text{sine AC} :: \angle\dot{A} : \text{sine } cn = \frac{\text{sine AC}}{\text{rad}} \times \angle\dot{A}.$$

Also, sine B*q* : sine BC :: sine *qr* : sine c*m*,

$$\text{viz. rad} : \text{sine BC} :: \angle\dot{B} : \text{sine } cm = \frac{\text{sine BC}}{\text{rad}} \times \angle\dot{B}.$$

II. In the small straight-lined triangle c*mn*.

1. rad : c*n* :: sine ∠ *m*c*n* : *mn*. (Y. 34.)

$$\text{Viz. rad} : \frac{\text{sine AC}}{\text{rad}} \times \angle\dot{A} :: \text{sine} \angle C : \dot{BC} = \frac{\text{sine AC} \times \text{sine} \angle C}{\text{rad}^2} \times \angle\dot{A}.$$

But sine ∠ C : sine AB :: sine ∠ B : sine AC. (G. 176.)

∵ sine ∠ C × sine AC = sine ∠ B × sine AB, consequently

$$\dot{BC} = \frac{\text{sine} \angle B \times \text{sine AB}}{\text{rad}^2} \times \angle\dot{A}.$$

2. rad : cos ∠ *mn*c :: c*n* : c*m*. (Y. 34.)

$$\text{Viz. rad} : \cos \angle C :: \frac{\text{sine AC}}{\text{rad}} \times \angle\dot{A} : \frac{\text{sine BC}}{\text{rad}} \times \angle\dot{B}.$$

(N) Hence
$$\angle\dot{B} = \frac{\cos \angle C \times \text{sine AC}}{\text{rad} \times \text{sine BC}} \times \angle\dot{A}.$$

3. Cot ∠ *mn*c : rad :: c*m* : *mn* (B. 34.)

(O)
$$\text{Viz. cot} \angle C : \text{rad} :: \frac{\text{sine BC}}{\text{rad}} \times \angle\dot{B} : \dot{BC} = \frac{\text{sine BC}}{\text{c} \angle C} \times \angle\dot{B}.$$

Again, by substituting AB for AC, and ∠ C for ∠ B, we obtain

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* La Lande's Astronomy, vol. iii. art. 4015 to 4034; Cagnoli, page 321, &c

(N. 353.) $\angle\dot{\mathrm{C}} = \dfrac{\cos\angle\mathrm{B}\times\text{sine AB}}{\text{rad}\times\text{sine BC}}\times\angle\dot{\mathrm{A}}$, and

(O. 353.) $\dot{\mathrm{BC}} = \dfrac{\text{sine BC}}{\cot\angle\mathrm{B}}\times\angle\dot{\mathrm{C}}$.

(P) By denoting the three sides of the triangle by a, b, c, and their opposite angles by A, B, C, the following proportions are deduced, where c and b are constant quantities.

1. $\dot{\mathrm{A}} : \dot{a} :: \text{rad}^2 : \text{sine } b\times\text{sine C}$.

2. $\dot{\mathrm{A}} : \dot{a} :: \text{rad}^2 : \text{sine } c\times\text{sine B}$.

But $\dfrac{\text{rad}^2}{\text{sine } b} = \text{cosec } b$, and $\dfrac{\text{rad}^2}{\text{sine } c} = \text{cosec } c$. (Q. 105.)

$\therefore\ \dot{\mathrm{A}} : \dot{a} :: \text{cosec } b : \text{sine C} :: \text{cosec C} : \text{sine } b$,

and $\dot{\mathrm{A}} : \dot{a} :: \text{cosec } c : \text{sine B} :: \text{cosec B} : \text{sine } c$.

3. $\dot{\mathrm{A}} : \dot{\mathrm{B}} :: \text{rad}\times\text{sine } a : \cos\mathrm{C}\times\text{sine } b$.

4. $\dot{\mathrm{A}} : \dot{\mathrm{C}} :: \text{rad}\times\text{sine } a : \cos\mathrm{B}\times\text{sine } c$.

5. $\dot{\mathrm{C}} : \dot{a} :: \cot\mathrm{B} : \text{sine } a$.

$\dot{\mathrm{B}} : \dot{a} :: \cot\mathrm{C} : \text{sine } a$.

6. $\therefore\ \dot{\mathrm{B}} : \dot{\mathrm{C}} :: \cot\mathrm{C} : \cot\mathrm{B}$,

and $\dot{\mathrm{B}} : \dot{\mathrm{C}} :: \text{tang B} : \text{tang C}$.

PROPOSITION VIII. (*Plate IV. Fig.* 6.)

(Q) *In any oblique-angled spherical triangle* ABC, *suppose the two angles* B *and* C *to remain constant**, *it is required to find the fluxions of the other parts.*

Take the supplemental triangle DEF (S. 136.) then DE, and DF will be constant, therefore by Proposition VII.

1. $\angle\dot{\mathrm{D}} : \dot{\mathrm{EF}} :: \text{cosec DF} : \text{sine}\angle\mathrm{F} :: \text{cosec}\angle\mathrm{F} : \text{sine DF}$,

Viz. $\dot{\mathrm{BC}} : \angle\dot{\mathrm{A}} :: \text{cosec}\angle\mathrm{C} : \text{sine AC} :: \text{cosec AC} : \text{sine C}$.

2. $\angle\dot{\mathrm{D}} : \dot{\mathrm{EF}} :: \text{cosec DE} : \text{sine}\angle\mathrm{E} :: \text{cosec}\angle\mathrm{E} : \text{sine DE}$,

Viz. $\dot{\mathrm{BC}} : \angle\dot{\mathrm{A}} :: \text{cosec}\angle\mathrm{B} : \text{sine AB} :: \text{cosec AB} : \text{sine B}$.

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* La Lande's Astronomy, art. 4034 to 4045; Cagnoli, page 325, &c.

3. $\angle\dot{D} : \angle\dot{E} :: \text{rad} \times \text{sine } EF : \cos\angle F \times \text{sine } FD$,

Viz. $B\dot{C} : A\dot{B} :: \text{rad} \times \text{sine}\angle A : \cos AC \times \text{sine}\angle C$.

4. $\angle\dot{D} : \angle\dot{F} :: \text{rad} \times \text{sine } EF : \cos\angle E \times \text{sine } DE$,

Viz. $B\dot{C} : A\dot{C} :: \text{rad} \times \text{sine}\angle A : \cos AB \times \text{sine}\angle B$.

5. $\angle\dot{F} : E\dot{F} :: \cot\angle E : \text{sine } EF$,

and $\angle\dot{E} : E\dot{F} :: \cot\angle F : \text{sine } EF$.

Viz. $A\dot{C} : \angle\dot{A} :: \cot AB : \text{sine}\angle A$,

and $A\dot{B} : \angle\dot{A} :: \cot AC : \text{sine}\angle A$.

6. $\because A\dot{C} : A\dot{B} :: \cot AB : \cot AC$,

and $A\dot{C} : A\dot{B} :: \text{tang } AC : \text{tang } AB$ (C. 103.)

(R) By denoting the three sides of the triangle by a, b, and c, and their opposite angles by A, B, and C, we derive the following proportions, wherein B and C are constant quantities.

1. $\dot{a} : \dot{A} :: \text{cosec } C : \text{sine } b :: \text{cosec } b : \text{sine } C$.
2. $\dot{a} : \dot{A} :: \text{cosec } B : \text{sine } c :: \text{cosec } c : \text{sine } B$.
3. $\dot{a} : \dot{c} :: \text{rad} \times \text{sine } A : \cos b \times \text{sine } C$.
4. $\dot{a} : \dot{b} :: \text{rad} \times \text{sine } A : \cos c \times \text{sine } B$.
5. $\dot{b} : \dot{A} :: \cot c : \text{sine } A$.
6. $\dot{c} : \dot{A} :: \cot b : \text{sine } A$.
7. $\dot{b} : \dot{c} :: \cot c : \cot b$.
8. $\dot{b} : \dot{c} :: \text{tang } b : \text{tang } c$.

SCHOLIUM.

(S) In all the preceding propositions if the sides of the triangle be diminished without limit, the triangle may be considered as rectilinear, and instead of the sines and tangents of the sides, we may substitute the sides themselves (Z. 193.) Hence the variations of plane triangles are readily deduced from those of spherical triangles, in every case where the fluxions are proportional to the sines or tangents of the sides. Thus, by the 4th proportion (P. 354.) we have shewn that $\dot{A} : \dot{C} :: \text{rad} \times \text{sine } a : \cos B \times \text{sine } C$, that is, (supposing th triangle straight-lined) $\dot{A} : \dot{C} :: \text{rad} \times a : \cos B \times c$.

Again, by the 8th proportion (R. 355.) $\dot{b} : \dot{c} ::$ tang b : tang c, that is, when the triangle is straight-lined, $\dot{b} : \dot{c} :: b : c$, and in the same manner the rest may be deduced.

The variations of rectilinear triangles may be deduced from the triangles themselves, without reference to spherical triangles, in a manner exactly similar to those deduced from the spherical triangles. *Vide* Traité de Trigonométrie, par M. Cagnoli, chapitre X.

THE USE OF THE FLUXIONAL ANALOGIES.*

PROPOSITION IX.

(T) *To find when that part of the equation of time dependent on the obliquity of the ecliptic is the greatest possible.*†

Here the sun's longitude will form the hypothenuse of a right-angled spherical triangle, his right ascension will be the base, and the obliquity of the ecliptic will be a constant angle.

Let the hypothenuse be denoted by b, and the base by c.

Then $\dot{b} : \dot{c} ::$ sine $2b$: sine $2c$ (A. 346.)

when $\dot{b} = \dot{c}$, then sine $2b =$ sine $2c$; but when two arcs have equal sines, the one must be the supplement of the other. (K. 31.) Consequently $b + c = 90°$, therefore when $\dot{b} - \dot{c} = 0$, that is when $b - c$ is a maximum, $b + c = 90°$.

The equation of time dependent on the obliquity of the ecliptic is therefore the greatest possible, when the sun's longitude and his right ascension together are equal to 90°. The sun being in the first quadrant of the ecliptic.

PROPOSITION X.

(U) *Given the parallax in altitude of a planet, to find its parallax in latitude and longitude.*

Let B represent the pole of the ecliptic, A the zenith, and C the place of the planet.

Then $\dot{b}$ will represent the parallax in altitude, $\dot{\text{B}}$ the parallax in longitude, and $\dot{a}$ the parallax in latitude.

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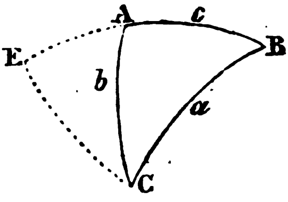
* A variety of examples will be met with in the perusal of La Lande's Astronomy, vol. iii.

† Simpson's Fluxions, page 550.



Now $\dot{\text{B}} = \frac{\text{sine C}}{\text{sine } a} \times \dot{b}$ and $\dot{a} = \frac{\text{cos C}}{\text{rad}} \times \dot{b}$ (I. 350.)

If EC be supposed to represent a part of the ecliptic, then BE and BC will be quadrants (H. 133.) and CEB and ECB will be right angles (I. 134.)

In the right-angled triangle CEA, making EA the middle part, we have $\frac{\text{sine ECA}}{\text{rad}} = \left(\frac{\text{cos C}}{\text{rad}} = \right) \frac{\text{sine EA}}{\text{sine AC}}$, hence by substitution, $\dot{a} = \frac{\text{sine EA}}{\text{sine } b} \times \dot{b}$, and because the altitude of the nonagesimal degree of the ecliptic, is an arc of a great circle comprehended between the zenith of any place of the pole of the ecliptic (R. 265.), we obtain the following proportion.

(W) *Sine of the zenith distance is to cosine of the altitude of the nonagesimal degree; as the parallax in altitude, is to the parallax in latitude,* viz. sine b : sine EA $=$ cos c :: $\dot{b}$. $\dot{a}$.

Again, in the right-angled triangle CEA, making the ∠ ECA the middle part, we have rad × cos ECA $=$ tang EC × cot b; but cot $b = \frac{\text{rad}^2}{\text{tang } b}$ (O. 104.) hence $\frac{\text{cos ECA}}{\text{rad}} = \left(\frac{\text{sine C}}{\text{rad}} = \right) \frac{\text{tang EC}}{\text{tang } b}$, and by substitution $\dot{\text{B}} = \frac{\text{tang EC}}{\text{tang } b} \times \dot{b}$ (because the planet is supposed to be in or very near to the ecliptic, sine $a =$ rad) hence,

(X) *Tangent of the planet's zenith distance, is to the tangent of its longitude from the nonagesimai degree; as the parallax in altitude, is to the parallax in longitude,* viz.

tang b : tang EC :: $\dot{b}$: $\dot{\text{B}}$.

PROPOSITION XI.

(Y) *Given the altitude of the nonagesimal degree of the ecliptic; the longitude of a planet from the nonagesimal degree, and its horizontal parallax, to find its parallax in latitude r longitude.*

It is shewn in the preceding proposition that $\dot{a} = \frac{\cos c}{\text{sine}\, b} \times \dot{b}$ and $\dot{\text{B}} = \frac{\text{tang EC}}{\text{tang}\, b} \times \dot{b}$. Now, if H represent the horizontal parallax of any planet, its parallax in altitude $\dot{b} = \frac{\text{sine}\, b}{\text{rad}} \times \text{H}$ nearly. (T. 96.) By substituting this value of $\dot{b}$ in the above equations $\dot{a} = \frac{\cos c}{\text{rad}} \times \text{H}$, and $\dot{\text{B}} = \frac{\text{tang EC} \times \text{sine}\, b}{\text{tang}\, b \times \text{rad}} \times \text{H}$.

In the right-angled triangle CEA, making b the middle part, $\text{rad} \times \cos b = \cos \text{EA} \times \cos \text{EC}$; but $\cos b = \frac{\text{rad} \times \text{sine}\, b}{\text{tang}\, b}$ and $\cos \text{EC} = \frac{\text{rad} \times \text{sine EC}}{\text{tang EC}}$ (M. 104.), hence $\frac{\text{sine}\, b}{\text{tang}\, b} = \frac{\cos \text{EA} \times \text{sine EC}}{\text{tang EC} \times \text{rad}}$ and by substitution $\dot{\text{B}} = \frac{\text{tang EC}}{\text{rad}} \times \frac{\cos \text{EA} \times \text{sine EC}}{\text{tang EC} \times \text{rad}} \times \text{H} = \frac{\cos \text{EA} \times \text{sine EC}}{\text{rad}^2} \times \text{H}$, but EC is the measure of the ∠B (L. 227.) and $\cos \text{EA} = \text{sine}\, a$, therefore $\dot{\text{B}} = \frac{\text{sine}\, c \times \text{sine B}}{\text{rad}^2} \times \text{H}$; hence are derived the following general rules.

1. *Radius, is to the cosine of the altitude of the nonagesimal degree of the ecliptic; as the horizontal parallax, is to the parallax in latitude.*

2. *The square of the radius, is to the rectangle of the sines of the altitude of the nonagesimal degree and the planet's longitude from thence; as the horizontal parallax, is to the parallax in longitude.**

PROPOSITION XII.

(Z) *To determine the correction for finding the time of apparent noon, from equal altitudes of the sun.*

It is obvious that if the sun's declination were invariable, half the interval of time between equal altitudes would shew the instant of noon; but by the variation in the sun's declination he will have the same altitude at different distances from the meridian: this variation will, in general, be very small, and can only affect the polar distance.

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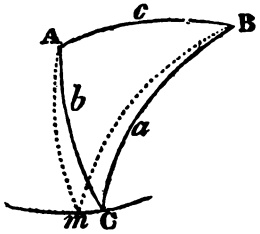
* Simpson's Fluxions, vol. ii. page 286.



If, therefore, we suppose B to represent the pole of the equinoctial, A the zenith, and C the place of the sun; AB and AC will be constant quantities, and BC variable.

Now we have shewn (P. 354.) that $\dot{a} : \dot{b} :: \text{sine } a : \text{cot c}$;
But $\text{cot c} = \frac{(\text{sine } a \times \text{cot } c) - (\cos a \times \cos \text{B})}{\text{sine B}}$ (O. 189.)
therefore by substitution, and dividing by sine of a,
$\dot{\text{B}} = \dot{a} \times \left(\frac{\text{c ot } c}{\text{sine B}} - [\frac{\cos a}{\text{sine } a} \times \frac{\cos \text{B}}{\text{sine B}}\right)$, but $\frac{\cos a}{\text{sine } a} = \frac{\cot a}{\text{rad}}$ and $\frac{\cos \text{B}}{\text{sine B}} = \frac{\cot \text{B}}{\text{rad}}$ (O. 104.) $= \frac{\text{rad}}{\text{tang B}}$, hence $\dot{\text{B}} = \dot{a} \times \left(\frac{\cot c}{\text{sine B}} - \frac{\cot a}{\text{tang B}}\right)$. The half of this expression, reduced into time, will be the correction required, where $\cot c =$ tangent of the latitude, and $\cot a =$ tangent of the declination.*

PROPOSITION XIII.

(A) *The error in taking the altitude of a star being given, to find the corresponding error in the hour angle.*

As in the preceding proposition, let B represent the pole of the equinoctial, A the zenith, and C the observed place of the star. Then c will be the co-latitude, a the star's co-declination, and b its co-altitude; the sides c and a will be constant quantities, and the hour angle B will be variable. It is shewn (P. 354.) that $\dot{\text{B}} : \dot{b} :: \text{rad}^2 : \text{sine A} \times \text{sine } c$, hence

$$\dot{\text{B}} = \frac{\text{rad}^2 \times \dot{b}}{\text{sine A} \times \text{sine } c} = \frac{\text{rad}^2}{\text{sine A} \times \text{sine } c} \times \dot{b}.$$

Since the variation of the angle B is the measure of the error in time, and that c, in the same latitude, and A (if the same error prevails in different observations, (are constant quantities; the error in time will not be altered whatever the altitude of the star may be; and this error will be the least if the altitude of the star be taken when it is on the prime vertical.*

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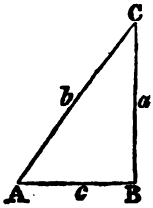
* Astronomie Nautique, par M. de Maupertuis, page 34. Vince's Trig metry, second edition, page 144, &c.

PROPOSITION XIV.

(B) *The error in the altitude of any tower, or other object, is to the error committed in taking the angle of elevation; as double the height of the observed object, is to the sine of double the angle of elevation.*



This proposition is designed to illustrate the scholium (S. 355.). From the first set of equations (C. 347.) we have $\dot{A} = \frac{\text{sine C}}{\text{sine } b} \times \dot{a}$, that is, $\dot{a} : \dot{A} :: \text{sine } b : \text{sine C}$, and considering the triangle as rectilinear (S. 355.) $\dot{a} : \dot{A} :: b : \text{sine C} = \cos A$, $\because \frac{\dot{a}}{\dot{A}} = b \times \frac{1}{\cos A}$.

But, sine $2A = 2 \text{ sine } A \times \cos A$, rad $= 1$ (O. 108.)

hence $\frac{1}{\cos A} = \frac{2 \text{ sine } A}{\text{sine } 2A}$ and $\frac{\dot{a}}{\dot{A}} = \frac{2\,b \,.\, \text{sine } A}{\text{sine } 2\,A}$.

Again, sine $A : a :: \text{rad} = 1 : b$ (Z. 34.)

And sine $A : 2a :: 1 : 2b \because 2b \,.\, \text{sine } A = 2a$, by substitution $\frac{\dot{a}}{\dot{A}} = \frac{2\,a}{\text{sine } 2\,A}$ that is, $\dot{a} : \dot{A} :: 2a : \text{sine } 2A$ which was to be shewn.

Corollary. Because $\dot{a} = \frac{\dot{A} \times 2a}{\text{sine } 2A}$ it is obvious that ($\dot{a}$) the error in altitude will be a minimum, when sine 2A is a maximum, that is sine of 90°. Hence if ($\dot{A}$) the error in the observed altitude be 1′, the sine or arc of which is ·0002909, and the observed angle be 45°, we have

$$\dot{a} = \frac{\dot{A} \times 2a}{\text{sine } 2A} = \frac{\cdot 0002909 \times 2a}{1} = \cdot 0005818 \times a = \frac{a}{1718 \cdot 8}.$$

See the note page 77.

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* See Dr. Mackay's Theory and Practice of finding the longitude at sea or land, vol. i. page 298, third edition.

CHAP. XIV.

MISCELLANEOUS PROPOSITIONS, &c.

(C) 1. *Of the French Division of the Circle.*

The modern French writers on Trigonometry divide the circumference of the circle into 400[*] equal parts or degrees, each degree into 100 equal parts or minutes, each minute into 100 equal parts or seconds, &c. which degrees, minutes, &c. they write in the usual manner, thus, $126^\circ.80'.\ 64''$, &c.

A French degree is therefore less than an English degree, in the ratio of 90 to 100, or of 9 to 10; a French minute is less than an English minute in the ratio of 90×60 to 100×100, or of 27 to 50; and a French second is less than an English second in the ratio of $90 \times 60 \times 60$ to $100 \times 100 \times 100$ or of 81 to 250.

Hence, if $n =$ any number of degrees, to turn English degrees into French, we have $9 : 10 :: n : \frac{10n}{9} = n + \frac{n}{9}$, and to turn French degrees into English, $10 : 9 :: n : \frac{9n}{10} = n - \frac{n}{10}$.

PROPOSITION I.

(D) *To turn French degrees, minutes, &c. into English.*

Rule. Consider the degrees as a whole number, after which place the minutes and seconds[†] as decimals; $\frac{1}{10}$ of this mixed decimal deducted from itself will give the English degrees corresponding to the French.

Example I. The latitude of Paris is $54^\circ.26'.36''$ in the

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[*] Elémens de Géométrie, par A. M. Legendre, 6th ed. page 328. Preface to Borda's Trigonometrical Tables (Paris, An. IX.), page 18, *et seq.*

[†] The minutes and seconds, if under 10, must have a cipher prefixed, thus $27^\circ.7'.35'$, must be written $27^\circ.07'.35''$, or $27^\circ.0735$; $45^\circ.18'.4'' = 45^\circ.18'.04'' = 45^\circ.1804$, &c.

French division of the circle, what is the corresponding latitude in the English division?

$$\begin{array}{rl}
54^\circ.26'.36''=54^\circ\cdot2636 & \text{French degrees.}\\
\tfrac{1}{10}\ =\ \ 5\ \cdot42636 & \\
\hline
48\ \cdot83724 & \text{English degrees.}\\
60 & \\
\hline
50\ \cdot23440 & \\
60 & \\
\hline
14\ \cdot06400 &
\end{array}$$

Answer. 48°.50'.14" English.

2. What number of degrees, &c. in the English division of the circle will correspond to 74°.4'.8" in the French division?

$$\begin{array}{r}
74^\circ.4'.8''=74^\circ.04'.08''=74^\circ\cdot0408\\
\tfrac{1}{10}\ =\ \ 7\ \cdot40408\\
\hline
66\ \cdot63672\\
60\\
\hline
38\ \cdot20320\\
60\\
\hline
12\ \cdot19200\\
\hline
\end{array}$$

Answer. 66°.38'.12" English.

PROPOSITION II.

(E) *To turn English degrees, minutes, &c. into French.*

Rule. Reduce the minutes and seconds, &c. to the decimal of a degree, and annex it to the given number of degrees; this mixed decimal increased by $\frac{1}{9}$ of itself will give the French degrees corresponding to the English.

Example. The latitude of Greenwich Observatory is 51°.28'.40" N. according to the English division of the circle, what is the corresponding latitude by the French division?

$$\begin{array}{rl}
51^\circ.28'.40''= & 51^\circ\cdot477777,\ \&c.\\
\tfrac{1}{9}\ = & 5\ \cdot719753,\ \&c.\\
\hline
 & 57\ \cdot197530=57^\circ.19'.75''.3.\\
\hline
\end{array}$$

Answer. 57°.19'.75".3 North, by the French division of the circle.

On the 22d March, 1813, the moon's distance from the sun, at midnight, was 114°.13′.21″ by the Nautical Almanac, what was the distance according to the French division of the circle?

$$\begin{aligned} 114°.13'.21'' &= 114°.2225 \\ \tfrac{1}{9} &= 12°.691388 \\ \hline &\ 126°.913888 = 126°.91'.39''. \end{aligned}$$

Answer. 126°.91′.39″.

(F) A table of arcs differing by 10 degrees according to the French division of the circle, or by 9 degrees by the English division, with the corresponding natural and logarithmical sines.

French Division.		*English Division.*		*Nat. sines.*		*Log. sines.*
sine 10° = cos 90°	-	sine 9° = cos 81°	-	·1564345	-	9·1943324
sine 20° = cos 80°	-	sine 18° = cos 72°	-	·3090170	-	9·4899824
sine 30° = cos 70°	-	sine 27° = cos 63°	-	·4539905	-	9·6570468
sine 40° = cos 60°	-	sine 36° = cos 54°	-	·5877853	-	9·7692187
sine 50° = cos 50°	-	sine 45° = cos 45°	-	·7071068	-	9·8494850
sine 60° = cos 40°	-	sine 54° = cos 36°	-	·8090170	-	9·9079576
sine 70° = cos 30°	-	sine 63° = cos 27°	-	·8910065	-	9·9498809
sine 80° = cos 20°	-	sine 72° = cos 18°	-	·9510565	-	9·9782063
sine 90° = cos 10°	-	sine 81° = cos 9°	-	·9876883	-	9·9946199
sine 100° = cos 0°	-	sine 90° = cos 0°	-	1·0000000	-	10·0000000

EXAMPLE.

The latitude of the Observatories of Paris and Pekin are 54°.26′.36″ N. and 44°.33′.73″ N., and their difference of longitude 126°.80′.56″, according to the French division of the circle, what is their distance? *

(G) SOLUTION BY THE FRENCH DIVISION OF THE CIRCLE†.

Here we have two sides a and b of a spherical triangle given, and the included angle c, to find the third side c.

By the formulæ (S. 191.) $\text{tang}\ \phi = \dfrac{\cos \text{c} \cdot \text{tang}\ b}{\text{rad}}$, and $\cos c = \dfrac{\cos b}{\cos \phi} \cdot \cos (a - \phi)$ where $a = (100° - 54°.26'.36'') = 45°.73'.64''$ $b = (100° - 44°.33'.73'') = 55°.66'.27''$, and $\text{c} = 126°.80'.56''$.

$\log \cos \text{c} = \log \cos 126°.80'.56'' = \log \cos 200° - 126°.80'.56'' = \log \cos 73°.19'.44''$.

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* Legendre's Geometry, page 403.

† By Borda's Tables, Paris, An.

$$\log\cos 73^\circ.19'.44'' = 9{\cdot}6114358$$
$$\log\text{tang } b = \log\text{tang } 55^\circ.66'.27'' = 10{\cdot}0776713$$

$$\log\text{tang } \varphi \;\;-\;\;- \;\;= 9{\cdot}6891071$$ The arc corresponding in the tables is $28^\circ.94'.23''$. But because the cosine of c is negative (K. 100.), it is plain, by the formula above, that tang φ is also negative, hence $\varphi = -28^\circ.94'.23''$ and $a - \varphi = 45^\circ.73'.64'' + 28^\circ.94'.23'' = 74^\circ.67'.87''$.

$$\log\cos(a-\varphi) = \log\cos 74^\circ.67'.87'' = 9{\cdot}5880938$$
$$\log\cos \;\; b = \log\cos 55^\circ.66'.27'' = 9{\cdot}8071949$$
$$19{\cdot}3952887$$
$$\log\cos\varphi = \log\cos 28^\circ.94'.23'' = 9{\cdot}9534823$$

$$\log\cos c \qquad = 9{\cdot}4418064.$$ The corresponding arc is $82^\circ.16'.05''$ the distance between Paris and Pekin $=821605$ seconds, or $821{\cdot}605$ myriametres; a myriametre being an arc of 10 minutes, and a metre an arc of $\frac{1}{10}$ of a second.*

(H) SOLUTION BY THE ENGLISH DIVISION OF THE CIRCLE.†

Lat of Paris $=54^\circ.26'.36''$ French $=48^\circ.50'.14''{\cdot}064$ English. (C. 360.)

Lat of Pekin $=44^\circ.33'.73''$ French $=39^\circ.54'.12''{\cdot}852$ English.

Diff. Long $=126^\circ.80'.56''$ French $=114^\circ.7'.30''{\cdot}144$ English.

Hence $a = (90^\circ - 48^\circ.50'.14''{\cdot}064 =)\; 41^\circ.9'.46''{\cdot}064$; $b = (90^\circ - 39^\circ.54'.12''{\cdot}852 =)\; 50^\circ.5'.47''{\cdot}148$.

and $c = 114^\circ.7'.30''{\cdot}144$.

$$\log\cos c = \log\cos 114^\circ.7'.30''{\cdot}144 = 9{\cdot}6114358$$
$$\log\text{tang} b = \log\text{tang } 50^\circ.5'.47''{\cdot}148 = 10{\cdot}0776713$$

$$\log\text{tang } \varphi = \log\text{tang } 26^\circ.2'.53'' = 9{\cdot}6891071$$

Because tang φ is negative, $a - \varphi = 41^\circ.9'.46'' + 26^\circ.2'.53'' = 67^\circ.12'.39''$.

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* According to the French mathematicians $\frac{1}{4}$ of the whole terrestrial meridian (viz. 100°) $=5130740$ toises in length, the *ten millionth part* of which is the *metre*; therefore the *metre* $=\frac{1}{10}$ of a second $=\cdot5130740$ toises $=3\,078444$ French feet; and because 107 French feet are equal to 114 English feet nearly, the *French metre* $=3{\cdot}28$ English feet. If a French toise $=6{\cdot}3945$ English feet, a metre will be $=3{\cdot}280852$ English feet.

† Tables Portatives, par Callet, édition Stéréotype, An. 3.

$$\begin{array}{rr}
\log\cos(a-\varphi)=\log\cos 67^{\circ}.12'.39''= & 9\cdot5880938\\
\log\cos b=\log\cos 50^{\circ}.5'.47''\cdot148= & 9\cdot8071949\\
\hline
 & 19\cdot3952887\\
\log\cos\varphi=\log\cos 26^{\circ}.2'.53''= & 9\cdot9534823\\
\hline
\log\cos c=\log\cos 73^{\circ}.56'.40''= & 9\cdot4418064\\
\hline
\end{array}$$

The length of a degree, by the trigonometrical survey of England and Wales, is 364950 feet*; hence, the distance between Paris and Pekin is 26986025 English feet.

73°.56′.40″ English=82°.16′.05″ French as above (E. 361.)

PROPOSITION III. (*Plate IV. Fig.* 7.)

(I) *If from the sum of the three angles of any spherical triangle two right angles be deducted, the remainder will be to two right angles, as the area of the triangle is to one-fourth of the surface of the sphere.*

Let ABC be the triangle; complete the circle ABED, and produce AC, BC to meet in the point F, cutting the circle ABED in the points E and D.

Then because BD and CF are each of them a semicircle (C. 133.) BC=DF, and for the same reason AC=EF.

The angle ACB= ∠DCE= ∠DFE (N. and Q. 135.) therefore the triangle ABC is equal to the triangle DEF (D. 141.); let the surface of each of these triangles be represented by *m*, and let *w*, *x*, and *y* represent the surfaces of the triangles in which they are situated. Now it is plain, that if any great circle of the sphere (as 1, 2, 3.) be divided into any number of equal parts, and through the points of division (1, 2, 3.) great circles be drawn so as to pass through the poles (B and D) of the divided great circle, the surface of the sphere will be divided into as many equal parts, or *lunulæ*, as there are parts in the divided great circle†; therefore, the whole circumference of the sphere, will be to any arc of this great circle, as the whole surface of the sphere, is to the surface of the lune comprehended between two great circles, whose greatest distance is measured by the aforesaid arc.

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* Vol. II. part ii. page 113.

† This may be farther illustrated by considering the manner in which the several meridians, on a terrestrial globe, divide the equator and the surface of the globe.

Hence $180^\circ : \angle A :: \frac{1}{2}$ surface of the sphere $: m+y$.

$180^\circ : \angle B :: \frac{1}{2}$ surface of the sphere $: m+w$.

$180^\circ : \angle C :: \frac{1}{2}$ surface of the sphere $: m+x$.

By Composition

$180^\circ : \angle A+\angle B+\angle C ::$ surface of the sphere $: 3m+y+w+x$; and, by Division,

$180^\circ : (\angle A+\angle B+\angle C)-180^\circ :: \frac{1}{2}$ surface of the sphere $: 2m$.

or, $180^\circ : (\angle A+\angle B+\angle C)-180^\circ :: \frac{1}{4}$ surface of the sphere $: m$.

Consequently $(\angle A+\angle B+\angle C)-180^\circ : 180^\circ :: m : \frac{1}{4}$ surface of the sphere. Q.E.D.

(K) Corollary. The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles. For if $s=\frac{1}{4}$ of the surface of the sphere, $180^\circ \times m = s \times (A+B+C-180^\circ)$.

$$\therefore m = \frac{s \times (A+B+C-180^\circ)}{180^\circ} = \frac{s}{180^\circ}.(A+B+C);-s.$$

But the whole surface of the sphere is equal to four times the area of one of its great circles*, and the area of a great circle $=\frac{1}{2}$ circumference × radius $= 180^\circ \times$ radius, if therefore radius $=1$, the surface of one fourth of the sphere may be expressed by 180°; therefore $m=(A+B+C)-180^\circ$.

PROPOSITION IV.

(L) The sum of the three angles of every spherical triangle being greater than 180° (T. 137.) the sum of the three observed angles of a triangle, on the surface of the earth†, ought to exceed 180°; which excess may be found by the following rule. *From the logarithm of the area of the triangle taken as a plane one, in feet, subtract the constant logarithm* 9·3267737, *and the remainder is the logarithm of the excess above* 180° *in seconds nearly.*‡

Put $\frac{1}{4}$ of the surface of the sphere $=s$, then

$(\angle A+\angle B+\angle C)-180^\circ = \dfrac{m \times 180^\circ}{s}$ (K. 365.) But, by mensuration, $\frac{1}{4}$ of the surface of the sphere $=\text{rad}^2 \times 3{\cdot}14159$, &c. therefore $(\angle A+\angle B+\angle C)-180^\circ = \dfrac{180^\circ \times m}{\text{rad}^2 \times 3{\cdot}14159, \&c.}$.

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* Simpson's Fluxions, page 189.

† The sides of the triangles which connect the successive stations of a trigonometrical survey, may be considered as formed by arcs of great circles, whose radii are equal to the radius of the sphere; and each observed angle, being formed by the tangents of two of these arcs, will be the measure of the spherical angle contained between them, at each successive station (R. 135.)

‡ Trigonometrical Survey of England and Wales, vol. i. page 138.

Now $3{\cdot}14159$, &c. : $\cdot5 = \text{rad} :: 360^\circ : \frac{180^\circ}{3{\cdot}14159,\ \&c.}$, and the length of a degree, supposing the earth to be a sphere, $= 60859{\cdot}1 \times 6$* feet, hence $\frac{180^\circ}{3{\cdot}14159,\ \&c.} \times 60859{\cdot}1 \times 6 =$ the earth's radius in feet $\because$

$$(\angle A + \angle B + \angle C) - 180^\circ = \frac{m \times 3{\cdot}14159,\ \&c.}{\overline{365154{\cdot}6}|^2 + 180^\circ}$$ the excess of the three angles above two right angles, and if A, B and C be expressed in seconds, the excess in seconds will be

$$\frac{m \times 3{\cdot}14159 \times 60 \times 60}{\overline{365154{\cdot}6}|^2 \times 180^\circ} = m \times \frac{62{\cdot}831853,\ \&c.}{\overline{365154{\cdot}6}|^2}$$, that is, log of $m \times$ (log of $62{\cdot}831853$, &c. $-$ log of $\overline{365154{\cdot}6}|^2 =$ log of $m + 1{\cdot}7981799 - 11{\cdot}1249536 = \log m - 9{\cdot}3267737$.
Q. E. D.

(M) EXAMPLE TO THE FOREGOING PROPOSITION. (*Trigonometrical Survey.*)

Names of the Stations.	*Observed angles.*	*Distance of Dunnose from*
† Butser-hill -	76°.12′.22″	Butser-hill 140580·4 feet.
Dean-hill -	48°. 4′.32·25	Dean-hill 183496·2 feet.
Dunnose -	55°.43′. 7	
	180°. 0′. 1.25	

Here we have two sides of a plane triangle, and the angle contained between them, given, to find the area, viz. $a = 140580{\cdot}4$ feet, $b = 183496{\cdot}2$ feet, and the contained $\angle C = 55^\circ.43'.7''$. By the common rule of mensuration $\frac{1}{2}\, a \times b \times$ natural sine of $\angle C =$ the area of the triangle.

$\frac{1}{2}a = 70290{\cdot}2$ log $= 4{\cdot}8468947$
$b = 183496{\cdot}2$ log $= 5{\cdot}2636271$
log sine $\angle C =$ log sine $55^\circ.43'.7'' = 9{\cdot}9171279$

log of $m = 10{\cdot}0276497$
Deduct the constant log $= 9{\cdot}3267737$

Log of the spherical excess $= 0{\cdot}7008760$ the number corresponding to which is $5''{\cdot}022$ or $5''$. Hence, the error in the three observed angles above, is $5'' - 1''{\cdot}25 = 3''{\cdot}75$.

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* Trigonometrical Survey, vol. i. page 138, Faden's edition.

† Philosophical Transactions, 1803, page 428. Dr. Rees's New Cyclopedia, second series of triangles, word degree. A variety of examples will be met with in the works here referred to, and in the Trigonometrical Survey, published by Mr. Faden.

(N) By the assistance of this spherical excess the observer will be enabled to examine the accuracy of his observations; and, having corrected the angles, the sides of the triangle, being arcs of great circles (supposing the earth to be a sphere) may be calculated by the rules of spherical trigonometry.

Legendre in his Geometry, page 416, has given the rule demonstrated in the following proposition for correcting the observed angles, and though the method be only an approximation, it is sufficiently accurate and commodious for practice.

PROPOSITION V.

(O) *A spherical triangle, whose sides are very small when compared with the radius of the sphere, being proposed; if from each of its angles you subtract one third of the excess of the sum of its three angles above two right angles, the angles thus diminished may be taken for the angles of a rectilinear triangle, whose sides are equal in length to those of the proposed spherical triangle.*

Let A, B and C represent the three angles of a plane triangle of which the opposite sides a, b and c are small relatively to the radius of the sphere.

Then,

$$a : b :: \text{sine } A : \text{sine } B \quad (\text{D. } 35.)$$

Let d be an arc indefinitely small, then considering the triangle to be spherical, $\text{sine } a : \text{sine } b :: \text{sine } (A+d) : \text{sine } (B+d)$ extremely near. But $\text{sine}(A+d) = (\text{sine } A \,.\, \cos d) + (\cos A \,.\, \text{sine } d)$ the radius being $= 1$ (D. 115.); now d being very small, by supposition, its sine is equal to the arc, and its cosine equal to the radius, therefore $\text{sine } (A+d) = \text{sine } A + \cos A \,.\, d$, and for the same reason $\text{sine}(B+d) = \text{sine } B + \cos B \,.\, d$; moreover it is shewn. (C. 127.)

that the sine of an arc $z = z - \dfrac{z^3}{2 \cdot 3 . r^2}$ $\because$ $\text{sine } a = a - \dfrac{a^3}{6}$ and $\text{sine } b = b - \dfrac{b^3}{6}$ the radius being 1; hence

$$a - \frac{a^3}{6} : b - \frac{b^3}{6} :: \text{sine } A + \cos A \,.\, d : \text{sine } B + \cos B \,.\, d,$$

but $b : a :: \text{sine } B : \text{sine } A$, and by compounding these proportions, we get

$$1 - \frac{a^2}{6} : 1 - \frac{b^2}{6} :: (\text{sine } A \,.\, \text{sine } B) + (\cos A \,.\, d \,.\, \text{sine } B) :$$

$(\text{sine } B \,.\, \text{sine } A) + (\cos B \,.\, d \,.\, \text{sine } A)$ but the antecedents and consequents of this proportion being nearly equal, by adding $\frac{a^2}{6}$ to the first and second terms, and deducting $\cos \text{A} . d . \text{sine B}$ from the third and fourth, the proportion becomes

$1 : 1+\frac{a^2-b^2}{6} :: \text{sine A} . \text{sine B} : (\text{sine B} . \text{sine A}) + (\cos \text{B} . d . \text{sine A}) - (\cos \text{A} . d . \text{sine B})$.

But $(\cos \text{B} . \text{sine A}) - (\cos \text{A} . \text{sine B}) . d = \text{sine}(\text{A}-\text{B}) . d$ (D. 115.)

$\because 1 : 1+\frac{a^2-b^2}{6} :: \text{sine A} . \text{sine B} : (\text{sine A} . \text{sine B}) + \text{sine}(\text{A}-\text{B}) . d$.

Consequently $1 : \frac{a^2-b^2}{6} :: \text{sine A} . \text{sine B} : \text{sine}\, \overline{\text{A}-\text{B}} . d$ (Euclid 17. *of* V.)

and $\text{sine}(\text{A}-\text{B}) . d = \text{sine A} . \text{sine B} . \frac{a^2-b^2}{6}$.

But since $a : b :: \text{sine A} : \text{sine B}$.

By composition $a+b : b :: \text{sine A} + \text{sine B} : \text{sine B}$.

And by division $a-b : b :: \text{sine A} - \text{sine B} : \text{sine B}$.

By multiplying the corresponding terms, we obtain

$$a^2-b^2 : b^2 :: \text{sine}^2 \text{A} - \text{sine}^2 \text{B} : \text{sine}^2 \text{B},$$

and because $\frac{a}{b} = \frac{\text{sine A}}{\text{sine B}}$, if the two consequents be multiplied by these quantities, we get $a^2-b^2 : ab :: \text{sine}^2 \text{A} - \text{sine}^2 \text{B} : \text{sine A} . \text{sine B}$.

$\because \text{sine A} . \text{sine B} . (a^2-b^2) = ab . (\text{sine}^2 \text{A} - \text{sine}^2 \text{B})$ hence by substitution, $\text{sine}(\text{A}-\text{B}) . d = \frac{ab}{6} . (\text{sine}^2 \text{A} - \text{sine}^2 \text{B})$.

But, $\text{sine}(\text{A}+\text{B}) . \text{sine}(\text{A}-\text{B}) = \text{sine}^2 \text{A} - \text{sine}^2 \text{B}$ (D. 115.)

Hence $\text{sine}(\text{A}-\text{B}) . d = \frac{ab}{6} . \text{sine}(\text{A}+\text{B}) . \text{sine}(\text{A}-\text{B})$,

$$\text{and } d = \frac{ab}{6} . \text{sine}(\text{A}+\text{B}).$$

Now because the sum of two of the angles (A + B) of a rectilinear triangle, is equal to the supplement of the third angle (C), and any arc and its supplement have the same sine, $d = \frac{1}{6} ab . \text{sine C}$, or $3d = \frac{1}{2} a . b . \text{sine C}$. But the three angles of the spherical triangle are $\text{A}+d$, $\text{B}+d$, $\text{C}+d$, and the three angles of the plane triangle are A, B, C; hence $(\text{A}+d)+(\text{B}+d)+(\text{C}+d) = \text{A}+\text{B}+\text{C}+3d$.

Or, $(\text{A}+d)+(\text{B}+d)+(\text{C}+d)-180° = 3d = \frac{1}{2} a . b . \text{sine C}$, t area of a plane triangle whereof the two sides are a and b, and the contained angle C, or the three sides a, b and c, and the three angles A, B, and C; and since the three sides a, b, c of the plane triangle differ insensibly from the three sides of the spherical triangle, the surface of the one may be taken from the surface of the other without material error: but the surface of the spherical triangle is the difference between the sum of its three angles and two right angles (K. 365.) therefore the truth of the proposition is evident.* Q. E. D.

(P) EXAMPLE TO THE PRECEDING PROPOSITION. *(Trigonometrical Survey.)*

Names of stations.	*Observed angles.*	*Distance of Hundred Acres from*	
Hundred Acres -	53°.58′35″·75		
Hanger-hill tower	68. 24. 44	Hanger-hill	71932·8 feet.
St. Ann's-hill - -	57. 36. 39· 5	St. Ann's-hill	79209·7 feet.
	179. 59. 59· 25		

Here a=71932·8, b=79209·7 and C=53°.58′.35″.75.

$\frac{1}{2}a$=35966·4 log=4·5558969

b=79209·7 log=4·8987783

log sine of C=log sine of 53°.58′.35″·75=9·9078287

log of the area of the triangle = 19·3625039

deduct the constant log=9·3267737

log of the spherical excess=·0357302 the number corresponding to which is 1·0857 seconds=3d, hence d= ·36″ nearly. But the observed angles are spherical angles, hence C corrected becomes (53°.58′.35″·75 − ·36″=) 53°.58′.35″·39 with this as an included angle, and the two sides given above, find the angles and the third side of a plane triangle. By the formulæ, Case III. page 43, tang $\phi = \frac{b}{a}$. rad, and $\frac{1}{2}$ (A ∼ B)=

$$\frac{\text{cot}\quad \text{C} \,.\, \text{tang}\,(\phi - 45^\circ)}{\text{rad}}$$

log b = 4·8987783
log a = 4·8569269

0·0418514
log rad=10·

tang ϕ=10·0418514

Hence ϕ=47°.45.23″·1
ϕ−45°= 2°.45.23″:1

tang (ϕ−45°) = tang 2°.45′·23″·1 = 8·6825576
cot $\frac{1}{2}$C = cot 26°.59′.17″.695 = 10·2930544

sum (rejecting rad=10) = 8·9756120
Hence $\frac{1}{2}$ (A ∼ B) = 5° 24′ 2″ 327
And $\frac{1}{2}$ (A + B) = 63. 0 42· 3

Therefore B = 68 24′ 44·627
And A = 57 36′ 39·973

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* This demonstration, which is more simple than Legendre's, is grounded on the same principles as that given in page 492 of Mr. LESLIE's Elements of Geometry, first edition, 1809.

The third side c may be found by the formula page 42, viz. $c = \frac{\text{sine } C \,.\, a}{\text{sine } A}$; by logarithms, $9{\cdot}9078281 + 4{\cdot}8569269 - 9{\cdot}9265646 = 4{\cdot}8381904$ log of c, hence $c = 68895{\cdot}43$.

This example is solved exactly by the same rules as the CVIIth example, page 417 of Legendre's Geometry, and by adding to each of the angles one-third of the spherical excess, a complete solution will be given to the spherical triangle.

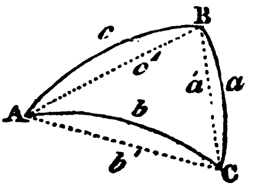
Angles of the spherical triangle.	Sides of the spherical triangle.
$C + d = 53°.58'.35''.75$	$c = 68895{\cdot}43$
$B + d = 68.\ 24.\ 44.\ 99$	$b = 79209{\cdot}7$
$A + d = 57.\ 36.\ 40.\ 33$	$a = 71932{\cdot}8$
180· 0· 1· 07	

(Q) The arcs and angles which occur in a trigonometrical survey may be calculated by the exact rules of spherical trigonometry in the manner of Boscovich: or by the approximating rules of Legendre; or the observed angles may be reduced to the angles formed by the chords of arcs, and an arc of the meridian may be considered as formed by the chords of curves. This last method has been successfully practised by Delambre in France, and by the late General Mudge in the trigonometrical survey of England and Wales.

PROPOSITION VI.

(R) *Given two sides of a spherical triangle, and the angle comprehended between them; to find the angle contained between the chords of these sides, supposing the chords not to differ materially from the arcs which they subtend.*

Let the three angles of the spherical triangle be represented by A, B, C, and their opposite sides by a, b, c; and let a', b', c' represent the chords of these sides, which chords are supposed not to differ essentially from the arcs.



Then $\cos a = (\cos A \,.\, \text{sine } b \,.\, \text{sine } c) + (\cos b \,.\, \cos c)$ the radius being 1. (Equation IV. D. 184.)

But $\cos a = 1 - 2\,\text{sine}^2\,\tfrac{1}{2}a$; $\cos b = 1 - 2\,\text{sine}^2\,\tfrac{1}{2}b$; and $\cos c = 1 - 2\,\text{sine}^2\,\tfrac{1}{2}c$ the radius being 1 (I. 117), therefore by substitution

$1-2\ \text{sine}^2\ \frac{1}{2}\ a = (\cos \text{A} . \text{sine}\ b . \text{sine}\ c) + (1-2\ \text{sine}^2\ \frac{1}{2}\ b) . (1-2\ \text{sine}^2\ \frac{1}{2}\ c)$. By transposition, &c. we obtain $2\text{sine}^2\frac{1}{2}b + 2\text{sine}^2\frac{1}{2}c - 2\text{sine}^2\frac{1}{2}\ a = (\cos \text{A}.\text{sine}\ b.\text{sine}\ c) + 4\text{sine}^2\frac{1}{2}\ b . \text{sine}^2\ \frac{1}{2}\ c$. But $a'^2 = c'^2 + b'^2 - (2\ c'\ b' . \cos \text{rectilinear} \angle \text{A}')$; (N. 92.) and because a', b', c' do not differ essentially from a, b, c (by hypothesis), and that small arcs are nearly equal to their sines, $a' = 2\ \text{sine}\ \frac{1}{2}\ a$; $b' = 2\ \text{sine}\ \frac{1}{2}\ b$; and $c' = 2\ \text{sine}\ \frac{1}{2}\ c$; therefore by substitution $4\ \text{sine}^2\ \frac{1}{2}\ a = (4\ \text{sine}^2\ \frac{1}{2}\ c + 4\ \text{sine}^2\ \frac{1}{2}\ b) - 4\ \text{sine}\ \frac{1}{2}\ c . 2\ \text{sine}\ \frac{1}{2}\ b . \cos \text{rectilinear} \angle \text{A}'$, or $2\ \text{sine}^2\ \frac{1}{2}\ a = (2\ \text{sine}^2\ \frac{1}{2}\ c + 2\ \text{sine}^2\ \frac{1}{2}\ b) - 2\ \text{sine}\ \frac{1}{2}\ c .\ 2\ \text{sine}\ \frac{1}{2}\ b . \cos \text{rectilinear} \angle \text{A}'$; by substituting this value of $2\ \text{sine}^2\ \frac{1}{2}\ a$ in the spherical equation above, we get, $2\ \text{sine}\frac{1}{2}c . 2\ \text{sine}\frac{1}{2}b . \cos \text{A}' = (\cos \text{A} . \text{sine}\ b . \text{sine}\ c) + (4\ \text{sine}^2\ \frac{1}{2}\ b . \text{sine}^2\ \frac{1}{2}\ c)$. Hence, $\cos \text{rectilinear} \angle \text{A}' =$

$$\frac{(\cos \text{A} . \text{sine}\ b . \text{sine}\ c) + (4\ \text{sine}^2\ \frac{1}{2}\ b . \text{sine}^2\ \frac{1}{2}\ c)}{2\ \text{sine}\ \frac{1}{2}\ c . 2\ \text{sine}\ \frac{1}{2}\ b} =$$

$$\cos \text{A} . \frac{\text{sine}\ b}{2\ \text{sine}\frac{1}{2}b} . \frac{\text{sine}\ c}{2\ \text{sine}\frac{1}{2}c} + (\text{sine}\ \tfrac{1}{2}\ b . \text{sine}\ \tfrac{1}{2}\ c).$$

But $\text{sine}\ b = 2 \cos \frac{1}{2}\ b . \text{sine}\ \frac{1}{2}\ b$, and $\text{sine}\ c = 2 \cos \frac{1}{2}\ c . \text{sine}\ \frac{1}{2}\ c$ (Z. 110.) therefore by substitution, $\cos \text{rectilinear} \angle \text{A}' = (\cos \text{A} . \cos \frac{1}{2}\ b . \cos \frac{1}{2}\ c) + (\text{sine}\ \frac{1}{2}\ b . \text{sine}\ \frac{1}{2}\ c)$.

(S) Scholium. In the investigation of the preceding formula, the radius 1 has been used in order to lessen the number of characters of symbols, but formulæ constructed for the radius 1, are easily converted into formulæ which involve a radius $= r$, by substituting for sine A, cos A, &c., $\frac{\text{sine A}}{r}$, $\frac{\cos \text{A}}{r}$, &c.

Hence, the foregoing formula by the introduction of r becomes $\cos \text{A}' = \frac{1}{r^2}[(\cos \text{A}.\cos \frac{1}{2} b.\cos \frac{1}{2}\ c) + (r . \text{sine}\ \frac{1}{2} b.\text{sine}\frac{1}{2}c)]$.

And this formula may be varied in the expression by auxiliary quantities as at Q. 190, &c.

(T) EXAMPLE.

At Calais, the correct angle between Watten and Fiennes, was 66°.30′.36″.875, the distance from Calais to Watten 15′.59″ of the earth's meridian; and from Calais to Fiennes 7′.26″, required the angle formed by the chords of these arcs.

log cos A = log cos 66°.30′.36″.875 = 9.6005211
log cos $\frac{1}{2}b$ = log cos 7′.59″. 5 = 9.9999988
log cos $\frac{1}{2}c$ = log cos 3′.43″. = 9.9999998
9.6005197

Reject the tens in the index, and subtract the resulting index from 10, prefixing the sign minus to the remainder.

The natural number answering to the common log -1.6005197 is $\cdot39858385$.

log sine $\frac{1}{2}b$ = log sine $7'.59''.5$ = $7\cdot3659128$
log sine $\frac{1}{2}c$ = log sine $3'.43''$ = $7\cdot0338796$
$4\cdot3997924$

Nat. num. answering to $-6\cdot3997924$ is $\cdot00000251$.

Hence $\cdot39858385+\cdot00000251=\cdot39858636$ the natural cosine of the rectilinear $\angle$ A′, answering to $66°.30'.36''.23$.

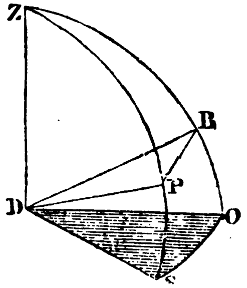
(U) The remainder of the calculation is performed by the rules of Plain Trigonometry, and the criterion of the accuracy of the observations is at once determined; for the sum of the three angles formed by the chords, and deduced from the spherical angles, will always be equal to two right angles, if the spherical angles have been correctly observed.

In all cases the rectilinear angle A′ differs but little from the spherical angle A, and to enable the calculator to obtain the necessary correction, with as little trouble as possible, M. Delambre has investigated a formula, from which he has constructed a set of tables for determining, in an easy manner, thr angles formed by the chords from the spherical angles.

PROPOSITION VII.

(W) *The angles of elevation of two distant objects on the surface of the earth being given, together with the oblique angle contained between the objects, to find the horizontal angle.*

Let SP and OB be two objects elevated above the surface of the earth; D the place of the observer, Z his zenith, and S and O two points on the surface of the earth.



The observed $\angle$ will be PDB, and the horizontal or required $\angle$ will be SDO, measured by the arc SO.

In the spherical triangle PZB; PZ, BZ and PB are given to find the $\angle$ PZB, the measure of which is the arc SO $=\angle$ SDO.

$$\text{sine } \tfrac{1}{2} \angle \text{ PZB} = \sqrt{\frac{\text{sine } \frac{1}{2}(\text{PB}+\text{PZ}-\text{BZ}) \,.\, \text{sine } \frac{1}{2}(\text{PB}+\text{ZB}-\text{ZP})}{\text{sine PZ} \,.\, \text{sine BZ}}}$$

(F. 184.)

But PB is the measure of the observed $\angle$ PDB; PZ is the complement of the arc SP, which is the measure of the $\angle$ PDS; BZ is the complement of the arc OB, which is the measure of the $\angle$ BDO; and the difference between any two arcs PZ and BZ is equal to the difference between their complements SP and OB. Therefore

$$\text{sine}\tfrac{1}{2}\angle BDO=\sqrt{\frac{\sin\frac{1}{2}(PDB+PDS-BDO).\sin\frac{1}{2}(PDB+BDO-PDS)}{\cos PDS\ .\ \cos PDO}}$$

the same as M. Delambre's formula already given at page 94, which formula is there illustrated by an example.

(X) In a trigonometrical survey, where a formula of the same nature as that given above is usually applied, ZP and ZB do not materially differ from 90°, or the altitudes or depressions PS and BO, do not exceed 2° or 3°; in such cases, the problem will admit of a convenient and useful approximation, such as that given by Legendre at page 413 of his Elements de Géométrie, sixième édition, and also by other authors.

Let PS and BO be represented by H and h, and let D be the observed angle BDP and x the correction, viz. let $D+x$ be the required angle $=$ ODS, for the $\angle$ ODS will always be greater than the $\angle$ BDP (H. 91.) Now $\cos z=\dfrac{\cos PB-(\cos PZ\ .\ \cos BZ)}{\text{sine } PZ\ .\ \text{sine } BZ}$ (D. 184.) the radius being 1, viz. $\cos D+x=\dfrac{\cos D-(\text{sine } H.\text{sine } h)}{\cos H\ .\ \cos h}$, but the cosine of $H=1-\dfrac{H^2}{2}+\dfrac{H^4}{2.3.4}-$&c. and cosine of $h=1-\dfrac{h^2}{2}+\dfrac{h^4}{2.3.4}-$&c. (E. 128.) and in very small arcs, the *sines* do not differ essentially from the arcs, hence by substitution $\cos D+x=\dfrac{\cos D-Hh}{1-\frac{1}{2}H^2.\ 1-\frac{1}{2}h^2}=\dfrac{\cos D-Hh}{1-\frac{1}{2}H^2-\frac{1}{2}h^2}$ rejecting all the powers of H and h above the second, as being small; but $\cos D+x=(\cos D\ .\ \cos x)-(\text{sine } D\ .\ \text{sine } x)$ the radius being 1, (2d Equation D. 115.) and as x is very small, by supposition, $\cos x=1$ the rad, and sine $x=x$.

$\because \cos D-\text{sine } D\ .\ x=\dfrac{\cos D-Hh}{1-\frac{1}{2}H^2-\frac{1}{2}h^2}=(\cos D-Hh)$ $\cdot(1+\frac{1}{2}H^2+\frac{1}{2}h^2)$ nearly, viz. $\cos D-\text{sine } D\ .\ x=\cos D+\frac{1}{2}\cos D$ $\cdot(H^2+h^2)-Hh$, by rejecting the smaller terms after multiplication, hence $x=\dfrac{Hh-\frac{1}{2}\cos D\ .\ (H^2+h^2)}{\text{sine } D}$ the correction of the observed angle D, the same as Legendre's formula.

(Y) The preceding formula may be simplified, by following the method of Legendre; thus, let $\frac{1}{2}(H+h)=p$ and $\frac{1}{2}(H-h)=q$, then $p^2-q'=Hh$, $p^2+q^2=\frac{1}{2}(H^2+h^2)$ and $x=$ $\dfrac{p^2-q^2}{\text{sine } D}-\dfrac{(p^2+q^2)\ .\ \cos D}{\text{sine } D}=\dfrac{p^2-(p^2\ .\ \cos D)-q^2+(q^2\ .\ \cos D)}{\text{sine } D}=$

$p^2 \cdot \frac{1 - \cos D}{\text{sine } D} - q^2 \cdot \frac{1 + \cos D}{\text{sine } D}$ by reduction. Again, $\text{tang} \frac{1}{2} D = \frac{1 - \cos D}{\text{sine } D}$ the radius being 1 (L. 111.), and $\cot \frac{1}{2} D = \frac{1 + \cos D}{\text{sine } D}$; therefore $x = (p^2 \cdot \text{tang} \frac{1}{2} D) - (q^2 \cdot \cot \frac{1}{2} D)$. In applying this formula to practice, p and q are generally given in seconds, x should therefore be expressed in seconds, hence if r be the number of seconds contained in the radius, the number of seconds contained in $x = \frac{p^2}{r} \cdot \text{tang} \frac{1}{2} D - \frac{q^2}{r} \cdot \cot \frac{1}{2} D$.

SCHOLIUM.

(Z) The five preceding propositions, viz. from the 3d to the 7th inclusive, comprehend the principal rules and formulæ used in the calculation of a trigonometrical survey, considering the earth as a sphere. The spheroidical form of the earth will occasion a small correction, and require formulæ somewhat different, though of no great difficulty, but they are here omitted because they do not properly belong to spherical trigonometry.

THE END OF SPHERICAL TRIGONOMETRY.

BOOK IV.

THE THEORY OF NAVIGATION.

CHAPTER I.

DEFINITIONS AND PLANE SAILING.

(A) NAVIGATION is the art of finding the latitude and longitude of a ship at sea, and her course and distance from that place to any other given place.

(B) *The Earth* is considered as a perfect sphere or globe, revolving on an imaginary line called its axis, from west to east, in twenty-four hours. This rotation towards the east causes all the heavenly bodies to have an apparent motion from east to west.

(C) *The equator*, generally called the line by seamen, divides the globe into two equal parts, called the northern and southern hemispheres.

(D) *Meridians* are great circles cutting the equator at right angles, and passing through its poles. Every point upon the surface of the earth is supposed to have a meridian passing through it. That meridian passing through *Greenwich* is called the *first** meridian.

(E) *Longitude* of places on the earth, is reckoned on the equator from the *first* meridian. If they be situated eastward of the first meridian, they are said to be in east longitude; if westward, they are in west longitude. The greatest longitude on the earth is 180 degrees.

(F) *The difference of longitude* between two places, is an arc of the equator, contained between the two meridians passing through these places.

(G) *The latitude* of a place on the earth, is reckoned from the equator, upon a meridian passing through the place. The greatest latitude a place can have is 90 degrees.

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* It is necessary for the purposes of Geography and Navigation, to call the meridian of some remarkable place, the *first* meridian, and to estimate the longitudes of all other places from that meridian. And, as all the tables in the Nautical Almanac, and other English astronomical tables, are adapted to the meridian passing through the Royal Observatory at Greenwich, our seamen always reckon their longitude from that meridian.

(H) *Parallels of latitude* are small circles parallel to the equator. Every place upon the surface of the earth is supposed to have a parallel of latitude passing through it.

(I) *The difference of latitude* between two places, is an arc of a meridian contained between the parallels of latitude which pass through these places.

(K) *Meridional distance* is the distance between the meridian sailed from, and that arrived at, and is reckoned on that parallel of latitude which the ship is in.

(L) *The Mariner's compass*, is a representation of the horizon; and is divided into 32 points, each point 11°.15′.

(M) *The variation of the compass* is the deviation of its points from the corresponding points of the horizon. When the north point of the compass is to the east of the true north point of the horizon, the variation is east; if it be to the west, the variation is west.

(N). If a ship be steered due north, or due south, her distance sailed is equal to her difference of latitude; and her track will be on some meridian.

(O) If a ship be steered due east or west, her track will be either on the equator, or some parallel of latitude; and the distance sailed will be equal to her departure, or meridional distance.

(P) If a ship be steered towards any point of the horizon between the north and east, north and west, south and east, or south and west; the track she describes will be a *Rhumb line.*

(Q) *A Rhumb line* is a curve upon the surface of the sphere, cutting all the meridians in equal angles.

(R) *The course* of a ship is the angle in which the track she describes cuts the meridians.

(S) *The bearing between two places* on the same parallel of latitude is east and west, on the same meridian north and south; in all other situations it is a rhumb line, continually approaching the pole.

(T) *The departure* is the whole easting, or westing, the ship makes in any single course.

PROPOSITION I. (*Plate III. Fig.* 2.)

(U) *In sailing upon a Rhumb line the differences of latitudes are proportional to the distances sailed.*

Let P represent the pole, WOQE a portion of the equator, A*bczeu*L a rhumb line, or the track described by a ship sailing from A to L; AP, *d*P, *f*P, *g*P, *q*P, &c. meridians; *ib*, *hc*, *kz*, &c.

parallels of latitude; and let the elementary triangles A*ib*, *bhc*, *ckz*, *zte*, &c. be conceived so indefinitely small as to differ insensibly from plane or rectilinear triangles.

Then, the angles *iAb*, *hbc*, *kcz*, &c. are equal (Q. 377); and the angles A*ib*, *bhc*, *ckz*, &c. are right angles, for the parallels of latitude cut the meridians at right angles. Therefore all the elementary triangles A*ib*, *bhc*, *ckz*, *zte*, &c. are equiangular and similar.

Hence, A*b* : A*i*::*bc* : *bh*::*cz* : *ck*::*ze* : *zt*, &c. (Euclid 4 *of* VI.) Therefore, A*b* : A*i*::A*b*+*bc*+*cz*+*ze*, &c. : A*i*+*bh*+*ck*+*zt*, &c. (Euclid 12 *of* V.) That is,

A*b* : A*i*::AL : A*l*; where A*b* and AL are distances, and A*i* and A*l* correspondent differences of latitude. Q. E. D.

PROPOSITION II. (*Plate III. Fig.* 2.)

(W) *In sailing upon a rhumb line, the departure correspondent to any course and distance, is equal to the sum of all the intermediate departures.*

For, as in the preceding proposition,

A*b* : *ib*::*bc* : *hc*::*cz* : *kz*::*ze* : *te*, &c. (Euclid 4 *of* VI.) therefore, A*b* : *ib*::A*b*+*bc*+*cz*+*ze*, &c. : *ib*+*hc*+*kz*+*te*, &c. (Euclid 12 *of* V.) But the whole distance AL is equal to the sum of all the intermediate distances A*b*+*bc*+*cz*, &c.; hence, A*b* : *ib*::AL : *ib*+*hc*+*kz*+*te*, &c. Q. E. D.

(X) Scholium. Hence it appears that the *meridional distance*, *departure*, and *difference of longitude*, are essentially different. Let a ship sail from A to L, when she arrives at L her meridional distance will be L*l*, her departure *ib*+*hc*+*kz*+*te*, &c. and her difference of longitude WE. But the meridional distance is evidently less than the departure (which is equal to the sum of all the arcs *ib*+*hc*+*kz*, &c.); because the several meridians converge towards the pole; and for the same reason the difference of longitude WE is greater than the departure. Again, let the ship return from L to A along the rhumb line LA, her meridional distance will then be A*a*, and her departure *xu*+*we*+*qz*+*gc*, &c. the same as before; for the elementary triangles are equal, an equal portion of the ship's track being the diagonal of each. Here the meridional distance A*a* is greater than the departure; hence in the same course, or track, backward and forward, the departure and difference of longitude remain the same, but the meridional distance is variable.

(Y) While the course remains the same, it has been shewn that the departure is greater than the meridional distance L*l*, and less than the meridional distance A*a*; yet it is very nearly equal to the meridional distance MN, in the middle latitude, between the latitude sailed from, and the latitude arrived at. This was probably a casual discovery; and when the places are near the equator, or when their parallels of latitude are not very far distant from each other; the nautical conclusions, drawn from a supposition that the departure is equal to the meridional distance, in the middle latitude, between the latitude sailed from, and the latitude arrived at, are very nearly the same as the conclusions derived from *Mercator's* sailing. But in high latitudes, or in a long run, when the course is not near some of the four cardinal points, this method is not sufficiently accurate. And, if a ship sail upon *several* courses, she makes a less departure near the pole, and a greater departure near the equator, from one place to another, than if she were to sail in a direct course; yet in such small distances, as a day's run, the difference is almost insensible.

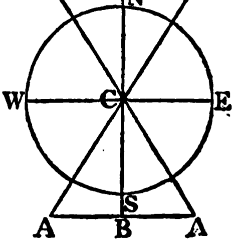
PROPOSITION III. (*Plate III. Fig.* 2.)

(Z) *Straight lines equal in length to the distance run, difference of latitude, and departure, from a right-angled plane triangle; having the angle opposite to the departure equal to the ship's course.*

For it is shewn (U. 377.) that,

$\mathrm{A}b : \mathrm{A}i :: \mathrm{AL} : \mathrm{A}l$; or $\mathrm{A}b : \mathrm{AL} :: \mathrm{A}i : \mathrm{A}l$; and by W. 378, we have $\mathrm{A}b : \mathrm{AL} :: ib : ib + hc + kz + te$, &c. therefore $\mathrm{A}i : \mathrm{A}l :: ib : ib + hc + kz + te$, &c. But the small elementary triangle is considered as straight lined, and is right-angled at i; therefore the triangle to which it is similar, may be considered as straight lined; AL will be the hypothenuse, $\mathrm{A}l$ the difference of latitude, and the departure $ib + hc + kz + te + yu + 50\,\mathrm{L}$.

(A) All problems solved by the preceding propositions are said to be in plane sailing; because the very same conclusions would be drawn if the earth were a plane, and all the meridians parallel to each other. Hence it appears that plane sailing is true, so far as course, distance, difference of latitude, and departure, are concerned.



(B) SCHOLIUM. Since, from the third proposition, the distance run, difference of latitude, and departure from a plane triangle; let CA in the annexed figure represent the distance, CB the difference of latitude, AB the departure, and the angle ACB the course (C. 52.), then will CAB be the complement of the course. Hence are deduced the following proportions for solving *all* the cases that can occur in the practice of plane sailing.

1. *Radius,*
: *distance;*
:: *sine of the course,*
: *departure.*

2. *Radius,*
: *distance;*
:: *cosine of the course,*
: *difference of latitude.*

3. *Cosine of course,*
: *diff. of latitude;*
:: *radius,*
: *distance.*

4. *Cosine of course,*
: *diff. of latitude;*
:: *sine of course,*
: *departure.*

5. *Distance,*
: *radius;*
:: *difference of latitude,*
: *cosine of course.*

6. *Distance,*
: *radius;*
:: *departure,*
: *sine of course.*

7. *Sine of course,*
: *departure;*
:: *radius,*
: *distance.*

8. *Sine of course,*
: *departure;*
:: *cosine of course,*
: *difference of latitude.*

9. *Difference of latitude,*
: *radius;*
:: *the departure,*
: *tangent of the course.*

For,

1. Rad : AC :: sine of C : AB	(Y. 34.)
2. Rad : AC :: sine of A = cos C : BC	
3. Cos C : BC :: rad : AC	(*From the 2d.*)
4. Cos C : BC :: sine C : AB	(1*st and* 2*d.*)
5. AC : rad :: BC : cos C	(2*d inverted.*)
6. AC : rad :: AB : sine of C	(1*st inverted.*)
7. Sine of C : AB :: rad : AC	(*From the first.*)
8. Sine of C : AB :: cos C : BC	(1*st and* 2*d.*)
9. BC : rad :: AB : tang C	(Y. 34.)

CHAP. II.

PARALLEL AND MIDDLE LATITUDE SAILING.

(C) WHEN a ship sails directly east or west, upon any parallel of latitude, the method of finding her difference of longitude, distance, &c. is called parallel sailing.

(D) Middle latitude sailing is founded on a *supposition*, that the meridional distance, half way between the latitude sailed from, and that bound to, is equal to the departure which the ship makes in sailing from the one latitude to the other. It is a compound of plane and parallel sailing.

PROPOSITION I. (*Plate III. Fig.* 2.)

(E) *In sailing upon any parallel of latitude, or directly east or west; the cosine of the latitude, is to radius, as the distance run, is to the difference of longitude.*

Let C be the centre of the sphere, P the north pole, WOQE an arc of the equator; P*m*W, P*n*O, meridians; *mn* the distance run on the parallel *mns*, and WO the difference of longitude.

In the plane of the parallel *mns*, draw *mr*, *nr*, meeting PC the axis of the sphere in *r*; and in the plane of the equator WOQE, draw WC, OC, meeting the axis of the sphere in C: then the angle *mrn* is equal to the angle WCO (R. 135.); therefore the arcs *mn* and WO are similar. Hence,

mn : parallel of lat. : : WO : equator.
Or, *mn* : WO : : parallel of lat. : equator.

But the circumferences of circles are as their radii (*Keith's* Geometry 196 *of* VII.); therefore

mr : WC : : parallel of lat. : equator.
Consequently *mr* : WC : : *mn* : WO.

But *mr* is the sine of the arc P*m*, or the cosine of the latitude W*m*, and WC is the radius of the sphere.

Therefore, cosine lat. : radius : : dist. : diff. long. Q. E. D.

(F) COROLLARY I. *Radius, is to the cosine of the latitude; as the difference of longitude is to the distance run.*

(G) Corollary II. *The distance between any two meridians, on one parallel of latitude, is to the distance between the same two meridians on any other parallel, as the cosine of the latitude of the first parallel, is to the cosine of the latitude of the second.*

(H) Scholium. This proposition and Corollary I. will solve all the cases that can occur in parallel sailing. Examples exercising these cases have been given already at page 296.

PROPOSITION II. (*Plate III. Fig. 2.*)

(I) *Cosine of the middle latitude (viz. the latitude half way between the latitude sailed from, and that bound to), is to radius; as the departure is to the difference of longitude (nearly).*

Let A*da* represent the parallel of latitude sailed from, *l*L that bound to, M*kzq*N the middle latitude between these parallels, and A*bczeu*L the rhumb line, or track of the ship, in sailing from A to L.

Now whether the ship sail from A to L, or from L to A, it has been shewn in the scholium, X. 378. that the departure will be $ib+hc+kz+te$, &c. and (Y. 378.) this departure is *nearly* equal to the meridional distance M*kzq*N. But,

Cosine WM : radius : : M*kzq*N : WOQE. (E. 381.)

Viz. Cosine mid. lat. : rad. : : departure : diff. long. (*nearly*). Q. E. D.

PROPOSITION III. (*Plate III. Fig. 2.*)

(K) *Difference of latitude, is to the difference of longitude; as the cosine of middle latitude, is to the tangent of the course (nearly).*

For, diff. lat. : radius : : dep. : tang. course (B. 379.)

And cos. mid. lat. : radius : : dep. : diff. long. (I. 382.)

Hence, diff. lat. × tang. course = radius × dep.

And, cos. mid. lat. × diff. long. = radius × dep.

Consequently, diff. lat. × tang. course = cos. mid. lat. × diff. long.

∴ Diff. lat. : diff. long. : : cos. mid. lat. : tang. course. Q. E. D.

(L) Corollary. *Cosine mid. lat. : tang. course : : diff. lat. diff. long. (nearly).*

PROPOSITION IV. (*Plate III. Fig. 2.*)

(M) *Distance sailed, is to the difference of longitude, as cosine of the middle latitude, is to sine of the course (nearly).*

For, radius : distance : : sine of course : dep. (B. 379.)

And, cos. mid. lat. : radius : : dep : diff. long. (I. 382.)

Hence, radius × dep. = distance × sine of course.

And radius × dep. = cos. mid. lat. × diff. long.

Consequently distance × sine of course = cos. mid. lat. × diff. long.

∵ Distance : diff. long. :: cos. mid. lat : sine of course. Q.E.D.

(N) SCHOLIUM. The second, third, and fourth propositions include all the cases that can occur in the practice of *middle latitude* sailing; the several proportions, for the sake of uniformity, are here collected.

1. *Radius,*
 : *difference of longitude ;*
 :: *cosine of middle latitude,*
 : *departure.*

2. *Cosine of middle latitude,*
 : *departure ;*
 :: *radius,*
 : *difference of longitude.*

3. *Difference of latitude,*
 : *difference of longitude ;*
 :: *cosine of middle latitude,*
 : *tangent of the course.*

4. *Cosine of middle latitude,*
 : *tangent of the course ;*
 :: *difference of latitude,*
 : *difference of longitude.*

5. *Cosine of middle latitude,*
 : *sine of the course ;*
 :: *distance,*
 : *difference of longitude.*

See pages 296 and 297.

CHAP. III.

MERCATOR'S SAILING.

(O) MERCATOR'S sailing is the art of finding on a plane surface, the motion of a ship upon any assigned course by the compass, which shall be true in latitude, longitude, and distance sailed.*

(P) In a Mercator's chart, from which this method of sailing is derived, the degrees of *longitude* are every where equal, the degrees of latitude increase as you approach the poles, and the rhumb line, or track the ship describes, is represented by a straight line.

(Q) On the globe the degrees of *latitude* are every where equal, and the degrees of longitude decrease as you approach the poles: that is, the distance between any two meridians in any latitude, is to the difference of longitude between these meridians, as the cosine of that latitude, is to the radius. (F. 381.) But, on a Mercator's chart, the distance between any two meridians is made equal to their difference of longitude, and consequently the parallels of latitude are enlarged, in the ratio of the cosine of the latitude to the radius: now, in order that the angle which the rhumb line makes with the several meridians on the chart, may correspond with the same angle on the globe, it will be necessary to increase the meridians in the same ratio, by which the parallels of latitude are increased, viz. as the cosine of the latitude is to the radius, or which is the same thing, as the radius is to the secant of the latitude. Hence the degrees of the meridians on a Mercator's chart increase towards the poles, as the secant of the latitude increases; likewise all the parallels of latitude, and every part of them, are larger than they are on the globe in the ratio of the radius to the secant of the latitude. Hence, though the latitudes, longitudes, and bearings of places are truly represented on a Mercator's chart, the distances are distorted in various proportions. To render these observations more clear, let us suppose WEL*l* (*Plate VI. Fig.* 24.) a Mercator's chart, constructed to represent the spherical surface WEL*l* (*Plate III. Fig.* 2.) Then w*l*; 10, 10; 20, 20; &c. are meridians; ABCZ*e*UL a rhumb line; IB, HC, KZ, T*e*, YU, &c. parallels of latitude, each straight line on the chart representing its corresponding arc on the spherical surface.

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* This includes the whole theory and practice of navigation, and if any method could be devised for measuring a ship's course and distance truly, nothing more would be wanted to complete the art.

The meridians in this projection being parallel to each other, IB is equal to w, 10; hence IB is the difference of longitude on the chart, corresponding with *ib* on the sphere. Therefore cos. w*i* : radius : : *ib* : IB (E. 381.)

Now to make the elementary triangle AIB on the chart, equiangular with A*ib* on the sphere, the difference of latitude A*i* must be increased in the same ratio with the departure *ib*; hence cosine w*i* : radius : : A*i* : AI. But the cosine of any arc is to the radius, as the radius is to the secant; therefore radius : secant of w*i* : : A*i* : AI.

(R) But A*i* is an indefinitely small portion of the sphere which is increased to AI on the chart. Let us suppose the radius of the sphere an unit, and the difference of latitude A*i* 1 minute[*]; then the last proportion will be 1 : secant w*i* : : 1′ : AI; consequently in this case, the increased minute will be equal to the secant of the latitude.

(S) Hence it follows that the natural secant of any latitude, will be equal to the increase of the next minute on the chart, nearly.

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[*] It is evident that if A*i* were to represent a smaller portion of the meridian than one minute, the conclusions derived from such a supposition would be more accurate.

(T) *The meridional line on a Mercator's chart (constructed on the above principles) is equal in length to the sum of all the natural secants of every minute of latitude contained in it.*

Thus suppose the difference of latitude A*l* on the sphere to be 6 minutes, then the enlarged meridian A*l* on the chart is equal to AI + BH + CK + ZT + *e*Y + U, 50; and these are the secants of WA; 10, B; 20, C; 30, Z; &c.

Hence is derived Mr. Wright's method of constructing a table of meridional parts.

(U) The difference of latitude (in geographical miles) between two places on the globe, is generally called the proper difference of latitude, to distinguish it from the difference of latitude (in geographical miles) on the chart, which is called the meridional difference of latitude.

(W) *A plane right-angled triangle formed by the ship's course, distance, difference of latitude, and departure* (Z. 379.) *will be similar to a plane triangle formed upon a Mercator's chart, by the enlarged distance, meridional difference of latitude and difference of longitude.*

For all the elementary right-angled triangles on the sphere, (*Plate III. Fig.* 2.) viz. A*ib*, *bhc*, *ckz*, *zte*, &c. are equiangular and similar (U. 377.); and from the very nature of the chart, the elementary triangles (*Plate VI. Fig.* 24.) AIB, BHC, CKZ, ZT*e*, &c. are likewise equiangular and similar, not only to each other, but to the elementary triangles A*ib*, *bhc*, &c. on the sphere.

Now the elementary triangles on the sphere may be truly represented by a plane triangle (Z. 379.); therefore the elementary triangles on the chart may likewise be truly represented by a plane triangle; equiangular and similar.

(X) I. MR. WRIGHT'S *method of constructing a table of meridional parts.*

Meridional parts of 1′ = Natural secant of 1′
M. P. of 2′ = Natural secants of 1′ + 2′
M. P. of 3′ = Natural secants of 1′ + 2′ + 3′
M. P. of 4′ = Natural secants of 1′ + 2′ + 3′ + 4′
M. P. of 5′ = Natural secants of 1′ + 2′ + 3′ + 4′ + 5′, &c.

Hence by a table of natural secants, as Sherwin's Tables, or Dr. Hutton's.

M. P. of 1′ = 1·0000000, Nat. sec. 1′ = 1·0000000.
M. P. of 2′ = 2·0000002, Nat. sec. 2′ = 1·0000002.
M. P. of 3′ = 3·0000006, Nat. sec. 3′ = 1·0000004.
M. P. of 4′ = 4·0000013, Nat. sec. 4′ = 1·0000007.
M. P. of 5′ = 5·0000024, Nat. sec. 5′ = 1·0000011, &c.

And by adding together the natural secants of a few pages, in either of the above-mentioned books, you will find

The meridional parts of 1 degree = 60·0031231.
M. P. of 2 degrees = 120·0246423.
M. P. of 3 degrees = 180·0829505.
M. P. of 4 degrees = 240·1963752.
M. P. of 5 degrees = 300·8833699.

(Y) In this manner (by a continual addition of the secants) Mr. Edward Wright formed the FIRST table of meridional parts, which is contained in his celebrated work intitled, *Certain Errors in Navigation Detected and Corrected.* This book was first published in 1599. Mr. William Oughtred was the next who constructed a table of meridional parts, by the continual addition of the intermediate secants of ½, 1½, 2½, &c. minutes; and these tables were again corrected, and extended, by Sir Jonas Moor.

(Z) It must be admitted that *Wright's* method of constructing a table of meridional parts is not *strictly* geometrical; for the cosines of any two parallels of latitude are not precisely the same, even if you suppose the common difference between these parallels, to be less than *one* minute of a degree, and the secants of such parallels do not increase in any regular ratio. A table constructed on *Wright's* principles, in high latitudes, will exceed the truth, a small matter. This, the learned author was sufficiently aware of; for he says he rejected some of the decimal parts of the secants in making his table. "Because that "indeed, at every point of latitude, a minute of the meridian "in this nautical planisphære [*Mercator's Chart*] hath some-"what lesse proportion to a minute of the parallel adjoyning "towards the *Æquinoctial*, than the secans of that parallels la-"titude hath to the whole sine. But in this table it was thought "sufficient to use such exactness as that thereby (in drawing "the lineaments of the nautical planisphære) sensible error "might be avoided. He that listeth to be more precise may "make the like tables to decades or tennes of seconds, out of "*Ioachimus Rhæticus* his *Canon magnus triangulorum.* Not-"withstanding the *Geometrician* that desireth *exact* truth, "cannot be so satisfied neither."

(A) II. ANOTHER *Method of constructing a table of meridional parts from the secant of an arc.*

It is shewn (G. 129.) that if a = the length of any arc, the radius being 1, the secant will be

$$1+\frac{a^2}{2}+\frac{5a^4}{24}+\frac{61a^6}{720}+\frac{277a^8}{8064}+\frac{50521a^{10}}{3628800}+\frac{540553a^{12}}{95800320}+\text{\&c.}$$

This multiplied by $\dot{a}$, the fluxion of the arc, will produce

$$\dot{a}+\frac{a^2\dot{a}}{2}+\frac{5a^4\dot{a}}{24}+\frac{61a^6\dot{a}}{720}+\frac{277a^8\dot{a}}{8064}+\frac{50521a^{10}\dot{a}}{3628800},$$

&c. the fluxion of the sum of the secants in the arc a, the fluent of which is

$$a+\frac{a^3}{6}+\frac{a^5}{24}+\frac{61a^7}{5040}+\frac{277a^9}{72576}+\frac{50521a^{11}}{3991680}+,$$

&c. the sum of all the secants contained in a.

EXAMPLE.

Let it be required to find the meridional parts correspondent to 5 degrees. The length of the arc of one minute being ·000290888208665, &c.*, supposing the radius an unit.

First, 5° × 60 × ·00029088820866. = ·087266462599, &c. = a the length of an arc of 5 degrees.

$a =$ + ·087266462599

$\frac{1}{6}a^3 =$ + 110762019

$\frac{1}{24}a^5 =$ + 210875

$\frac{61}{5040}a^7 =$ + 466

Hence $a+\frac{1}{6}a^3+\frac{1}{24}a^5+\frac{61}{5040}a^7$, &c. = ·087377435959

This result, divided by ·000290888208665, &c. the length of the arc of 1′ will give the meridional parts of 5° = 300·381498, true to the last place of decimals.

By a similar process you will find the meridional parts of 10° to be 603·0695795. Proceed in the same manner to find the meridional parts of any other degree of latitude.

(B) III. *The method of constructing a table of meridional parts, by a table of logarithmical tangents.*

PROPOSITION I. (*Plate VI. Fig. 25.*)

If the logarithmical tangent of half the complement of any degree of latitude be subtracted from 10, *and the remainder be multiplied by* 7915.7044679, &c.; *the product will give the meridional parts (in minutes), correspondent to that latitude.*

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* Gardiner's edition of Sherwin's Logarithms, page 44 (1742). Traité de Trigonométrie, par Cagnoli (Table AA), page 474.

DEMONSTRATION. Let P be the pole, EQ a portion of the equator; Q*v* the latitude, O*v* its sine and OE $=$ *vw* its cosine; and E*v* the radius of the sphere.

Put $x =$ Q*v* the latitude, $m =$ the length of Q*v*, on a Mercator's chart, called the meridional parts; $y =$ O*v* $=$ E*w* the sine of the latitude, and $r =$ E*v* the radius of the sphere. Then will EO $=$ *wv* $= \sqrt{r^2-y^2}$, and because the degrees of latitude on a Mercator's chart increase as the cosine of latitude is to radius, we have

$$\sqrt{r^2-y^2} : r :: \dot{x} : \dot{m}, \text{ hence } \dot{m} = \frac{r\dot{x}}{\sqrt{r^2-y^2}}$$

$$\text{But, } \sqrt{r^3-y^2} : r :: \dot{y} : \dot{x}, \text{ conseq. } \dot{x} = \frac{r\dot{y}}{\sqrt{r^2-y^2}}$$

For $\dot{x}$ write its value, then $\dot{m} = \frac{r\dot{y}}{r^2-y^2}$ the fluent of which is $m = r \times 2{\cdot}30258509299404$, &c. $\times \frac{1}{2} \times \log.\frac{r+y}{r-y} +$ the correction, $= r \times 2{\cdot}302585$, &c. $\times \log.\sqrt{\frac{r+y}{r-y}} +$ the correction.

But by Plane Trigonometry, in the triangle *vws*,

vw : rad :: *ws* : tang *wvs*, or co-tang *wsv*.

Viz. $\sqrt{r^2-y^2} : r :: r+y : r \times \frac{r+y}{\sqrt{r^2-y^2}} = r \times \sqrt{\frac{r+y}{r-y} \times \frac{r+y}{r+y}}$ $= r \times \sqrt{\frac{r+y}{r-y}}$ the co-tangent of *wsv*, measured by half the arc P*v*, the complement of the latitude. Hence we have $m =$ $r \times 2{\cdot}302585$, &c. $\times \log.\frac{\text{co-tang. } \frac{1}{2} \text{ comp. lat.}}{r} +$ the correction

But when $m = o$ the co-tangent of half the complement of latitude $= r$ (L. 31.); and hence the correction is nothing, therefore $m = r \times 2{\cdot}302585$, &c. $\times \log.\frac{\text{co-tang. } \frac{1}{2} \text{ comp. lat.}}{r} =$ $r \times 2{\cdot}3025$; &c. $\times \log.\frac{r}{\text{tang. } \frac{1}{2} \text{ comp. lat.}}$; for the co-tangent of an arc divided by the radius, is equal to the radius divided by the tangent (Z. 103.)

But the tables of meridional parts are generally expressed in geographical miles, therefore we must express the radius of the sphere in geographical miles. If the diameter of the earth be 1, the circumference will be 3·14159265589, &c., hence

3·141592653589 : $\frac{1}{2}$ :: 360° × 60 : 3437·74677, &c. consequently $m = 2{\cdot}3025$, &c. × 3437·74677, &c. × log.

$$\frac{r}{\text{tang.}\frac{1}{2}\text{comp. lat.}} = 7915{\cdot}7044679 \times \log.\frac{r}{\text{tang.}\frac{1}{2}\text{comp. lat.}}.$$ Q. E. D.

Note. It would be improper to make the value of r in the latter part of the expression = 3437·7467, &c., unless the table of logarithmical tangents was calculated to the same radius.

EXAMPLE.

Let it be required to find the meridional parts correspondent to 5 degrees.

$$\begin{array}{r} 90^\circ \\ 5 \\ \hline 2\,|\,85 \end{array}$$

Log. radius = 10·

Halfcomp. lat. 42·30 the tang. by Ulacq's tables, = 9·9620524617

·0379475383

this multiplied by 7915·7044679 produces 300·38149846, &c. exactly agreeing with the answer before found by the secants.

In the same manner you will find the meridional parts of 10° to be 603·069579, &c., the logarithmical tangent of 40° being 9·9238135302.

(C) The following table comprehends the meridional parts answering to every five degrees of latitude. The first column contains the degrees, the second the meridional parts copied from *Wright's* original treatise, and the third the meridional parts truly calculated by this problem.—As our tables of meridional parts are seldom carried to decimals, the reader will readily perceive that *Wright's* tables may be used without sensible error.

Degrees.	*Mr. Wright's* Merid. Pts.	*True* Merid. Parts.	Degrees.	*Mr. Wright's* Merid. Pts.	*True* Merid. Parts.	Degrees.	*Mr. Wright's* Merid. Pts.	*True* Merid. Parts.
5	300·3694	300·3815	35	2244·3047	2244·2868	65	5179·3079	5178·8081
10	603·0475	603·0696	40	2622·7559	2622·6902	70	5966·6811	5965·9179
15	910·4325	910·4606	45	3030·1271	3029·9392	75	6971·5485	6970·3390
20	1225·1292	1225·1390	50	3474·6045	3474·4720	80	8377·3416	8375·1970
25	1549·9878	1549·9952	55	3968·1879	3967·9661	85	10769·6200	10764 6210
30	1888·3768	1888·3754	60	4527·7106	4527·3677	89	16317·5324	16299·5563

(D) A table of meridional-parts being formed, by any of the preceding methods, to every degree and minute of the quadrar a Mercator's chart may be readily constructed therefrom.

EXAMPLE. Let it be required to construct a Mercator's c containing 60 degrees of east longitude, and extending from the equator to 40 degrees of north latitude?

I. Draw WE to represent the equator (*Plate VI. Fig.* 24.), on which set off WE = 3600 miles, (the number of miles contained in 60 degrees, the extent of the chart,) from any scale of equal parts.

II. Divide the line WE into 6 equal parts in the points 10, 20, 30, 40, &c., each part containing 10 degrees; and each of these parts may be subdivided into single degrees, half degrees, &c. if necessary. Then straight lines drawn through the points W, 10, 20, 30, &c. perpendicular to WE, will represent the meridians passing through every 10 degrees.

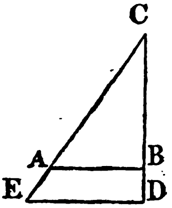
III. From the same scale of equal parts take 2622·7 (the meridional parts answering to 40 degrees) and set them off from W to *l* and from E to L, and join L*l*, which will represent the parallel of 40 degrees of latitude, and determine the extent of the chart northward.

IV. From the scale of equal parts take 603 (the meridional parts answering to 10 degrees) and set them off from W to A, and from E to 10, and join A, 10, this will represent the parallel of 10 degrees. In the same manner W, 20, = 1225; and W, 30, = 1888; and draw the parallels of latitude; or make A, 20, = 1225 − 603; 20, 30, = 1888 − 1225; &c.

If the chart does not being at the equator, it is evident that the meridional parts correspondent to the least latitude contained in it, must be subtracted from the meridional parts of each point of greater latitude.

PROPOSITION II.

(E) *Radius, is to the tangent of the course; as the meridional difference of latitude, is to the difference of longitude.*



In the annexed figure BC is the proper difference of latitude, DC the meridional difference of latitude, the angle ACB the course, AC the distance sailed, AB the departure, and ED the difference of longitude. Then EDC is a right-angled triangle, and similar to ABC (W. 385.) Now DC : radius :: ED : tangent of ACB; therefore radius : tangent of ACB :: DC : ED.

PROPOSITION III.

(F) *The number* ·00012633114, &c.
Is to the natural tangent of the course;
As the difference between the logarithmical tangents of half the complements of the latitudes sailed from, and bound to,

Is to the differences of longitudes between these places, in geographical miles.

Let the less latitude be represented by l, and the greater by L; then the meridional parts correspondent to $l =$

$\log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. } l} \times 7915{\cdot}7044$, &c. (B. 387.) and by the same rule the meridional parts correspondent to $\text{L} =$

$\log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. L}} \times 7915{\cdot}7044$, &c. The difference between these quantities is

$\left(\log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. } l} \sim \log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. L}}\right) \times 7915{\cdot}7044$, &c. the meridional difference of latitude in miles. Now (by E. 390.)

rad. : tang. course :: $\left(\log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. } l} \sim \log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. L}}\right)$ $\times\, 7915{\cdot}7044$, &c. : diff. longitude.

But, $\log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. } l} \sim \log.\dfrac{r}{\text{tang. } \frac{1}{2} \text{ comp. L}}$ is evidently $=$ log. tang. $\frac{1}{2}$ comp. $l \sim$ log. tang. $\frac{1}{2}$ comp. L; hence rad. : tang. course :: (log. tang. $\frac{1}{2}$ comp. $l \sim$ log. tang. $\frac{1}{2}$ comp. L.) $\times\, 7915{\cdot}7044$, &c. : diff. longitude.

Divide the antecedents by $7915{\cdot}7044$, &c. then

$\dfrac{\text{rad}}{7915{\cdot}7044, \text{ \&c.}}$: tang. course :: (log. tang. $\frac{1}{2}$ comp. $l \sim$ log. tang. $\frac{1}{2}$ comp. L.) : diff. longitude, but $\dfrac{\text{rad.}}{7915{\cdot}7044, \text{\&c.}} = \dfrac{1}{7915{\cdot}7044, \text{ \&c.}}$ $= {\cdot}000126331143874$, &c. hence it follows that

${\cdot}000126331143874$, &c. : tang. course :: (long. tang. $\frac{1}{2}$ comp. $l \sim$ log. tang. $\frac{1}{2}$ comp. L) : diff. longitude.

(G) COROLLARY. *Find the logarithmical tangent of half the complement of the latitude sailed from, the logarithmical tangent of half the complement of the latitude bound to, multiply their difference by* 10000, *and find the common logarithm of the product.*

THEN,

The logarithmical tangent of the angle 51°.38′.9″.14‴ $=$ 10·1015103,

Is to the logarithm found above;

As the logarithm tangent of the course,

Is to the logarithm of the difference of longitude in miles.

For, ·00012633114, &c. : the natural tang. course :: (log. tang. $\frac{1}{2}$ comp. $l \sim$ log. tang. $\frac{1}{2}$ comp. L.) : diff. longitude in miles.

Multiply the antecedents by 10000, then

1·2633114, &c. : the natural tang. course :: (log. tang. $\frac{1}{2}$ comp.

$l. \sim$ log.tang.$\frac{1}{2}$comp. L.) + 10000 : diff. longitude, but 1·2633114, &c. is the natural tangent of 51°.38′.9″.14‴, the logarithm of which is 10·1015103; hence by taking the logarithms of all the terms in the last proportion, we deduce the above rule.

(H) SCHOLIUM.

The second proposition (E. 390.) and the following, which is immediately deduced therefrom, (viz. Meridional difference of latitude, is to difference of longitude; as radius is to the tangent of the course,) will solve all the cases that can occur in Mercator's sailing by the help of a table of meridional parts. And the corollary (G. 391.) will solve the whole with the assistance of a table of logarithmical tangents (independent of a table of meridional parts) by varying the proportion.

THE END OF NAVIGATION.

TABLES.

I. A table of *the Logarithms* of Numbers, from an unit to ten thousand.

II. A table of *natural sines* to every degree and minute of the quadrant.

III. A table of *Logarithmical sines and tangents* to every degree and minute of the quadrant.

IV. A table of the Refraction in altitude, of the heavenly bodies.

V. A table of the depression, or dip, of the horizon of the sea.

VI. A table of the sun's parallax in altitude.

VII. A table of the augmentation of the moon's semidiameter.

VIII. A table of the right ascensions and declinations of thirty-six principal fixed stars, corrected to the beginning of the year 1822.

TABLE I. LOGARITHMS OF NUMBERS FROM 1 TO 10,000.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	00000	21	32222	41	61278	61	78533	81	90849
2	30103	22	34242	42	62325	62	79239	82	91381
3	47712	23	36173	43	63347	63	79934	83	91908
4	60206	24	38021	44	64345	64	80618	84	92428
5	69897	25	39794	45	65321	65	81291	85	92942
6	77815	26	41497	46	66276	66	81954	86	93450
7	84510	27	43136	47	67210	67	82607	87	93952
8	90309	28	44716	48	68124	68	83251	88	94448
9	95424	29	46240	49	69020	69	83885	89	94939
10	00000	30	47712	50	69897	70	84510	90	95424
11	04139	31	49136	51	70757	71	85126	91	95904
12	07918	32	50515	52	71600	72	85733	92	96379
13	11394	33	51851	53	72428	73	86332	93	96848
14	14613	34	53148	54	73239	74	86923	94	97313
15	17609	35	54407	55	74036	75	87506	95	97772
16	20412	36	55630	56	74819	76	88081	96	98227
17	23045	37	56820	57	75587	77	88649	97	98677
18	25527	38	57978	58	76343	78	89209	98	99123
19	27875	39	59106	59	77085	79	89763	99	99564
20	30103	40	60206	60	77815	80	90309	100	00000

No.	0	1	2	3	4	5	6	7	8	9
100	00000	00043	00087	00130	00173	00217	00260	00303	00346	00389
101	0432	0475	0518	0561	0604	0647	0689	0732	0775	0817
102	0860	0903	0945	0988	1030	1072	1115	1157	1199	1242
103	1284	1326	1368	1410	1452	1494	1536	1578	1620	1662
104	1703	1745	1787	1828	1870	1912	1953	1995	2036	2078
105	2119	2160	2202	2243	2284	2325	2366	2407	2449	2490
106	2531	2572	2612	2653	2694	2735	2776	2816	2857	2898
107	2938	2979	3019	3060	3100	3141	3181	3222	3262	3302
108	3342	3383	3423	3463	3503	3543	3583	3623	3663	3703
109	3743	3782	3822	3862	3902	3941	3981	4021	4060	4100
110	04139	04179	04218	04258	04297	04336	04376	04415	04454	04493
111	4532	4571	4610	4650	4689	4727	4766	4805	4844	4883
112	4922	4961	4999	5038	5077	5115	5154	5192	5231	5269
113	5308	5346	5385	5423	5461	5500	5538	5576	5614	5652
114	5690	5729	5767	5805	5843	5881	5918	5956	5994	6032
115	6070	6108	6145	6183	6221	6258	6296	6333	6371	6408
116	6446	6483	6521	6558	6595	6633	6670	6707	6744	6781
117	6819	6856	6893	6930	6967	7004	7041	7078	7115	7151
118	7188	7225	7262	7298	7335	7372	7408	7445	7482	7518
119	7555	7591	7628	7664	7700	7737	7773	7809	7846	7882
120	07918	07954	07990	08027	08063	08099	08135	08171	08207	08243
121	8279	8314	8350	8386	8422	8458	8493	8529	8565	8600
122	8636	8672	8707	8743	8778	8814	8849	8884	8920	8955
123	8991	9026	9061	9096	9132	9167	9202	9237	9272	9307
124	9342	9377	9412	9447	9482	9517	9552	9587	9621	9656
125	9691	9726	9760	9795	9830	9864	9899	9934	9968	10003
126	10037	10072	10106	10140	10175	10209	10243	10278	10312	0346
127	0380	0415	0449	0483	0517	0551	0585	0619	0653	0687
128	0721	0755	0789	0823	0857	0890	0924	0958	0992	1025
129	1059	1093	1126	1160	1193	1227	1261	1294	1327	1361
130	11394	11428	11461	11494	11528	11561	11594	11628	11661	11694
131	1727	1760	1793	1826	1860	1893	1926	1959	1992	2024
132	2057	2090	2123	2156	2189	2222	2254	2287	2320	2352
133	2385	2418	2450	2483	2516	2548	2581	2613	2646	2678
134	2710	2743	2775	2808	2840	2872	2905	2937	2969	3001
135	3033	3066	3098	3130	3162	3194	3226	3258	3290	3322
136	3354	3386	3418	3450	3481	3513	3545	3577	3609	3640
137	3672	3704	3735	3767	3799	3830	3862	3893	3925	3956
138	3988	4019	4051	4082	4114	4145	4176	4208	4239	4270
139	4301	4333	4364	4395	4426	4457	4489	4520	4551	4582
140	14613	14644	14675	14706	14737	14768	14799	14829	14860	14891
141	4922	4953	4983	5014	5045	5076	5106	5137	5168	5198
142	5229	5259	5290	5320	5351	5381	5412	5442	5473	5503
143	5534	5564	5594	5625	5655	5685	5715	5746	5776	5806
144	5836	5866	5897	5927	5957	5987	6017	6047	6077	6107
145	6137	6167	6197	6227	6256	6286	6316	6346	6376	6406
146	6435	6465	6495	6524	6554	6584	6613	6643	6673	6702
147	6732	6761	6791	6820	6850	6879	6909	6938	6967	6997
148	7026	7056	7085	7114	7143	7173	7202	7231	7260	7289
149	7319	7348	7377	7406	7435	7464	7493	7522	7551	7580
150	17609	17638	17667	17696	17725	17754	17782	17811	17840	17869
151	7898	7926	7955	7984	8013	8041	8070	8099	8127	8156
152	8184	8213	8241	8270	8298	8327	8355	8384	8412	8441
153	8469	8498	8526	8554	8583	8611	8639	8667	8696	8724
154	8752	8780	8808	8837	8865	8893	8921	8949	8977	9005
155	9033	9061	9089	9117	9145	9173	9201	9229	9257	9285
156	9312	9340	9368	9396	9424	9451	9479	9507	9535	9562
157	9590	9618	9645	9673	9700	9728	9756	9783	9811	9838
158	9866	9893	9921	9948	9976	20003	20030	20058	20085	20112
159	20140	20167	20194	20222	20249	0276	0303	0330	0358	0385

No.	0	1	2	3	4	5	6	7	8	9
160	20412	20439	20466	20493	20520	20548	20575	20602	20629	20656
161	0683	0710	0737	0763	0790	0817	0844	0871	0898	0925
162	0952	0978	1005	1032	1059	1085	1112	1139	1165	1192
163	1219	1245	1272	1299	1325	1352	1378	1405	1431	1458
164	1484	1511	1537	1564	1590	1617	1643	1669	1696	1722
165	1748	1775	1801	1827	1854	1880	1906	1932	1958	1985
166	2011	2037	2063	2089	2115	2141	2167	2194	2220	2246
167	2272	2298	2324	2350	2376	2401	2427	2453	2479	2505
168	2531	2557	2583	2608	2634	2660	2686	2712	2737	2763
169	2789	2814	2840	2866	2891	2917	2943	2968	2994	3019
170	23045	23070	23096	23121	23147	23172	23198	23223	23249	23274
171	3300	3325	3350	3376	3401	3426	3452	3477	3502	3528
172	3553	3578	3603	3629	3654	3679	3704	3729	3754	3779
173	3805	3830	3855	3880	3905	3930	3955	3980	4005	4030
174	4055	4080	4105	4130	4155	4180	4204	4229	4254	4279
175	4304	4329	4353	4378	4403	4428	4452	4477	4502	4527
176	4551	4576	4601	4625	4650	4674	4699	4724	4748	4773
177	4797	4822	4846	4871	4895	4920	4944	4969	4993	5018
178	5042	5066	5091	5115	5139	5164	5188	5212	5237	5261
179	5285	5310	5334	5358	5382	5406	5431	5455	5479	5503
180	25527	25551	25575	25600	25624	25648	25672	25696	25720	25744
181	5768	5792	5816	5840	5864	5888	5912	5935	5959	5983
182	6007	6031	6055	6079	6102	6126	6150	6174	6198	6221
183	6245	6269	6293	6316	6340	6364	6387	6411	6435	6458
184	6482	6505	6529	6553	6576	6600	6623	6647	6670	6694
185	6717	6741	6764	6788	6811	6834	6858	6881	6905	6928
186	6951	6975	6998	7021	7045	7068	7091	7114	7138	7161
187	7184	7207	7231	7254	7277	7300	7323	7346	7370	7393
188	7416	7439	7462	7485	7508	7531	7554	7577	7600	7623
189	7646	7669	7692	7715	7738	7761	7784	7807	7830	7852
190	27875	27898	27921	27944	27967	27989	28012	28035	28058	28081
191	8103	8126	8149	8171	8194	8217	8240	8262	8285	8307
192	8330	8353	8375	8398	8421	8443	8466	8488	8511	8533
193	8556	8578	8601	8623	8646	8668	8691	8713	8735	8758
194	8780	8803	8825	8847	8870	8892	8914	8937	8959	8981
195	9003	9026	9048	9060	9092	9115	9137	9159	9181	9203
196	9226	9248	9270	9292	9314	9336	9358	9380	9403	9425
197	9447	9469	9491	9513	9535	9557	9579	9601	9623	9645
198	9667	9688	9710	9732	9754	9776	9798	9820	9442	9863
199	9885	9907	9929	9951	9973	9994	30016	30038	30060	30081
200	30103	30125	30146	30168	30190	30211	30233	30255	30276	30298
201	0320	0341	0363	0384	0406	0428	0449	0471	0492	0514
202	0535	0557	0578	0600	0621	0643	0664	0685	0707	0728
203	0750	0771	0792	0814	0835	0856	0878	0899	0920	0942
204	0923	0984	1006	1027	1048	1069	1091	1112	1133	1154
205	1175	1197	1218	1239	1260	1281	1302	1323	1345	1366
206	1387	1408	1429	1450	1471	1492	1513	1534	1555	1576
207	1597	1618	1639	1660	1681	1702	1723	1744	1765	1785
208	1806	1827	1848	1869	1890	1911	1931	1952	1973	1994
209	2015	2035	2056	2077	2098	2118	2139	2160	2181	2201
210	32222	32243	32263	32284	32305	32325	32346	32366	32387	32408
211	2428	2449	2469	2490	2510	2531	2552	2572	2593	2613
212	2634	2654	2675	2695	2715	2736	2756	2777	2797	2818
213	2838	2858	2879	2899	2919	2940	2960	2980	3001	3021
214	3041	3062	3082	3102	3122	3143	3163	3183	3203	3224
215	3244	3264	3284	3304	3325	3345	3365	3385	3405	3425
216	3445	3465	3486	3506	3526	3546	3566	3586	3606	3626
217	3646	3666	3686	3706	3726	3746	3766	3786	3806	3826
218	3846	3866	3885	3905	3925	3945	3965	3985	4005	4025
219	4044	4064	4084	4104	4124	4143	4163	4183	4203	4223

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220	34242	34262	34282	34301	34321	34341	34361	34380	34400	34420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	35005
224	35025	35044	35064	35083	35102	35122	35141	35160	35180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	36003	36021	36040	36059	36078	36097	36116	36135	36154
230	36173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	37014	37033	37051	37070	37088
235	37107	37125	37144	37162	37181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	38003
240	38021	38039	38057	38075	38093	38112	38130	38148	38166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	39005	39023	39041	39058	39076
246	39094	39111	39129	39146	39164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
251	967	985	40002	40019	40037	40054	40071	40088	40106	40123
252	40140	40157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	41010	41027	41044	41061	41078	41095	41111	41128	41145
258	41162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	42012	42029	42045	42062	42078	42095	42111	42127	42144
264	42160	177	193	210	226	243	259	275	292	398
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	43008	43024	43040	43056	43072	43088	43104	43120
270	43136	43152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	44012	44028	44044	44059	44075
276	44091	44107	44122	44138	44154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700

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280	44716	44731	44747	44762	44778	44793	44809	44824	44840	44855
281	871	886	902	917	932	948	963	979	994	45010
282	45025	45040	45056	45071	45086	45102	45117	45133	45148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	46000	46015	46030	46045	46060	46075
289	46090	46105	46120	46135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	47012	47026	47041	47056	47070	47085	47100	47114
296	47129	47144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48001	48015	48029	48044	48058	48073	48087	48101	48116	48130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	49010	49024	49038	49052	49066	49080	49094	49108	49122
310	49136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	50010	50024	50037	50051	50065	50079	50092
317	50106	50120	50133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	51001	51014	51028	51041
324	51055	51068	51081	51095	51108	51121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	52009	52022	52035	52048	52061	52075	52088	52101
332	52114	52127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	53007
339	53020	53033	53046	53058	53071	53084	53097	53110	53122	135

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340	53148	53161	53173	53186	53199	53212	53224	53237	53250	53263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	54008	54020
347	54033	54045	54058	54070	54083	54095	54108	54120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	55011
355	55023	55035	55047	55060	55072	55084	55096	55108	55121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	56003	56015	56027	56038	56050	56062	56074	56086	56098
364	56110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	57008	57019	57031	57043
372	57054	57066	57078	57089	57101	57113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	58001	58013	58024	58035	58047	58058	58070	58081
381	58092	58104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	59006	59017	59028	59040	59051	59062	59073	59084	59095
390	59106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	60010	60021	60032	60043	60054	60065	60076	60086
399	60097	60108	119	130	141	152	163	173	184	195

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400	60206	60217	60228	60239	60249	60260	60271	60282	60293	60304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	61002	61013	61023	61034	61045	61055
408	61066	61077	61087	61098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	263
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	62003
417	62014	62024	62034	62045	62055	62066	62076	62086	62097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	63002	63012	63022	63033
427	63043	63053	63063	63073	63083	63094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	64008	64018	64028	64038
437	64048	64058	64068	64078	64088	64098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	65002	65011	65021
447	65031	65040	65050	65060	65070	65079	65089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	639	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	66001	66011	66020	66030	66039	66049	66058	66068	66077
458	66087	096	106	115	124	134	143	153	162	172
859	181	191	200	210	219	229	238	247	257	266

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460	66276	66285	66295	66304	66314	66323	66332	66342	66351	66361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	67006	67015
468	67025	67034	67043	67052	67062	67071	67080	67089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	68006	68015	68024
479	68034	68043	68052	68061	68070	68079	68088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	706	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	69002	69011
490	69020	69028	69037	69046	69055	69064	69073	69082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
501	984	992	70001	70010	70018	70027	70036	70044	70053	70062
502	70070	70079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	251	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	71003
513	71012	71020	71029	71037	71046	71054	71063	71071	71079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	218	257
516	265	273	282	290	299	307	315	324	332	341
517	319	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592

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521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	72008
525	72016	72024	72032	72041	72049	72057	72066	72074	72082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	73006	73014	73022	73030	73038	73046	73054	73062	73070
538	73078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	74005	74013	74020	74028
550	74036	74044	74052	74060	74068	74076	084	092	099	107
551	215	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	75005	75012	75020	75028	75035	75043
563	75051	75059	75066	75074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
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571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	76005	76012	76020	76027	76035
576	76042	76050	76057	76065	76072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	165
578	193	200	208	215	223	230	288	245	253	260
579	268	275	283	290	298	305	313	320	328	335

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580	76343	76350	76358	76365	76373	76380	76388	76395	76403	76410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	77004
589	77012	77019	77026	77034	77041	77048	77056	77063	77070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	78003	78010	78017	78025
603	78032	78039	78046	78053	78061	78068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	79000	79007	79014	79021
617	79029	79036	79043	79050	79057	79064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
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631	80003	80010	80017	80024	80030	80037	80044	80051	80058	80065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611

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642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	81003	81010	81017
646	81023	81030	81037	81043	81050	81057	81064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
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659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	82000	82007	82014
661	82020	82027	82033	82040	82046	82053	82060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	83001	83008	83014	83020	83027	83033	83040	83046	83052
677	83059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
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686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
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691	948	954	960	967	973	979	985	992	998	84004
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693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	278	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504

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701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
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704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85003	85009	85016	85022	85028	85034	85040	85046	85052	85058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	86004	86010	86016	86022	86028
725	86034	86040	86046	86052	86058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
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742	87040	87046	87052	87058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
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746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
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749	448	454	460	466	471	477	483	489	495	500
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751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
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759	88024	88030	88036	88041	88047	88053	058	064	070	076

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763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
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770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	89003	89009	89014	89020	89025	89031	89037
777	89042	89048	89053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
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782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
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786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
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792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	90004	90009	90015	90020	90026	90031
795	90037	90042	90048	90053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	91004
813	91009	91014	91020	91025	91030	91036	91041	91046	91052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376

No.	0	1	2	3	4	5	6	7	8	9
820	91381	91387	91392	91397	91403	91408	91413	91418	91424	91429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
831	960	965	971	976	981	986	991	997	92002	92007
832	92012	92018	92023	92028	92033	92038	92044	92049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
851	993	998	93003	93008	93013	93018	93024	93029	93034	93039
852	93044	93049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94002	94007	94012	94017	94022	94027	94032	94037	94042	94047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443

No.	0	1	2	3	4	5	6	7	8	9
880	94448	94453	94458	94463	94468	94473	94478	94483	94488	94493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	95002	95007	95012	95017	95022	95027	95032
892	95036	95041	95046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	96004	96009	96014	96019	96023	96028	96033	96038	96042
913	96047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	97002	97007	97011	97016	97021	97025	97030
934	97035	97039	97044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308

No.	0	1	2	3	4	5	6	7	8	9
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941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98000	98005	98009	98014	98019	98023	98028	98032	98037	98041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	99003	99007	99012	99016	99021	99025	99029
978	99034	99038	99043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996

	0 Degrees.		1 Degree.		2 Degrees.		3 Degrees.		4 Degrees.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	00000	1·00000	01745	99985	03490	99939	05234	99863	06976	99756	60
1	29	00000	774	84	519	938	263	61	07005	54	59
2	58	00000	803	84	548	937	292	60	034	52	58
3	87	00000	832	83	577	936	321	58	063	50	57
4	00116	00000	862	83	606	935	350	57	092	48	56
5	145	00000	891	82	635	934	379	55	121	46	55
6	175	00000	920	82	664	933	408	54	150	44	54
7	204	00000	949	81	693	932	437	52	179	42	53
8	233	00000	978	80	723	931	466	51	208	40	52
9	262	00000	02007	80	752	930	495	49	237	38	51
10	00291	1·00000	036	99979	781	99929	524	99847	266	99736	50
11	320	·99999	065	79	810	927	553	46	295	34	49
12	349	999	094	78	839	926	582	44	324	31	48
13	378	999	123	77	868	925	611	42	353	29	47
14	407	999	152	77	897	924	640	41	382	28	46
15	436	999	181	76	926	923	669	39	411	25	45
16	465	999	211	76	955	922	698	38	440	23	44
17	495	999	240	75	984	921	727	36	469	21	43
18	524	999	269	74	04013	919	756	34	498	19	42
19	553	998	298	74	042	918	785	33	07527	16	41
20	00582	99998	02327	99973	071	99917	814	99831	556	99714	40
21	611	998	356	72	100	916	844	29	585	12	39
22	640	998	385	72	129	915	873	27	614	10	38
23	669	998	414	71	159	913	902	26	643	08	37
24	698	998	443	70	188	912	931	24	672	05	36
25	727	997	472	69	217	911	960	22	701	03	35
26	756	997	501	69	246	910	989	21	730	01	34
27	785	997	530	68	275	909	06018	19	759	99699	33
28	814	997	560	67	304	907	047	17	788	96	32
29	844	996	589	66	333	906	076	15	817	94	31
30	00873	99996	618	99966	362	99905	105	99813	846	92	30
31	902	996	647	65	391	904	134	12	875	89	29
32	931	996	676	64	420	902	163	10	904	87	28
33	960	995	705	63	449	901	192	08	933	85	27
34	989	995	734	63	478	900	221	06	962	83	26
35	01018	995	763	62	507	898	250	04	991	80	25
36	047	995	792	61	536	897	279	03	08020	78	24
37	076	994	821	60	565	896	308	01	049	76	23
38	105	994	850	59	594	894	337	99799	078	73	22
39	134	994	879	59	623	893	366	97	107	71	21
40	01164	99993	908	99958	04653	99892	395	95	136	99668	20
41	193	993	938	57	682	890	424	93	165	66	19
42	222	993	967	56	711	889	453	92	194	64	18
43	251	992	996	55	740	888	482	90	223	61	17
44	280	992	03025	54	769	886	511	88	252	59	16
45	309	991	054	53	798	885	540	86	281	57	15
46	338	991	083	52	827	883	569	84	310	54	14
47	367	991	112	52	856	882	598	82	339	52	13
48	396	990	141	51	885	881	627	80	368	49	12
49	425	990	170	50	914	879	06656	78	397	47	11
50	01454	99989	199	99949	04943	99878	685	99776	426	99644	10
51	483	989	228	48	972	876	714	74	455	42	9
52	513	989	257	47	05001	875	743	72	484	39	8
53	542	988	286	46	030	873	773	70	513	37	7
54	571	988	316	45	059	872	802	68	542	35	6
55	600	987	345	44	088	870	831	66	571	32	5
56	629	987	374	43	117	869	860	64	600	30	4
57	658	986	403	42	146	867	889	62	629	27	3
58	687	986	432	41	175	866	918	60	658	25	2
59	716	985	461	40	205	864	947	58	687	22	1
60	745	985	490	39	234	863	976	56	716	19	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	89 Degrees.		88 Degrees.		87 Degrees.		86 Degrees.		85 Degrees.		

	5 Deg.		6 Deg.		7 Deg.		8 Deg.		9 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	08716	99619	10453	99452	12187	99255	13917	99027	15643	98769	60
1	745	17	482	49	12216	251	946	023	672	764	59
2	774	14	511	46	45	248	975	019	701	760	58
3	803	12	540	43	74	244	14004	015	730	755	57
4	831	09	569	40	12302	240	033	011	758	751	56
5	860	07	597	37	31	237	061	006	787	746	55
6	889	04	626	34	60	233	090	002	816	741	54
7	918	02	655	31	89	230	119	98998	845	737	53
8	947	99599	684	28	12418	226	148	994	873	732	52
9	976	96	713	24	47	222	177	990	902	728	51
10	09005	94	10742	99421	76	99219	14205	986	931	98723	50
11	034	91	771	18	12504	215	234	982	959	718	49
12	063	88	800	15	33	211	263	978	988	714	48
13	092	86	829	12	62	208	292	973	16017	709	47
14	121	83	858	09	91	204	320	969	046	704	46
15	150	80	887	06	12620	200	349	965	074	700	45
16	179	78	916	02	49	197	378	961	103	695	44
17	208	75	945	99399	78	193	407	957	132	690	43
18	237	72	973	96	12706	189	436	953	160	686	42
19	266	70	11002	93	35	186	464	948	189	681	41
20	09295	99567	031	90	64	99182	14493	98944	16218	98676	40
21	324	64	060	86	93	178	522	940	246	671	39
22	353	62	089	83	12822	175	551	936	275	667	38
23	382	59	118	80	51	171	580	931	304	662	37
24	411	56	147	77	80	167	608	927	333	657	36
25	440	53	176	74	12908	163	637	923	361	652	35
26	469	51	205	70	37	160	666	919	390	648	34
27	498	48	234	67	66	156	695	914	419	643	33
28	527	45	263	64	95	152	723	910	447	638	32
29	556	42	291	60	13024	148	752	906	476	633	31
30	09585	99540	11320	99357	053	99144	14781	98902	16505	98629	30
31	614	37	349	54	081	141	810	897	533	624	29
32	642	34	378	51	110	137	838	893	562	619	28
33	671	31	407	47	139	133	867	889	591	614	27
34	700	28	436	44	168	129	896	884	620	609	26
35	729	26	465	41	197	125	925	880	648	604	25
36	758	23	494	37	226	122	954	876	677	600	24
37	787	20	523	34	254	118	982	871	706	595	23
38	816	17	552	31	283	114	15011	867	734	590	22
39	845	14	580	27	312	110	040	863	763	585	21
40	09874	99511	11609	99324	13341	99106	069	98858	16792	98580	20
41	903	08	638	20	370	102	097	854	820	575	19
42	932	06	667	17	399	098	126	849	849	570	18
43	961	03	696	14	427	094	155	845	878	565	17
44	990	00	725	10	456	091	184	841	906	561	16
45	10019	99497	754	07	485	087	212	836	935	556	15
46	048	94	783	03	514	083	241	832	964	551	14
47	077	91	812	00	543	079	270	827	992	546	13
48	106	88	840	99297	572	075	299	823	17021	541	12
49	135	85	869	93	600	071	327	818	050	536	11
50	164	99482	11898	90	13629	99067	15356	98814	078	98531	10
51	192	79	927	86	658	063	385	809	107	526	9
52	221	76	956	83	687	059	414	805	136	521	8
53	250	73	985	79	716	055	442	800	164	516	7
54	279	70	12014	76	744	051	471	796	193	511	6
55	308	67	043	72	773	047	500	791	222	506	5
56	337	64	071	69	802	043	529	787	250	501	4
57	366	61	100	65	831	039	557	782	279	496	3
58	395	58	129	62	860	035	586	778	308	491	2
59	424	55	158	58	889	031	615	773	336	486	1
60	453	52	187	55	917	027	643	769	365	481	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	84 Deg.		83 Deg.		82 Deg.		81 Deg.		80 Deg.		

	10 Deg.		11 Deg.		12 Deg.		13 Deg.		14 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	17365	98481	19081	98163	20791	97815	22495	97437	24192	97030	60
1	393	76	109	157	820	809	523	430	220	023	59
2	422	71	138	152	848	803	552	424	249	015	58
3	451	66	167	146	877	797	580	417	277	008	57
4	479	61	195	140	905	791	608	411	305	001	56
5	508	55	224	135	933	784	637	404	333	96994	55
6	537	50	252	129	962	778	665	398	362	987	54
7	565	45	281	124	990	772	693	391	390	980	53
8	594	40	309	118	21019	766	722	384	418	973	52
9	623	35	338	112	047	760	750	378	446	966	51
10	17651	98430	19366	98107	21076	97754	22778	97371	24474	96959	50
11	680	25	395	101	104	748	807	365	503	952	49
12	708	20	423	096	132	742	835	358	531	945	48
13	737	14	452	090	161	735	863	351	559	937	47
14	766	09	481	084	189	729	892	345	587	930	46
15	794	04	509	079	218	723	920	338	615	923	45
16	823	98399	538	073	246	717	948	331	644	916	44
17	852	94	566	067	275	711	977	325	672	909	43
18	880	89	595	061	303	705	23005	318	700	902	42
19	909	83	623	056	331	698	033	311	728	894	41
20	17937	98378	19652	98050	21360	97692	23062	97304	24756	96887	40
21	966	73	680	044	388	686	090	298	784	880	39
22	995	68	709	039	417	680	118	291	813	873	38
23	18023	62	737	033	445	673	146	284	841	866	37
24	052	57	766	027	474	667	175	278	869	858	36
25	081	52	794	021	502	661	203	271	897	851	35
26	109	47	823	016	530	655	231	264	925	844	34
27	138	41	851	010	559	648	260	257	954	837	33
28	166	36	880	004	587	642	288	251	982	829	32
29	195	31	908	97998	616	636	316	244	25010	822	31
30	18224	98325	19937	992	21644	97630	23345	97237	038	96815	30
31	252	20	965	987	672	623	373	230	066	807	29
32	281	15	994	981	701	617	401	223	094	800	28
33	309	10	20022	975	729	611	429	217	122	793	27
34	338	04	051	969	758	604	458	210	151	786	26
35	367	98299	079	963	786	598	486	203	179	778	25
36	595	94	108	958	814	592	514	196	207	771	24
37	424	88	136	952	843	585	542	189	235	764	23
38	452	83	165	946	871	579	571	182	263	756	22
39	481	77	193	940	899	573	599	176	291	749	21
40	18509	98272	20222	97934	21928	97566	23627	97169	25320	96742	20
41	538	67	250	928	956	560	656	162	348	734	19
42	567	61	279	922	985	553	684	155	376	727	18
43	595	56	307	916	22013	547	712	148	404	719	17
44	624	50	336	910	041	541	740	141	432	712	16
45	652	45	364	905	070	534	769	134	460	705	15
46	681	40	393	899	098	528	797	127	488	697	14
47	710	34	421	893	126	521	825	120	516	690	13
48	738	29	450	887	155	515	853	113	545	682	12
49	767	23	478	881	183	508	882	106	573	675	11
50	18795	98218	20507	97875	22212	97502	23910	97100	25601	96667	10
51	824	12	535	869	240	496	938	093	629	660	9
52	852	07	563	863	268	489	966	086	657	653	8
53	881	01	592	857	297	483	995	079	685	645	7
54	910	98196	620	851	325	476	24023	072	713	638	6
55	938	90	649	845	353	470	051	065	741	630	5
56	967	85	677	839	382	463	079	058	769	623	4
57	995	79	706	833	410	457	108	051	798	615	3
58	19024	74	734	827	438	450	136	044	826	608	2
59	052	68	763	821	467	444	164	037	854	600	1
60	081	63	791	815	495	437	192	030	882	593	0
M	N. cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	79 Deg.		78 Deg.		77 Deg.		76 Deg.		75 Deg.		

	15 Deg.		16 Deg.		17 Deg.		18 Deg.		19 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	25882	96593	27564	96126	29237	95630	30902	95106	32557	94552	60
1	25910	85	592	118	265	622	929	097	584	542	59
2	38	78	620	110	293	613	957	088	612	533	58
3	66	70	648	102	321	605	985	079	639	523	57
4	94	62	676	094	348	596	31012	070	667	514	56
5	26022	55	704	086	376	588	040	061	694	504	55
6	050	47	731	078	404	579	068	052	722	495	54
7	079	40	759	070	432	571	095	043	749	485	53
8	107	32	787	062	460	562	123	033	777	476	52
9	135	24	815	054	487	554	151	024	804	466	51
10	26163	96517	27843	96046	29515	95543	31178	015	32832	94457	50
11	191	09	871	037	543	536	206	006	859	447	49
12	219	02	899	029	571	528	233	94997	887	438	48
13	247	96494	927	021	599	519	261	988	914	428	47
14	275	86	955	013	626	511	289	979	942	418	46
15	303	79	983	005	654	502	316	970	969	409	45
16	331	71	28011	95997	682	493	344	961	997	399	44
17	359	63	039	989	710	485	372	952	33024	390	43
18	387	56	067	981	737	476	399	943	051	380	42
19	415	48	095	972	765	467	427	933	079	370	41
20	26443	96440	28123	95964	29793	95459	31454	94924	33106	94361	40
21	471	33	150	956	821	450	482	915	134	351	39
22	500	25	178	948	849	441	510	906	161	342	38
23	528	17	206	940	876	433	537	897	189	332	37
24	556	10	234	931	904	424	565	888	216	322	36
25	584	02	262	923	932	415	593	878	244	313	35
26	612	96394	290	915	960	407	620	869	271	303	34
27	640	86	318	907	987	398	648	860	298	293	33
28	668	79	346	898	30015	389	675	851	326	284	32
29	696	71	374	890	043	380	703	842	353	274	31
30	26724	96363	28402	95882	30071	95372	31730	94832	33381	94264	30
31	752	55	429	74	098	363	758	823	408	254	29
32	780	47	457	65	126	354	786	814	436	245	28
33	808	40	485	57	154	345	813	805	463	235	27
34	836	32	513	49	182	337	841	795	490	225	26
35	864	24	541	41	209	328	868	786	518	216	25
36	892	16	569	32	237	319	896	777	545	206	24
37	920	08	597	24	265	310	923	768	573	196	23
38	948	01	625	16	292	301	951	758	600	186	22
39	976	96293	652	07	320	293	979	749	627	176	21
40	27004	96285	28680	95799	30348	95284	32006	94740	33655	94167	20
41	032	77	708	91	376	275	034	730	682	157	19
42	060	69	736	82	403	266	061	721	710	147	18
43	088	61	764	74	431	257	089	712	737	137	17
44	116	53	792	66	459	248	116	702	764	127	16
45	144	46	820	57	486	240	144	693	792	118	15
46	172	38	847	49	514	231	171	684	819	108	14
47	200	30	875	40	542	222	199	674	846	098	13
48	228	22	903	32	570	213	227	665	874	088	12
49	256	14	931	24	597	204	254	656	901	078	11
50	27284	96206	28959	95715	30625	95195	94646	94646	33929	94068	10
51	312	96198	987	07	653	186	309	637	956	058	9
52	340	90	29015	95698	680	177	337	627	983	049	8
53	368	82	042	90	708	168	364	618	34011	039	7
54	396	74	070	81	736	159	392	609	038	029	6
55	424	66	098	73	763	150	419	599	065	019	5
56	452	58	126	64	791	142	447	590	093	009	4
57	480	50	154	56	819	133	474	580	120	93999	3
58	508	42	182	47	846	124	502	571	147	989	2
59	536	34	209	39	874	115	529	561	175	979	1
60	564	26	237	30	902	106	557	552	202	969	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N. cos.	N.sine	N.cos.	N.sine	M
	74 Deg.		73 Deg.		72 Deg.		71 Deg.		70 Deg.		

	20 Deg.		21 Deg.		22 Deg.		23 Deg.		24 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	34202	93969	35837	93358	37461	92718	39073	92050	40674	91355	60
1	229	959	864	348	488	707	100	039	700	343	59
2	257	949	891	337	515	697	127	028	727	331	58
3	284	939	918	327	542	686	153	016	753	319	57
4	311	929	945	316	569	675	180	005	780	307	56
5	339	919	973	306	595	664	207	91994	806	295	55
6	366	909	36000	295	622	653	234	982	833	283	54
7	393	899	027	285	649	642	260	971	860	272	53
8	421	889	054	274	676	631	287	959	886	260	52
9	448	879	081	264	703	620	314	948	913	248	51
10	34475	93869	36108	93253	37730	92609	39341	91936	40939	91236	50
11	503	859	135	243	757	598	367	925	966	224	49
12	530	849	162	232	784	587	394	914	992	212	48
13	557	839	190	222	811	576	421	902	41019	200	47
14	584	829	217	211	838	565	448	891	045	188	46
15	612	819	244	201	865	554	474	879	072	176	45
16	639	809	271	190	892	543	501	868	098	164	44
17	666	799	298	180	919	532	528	856	125	152	43
18	694	789	325	169	946	521	555	845	151	140	42
19	721	779	352	159	973	510	581	833	178	128	41
20	34748	93769	36379	93148	37999	92499	39608	91822	41204	91116	40
21	775	759	406	137	38026	488	635	810	231	104	39
22	803	748	434	127	053	477	661	799	257	092	38
23	830	738	461	116	080	466	688	787	284	080	37
24	857	728	488	106	107	455	715	775	310	068	36
25	884	718	515	095	134	444	741	764	337	056	35
26	912	708	542	084	161	432	768	752	363	044	34
27	939	698	569	074	188	421	795	741	390	032	33
28	966	688	596	063	215	410	822	729	416	020	32
29	993	677	623	052	241	399	848	718	443	008	31
30	35021	93667	36650	93042	38268	92388	39875	91706	41469	90996	30
31	048	657	677	031	295	377	902	694	496	984	29
32	075	647	704	020	322	366	928	683	522	972	28
33	102	637	731	010	349	355	955	671	549	960	27
34	130	626	758	92999	376	343	982	660	575	948	26
35	157	616	785	988	403	332	40008	648	602	936	25
36	184	606	812	978	430	321	035	636	628	924	24
37	211	596	839	967	456	310	062	625	655	911	23
38	239	585	867	956	483	299	088	613	681	899	22
39	266	575	894	945	510	287	115	601	707	887	21
40	35293	93565	36921	92935	38537	92276	40141	91590	41734	90875	20
41	320	555	948	924	564	265	168	578	760	863	19
42	347	544	975	913	591	254	195	566	787	851	18
43	375	534	37002	902	617	243	221	555	813	839	17
44	402	524	029	892	644	231	248	543	840	826	16
45	429	514	056	881	671	220	275	531	866	814	15
46	456	503	083	870	698	209	301	519	892	802	14
47	484	493	110	859	725	198	328	508	919	790	13
48	511	483	137	849	752	186	355	496	945	778	12
49	538	472	164	838	778	175	381	484	972	766	11
50	35565	93462	37191	92827	38805	92164	40408	91472	41998	90753	10
51	592	452	218	816	832	152	434	461	42024	741	9
52	619	441	245	805	859	141	461	449	051	729	8
53	647	431	272	794	886	130	488	437	077	717	7
54	674	420	299	784	912	119	514	425	104	704	6
55	701	410	326	773	939	107	541	414	130	692	5
56	728	400	353	762	966	096	567	402	156	680	4
57	755	389	380	751	993	085	594	390	183	668	3
58	782	379	407	740	39020	073	621	378	209	655	2
59	810	368	434	729	046	062	647	366	235	643	1
60	837	358	461	718	073	050	674	355	262	631	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
'	69 Deg.		68 Deg.		67 Deg.		66 Deg.		65 Deg.		

	25 Deg.		26 Deg.		27 Deg.		28 Deg.		29 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	42262	90631	43837	89879	45399	89101	46947	88295	48481	87462	60
1	288	618	863	867	425	087	973	281	506	448	59
2	315	606	889	854	451	074	999	267	532	434	58
3	341	594	916	841	477	061	47024	254	557	420	57
4	367	582	942	828	503	048	050	240	583	406	56
5	394	569	968	816	529	035	076	226	608	391	55
6	420	557	994	803	554	021	101	213	634	377	54
7	446	545	44020	790	580	008	127	199	659	363	53
8	473	532	046	777	606	88995	153	185	684	349	52
9	499	520	072	764	632	981	178	172	710	335	51
10	42525	90507	44098	89752	45658	88968	47204	88158	48735	87321	50
11	552	495	124	739	684	955	229	144	761	306	49
12	578	483	151	726	710	942	255	130	786	292	48
13	604	470	177	713	736	928	281	117	811	278	47
14	631	458	203	700	762	915	306	103	837	264	46
15	657	446	229	687	787	902	332	089	862	250	45
16	683	433	255	674	813	888	358	075	888	235	44
17	709	421	281	662	839	875	383	062	913	221	43
18	736	408	307	649	865	862	409	048	938	207	42
19	762	396	333	636	891	848	434	034	964	193	41
20	42788	90383	44359	89623	45917	88835	47460	88020	48989	87178	40
21	815	371	385	610	942	822	486	006	49014	164	39
22	841	358	411	597	968	808	511	87993	040	150	38
23	867	346	437	584	994	795	537	979	065	136	37
24	894	334	464	571	46020	782	562	965	090	121	36
25	920	321	490	558	046	768	588	951	116	107	35
26	946	309	516	545	072	755	614	937	141	093	34
27	972	296	542	532	097	741	639	923	166	079	33
28	999	284	568	519	123	728	665	909	192	064	32
29	43025	271	594	506	149	715	690	896	217	050	31
30	051	90259	44620	89493	46175	88701	47716	87882	49242	87036	30
31	077	246	646	480	201	688	741	868	268	021	29
32	104	233	672	467	226	674	767	854	293	007	28
33	130	221	698	454	252	661	793	840	318	86993	27
34	156	208	724	441	278	647	818	826	344	978	26
35	182	196	750	428	304	634	844	812	369	964	25
36	209	183	776	415	330	620	869	798	394	949	24
37	235	171	802	402	355	607	895	784	419	935	23
38	261	158	828	389	381	593	920	770	445	921	22
39	287	146	854	376	407	580	946	756	470	906	21
40	43313	90133	44880	89363	46433	88566	47971	87743	49495	86892	20
41	340	120	906	350	458	553	997	729	521	878	19
42	366	108	932	337	484	539	48022	715	546	863	18
43	392	095	958	324	510	526	048	701	571	849	17
44	418	082	984	311	536	512	073	687	596	834	16
45	445	070	45010	298	561	499	099	673	622	820	15
46	471	057	036	285	587	485	124	659	647	805	14
47	497	045	062	272	613	472	150	645	672	791	13
48	523	032	088	259	639	458	175	631	697	777	12
49	549	019	114	245	664	445	201	617	723	762	11
50	43575	90007	45140	89232	46690	88431	48226	87603	49748	86748	10
51	602	89994	166	219	716	417	252	589	773	733	9
52	628	981	192	206	742	404	277	575	798	719	8
53	654	968	218	193	767	390	303	561	824	704	7
54	680	956	243	180	793	377	328	546	849	690	6
55	706	943	269	167	819	363	354	532	874	675	5
56	733	930	295	153	844	349	379	518	899	661	4
57	759	918	321	140	870	336	405	504	924	646	3
58	785	905	347	127	896	322	430	490	950	632	2
59	811	892	373	114	921	308	456	476	975	617	1
60	837	879	399	101	947	295	481	462	50000	603	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	64 Deg.		63 Deg.		62 Deg.		61 Deg.		60 Deg.		

	30 Deg.		31 Deg.		32 Deg.		33 Deg.		34 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	50000	86603	51504	85717	52992	84805	54464	83867	55919	82904	60
1	025	588	529	702	53017	789	488	851	943	887	59
2	050	573	554	687	041	774	513	835	968	871	58
3	076	559	579	672	066	759	537	819	992	855	57
4	101	544	604	657	091	743	561	804	56016	839	56
5	126	530	628	642	115	728	586	788	040	822	55
6	151	515	653	627	140	712	610	772	064	806	54
7	176	501	678	612	164	697	635	756	088	790	53
8	201	486	703	597	189	681	659	740	112	773	52
9	227	471	728	582	214	666	683	724	136	757	51
10	50252	86457	51753	85567	53238	84650	54708	83708	56160	82741	50
11	277	442	778	551	263	635	732	692	184	724	49
12	302	427	803	536	288	619	756	676	208	708	48
13	327	413	828	521	312	604	781	660	232	692	47
14	352	398	852	506	337	588	805	645	256	675	46
15	377	384	877	491	361	573	829	629	280	659	45
16	403	369	902	476	386	557	854	613	305	643	44
17	428	354	927	461	411	542	878	597	329	626	43
18	453	340	952	446	435	526	902	581	353	610	42
19	478	325	977	431	460	511	927	565	377	593	41
20	50503	86310	52002	85416	53484	84495	54951	83549	56401	82577	40
21	528	295	026	401	509	480	975	533	425	561	39
22	553	281	051	385	534	464	999	517	449	544	38
23	578	266	076	370	558	448	55024	501	473	528	37
24	603	251	101	355	583	433	048	485	497	511	36
25	628	237	126	340	607	417	072	469	521	495	35
26	654	222	151	325	632	402	097	453	545	478	34
27	679	207	175	310	656	386	121	437	569	462	33
28	704	192	200	294	681	370	145	421	593	446	32
29	729	178	225	279	705	355	169	405	617	429	31
30	50754	86163	52250	85264	53730	84339	55194	83389	56641	82413	30
31	779	148	275	249	754	324	218	373	665	396	29
32	804	133	299	234	779	308	242	356	689	380	28
33	829	119	324	218	804	292	266	340	713	363	27
34	854	104	349	203	828	277	291	324	736	347	26
35	879	089	374	188	853	261	315	308	760	330	25
36	904	074	399	173	877	245	339	292	784	314	24
37	929	059	423	157	902	230	363	276	808	297	23
38	954	045	448	142	926	214	388	260	832	281	22
39	979	030	473	127	951	198	412	244	856	264	21
40	51004	86015	52498	85112	53975	84182	55436	83228	56880	82248	20
41	029	000	522	096	54000	167	460	212	904	231	19
42	054	85985	547	081	024	151	484	195	928	214	18
43	079	970	572	066	049	135	509	179	952	198	17
44	104	956	597	051	073	120	533	163	976	181	16
45	129	941	621	035	097	104	557	147	57000	165	15
46	154	926	646	020	122	088	581	131	024	148	14
47	179	911	671	005	146	072	605	115	047	132	13
48	204	896	696	84989	171	057	630	098	071	115	12
49	229	881	720	974	195	041	654	082	095	098	11
50	51254	85866	52745	84959	54220	84025	55678	83066	57119	82082	10
51	279	851	770	943	244	009	702	050	143	065	9
52	304	836	794	928	269	83994	726	034	167	048	8
53	329	821	819	913	293	978	750	017	191	032	7
54	354	806	844	897	317	962	775	001	215	015	6
55	379	792	869	882	342	946	799	82985	238	81999	5
56	404	777	893	866	366	930	823	969	262	982	4
57	429	762	918	851	391	215	847	953	286	865	3
58	454	747	943	836	415	899	871	936	310	949	2
59	479	732	967	820	440	883	895	920	334	932	1
60	504	717	992	805	464	867	919	904	358	915	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	59 Deg.		58 Deg.		57 Deg.		56 Deg.		55 Deg.		

	35 Deg.		36 Deg.		37 Deg.		38 Deg.		39 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos	N.sine	N.cos.	N.sine	N.cos.	M
0	57358	81915	58779	80902	60182	79864	61566	78801	62932	77715	60
1	381	899	802	885	205	846	589	783	955	696	59
2	405	882	826	867	228	829	612	765	977	678	58
3	429	865	849	850	251	811	635	747	63000	660	57
4	453	848	873	833	274	793	658	729	022	641	56
5	477	832	896	816	298	776	681	711	045	623	55
6	501	815	920	799	321	758	704	694	068	605	54
7	524	798	943	782	344	741	726	676	090	586	53
8	548	782	967	765	367	723	749	658	113	568	52
9	572	765	990	748	390	706	772	640	135	550	51
10	57596	81748	59014	80730	60414	79688	61795	78622	63158	77531	50
11	619	731	037	713	437	671	818	604	180	513	49
12	643	714	061	696	460	653	841	586	203	494	48
13	667	698	084	679	483	635	864	568	225	476	47
14	691	681	108	662	506	618	887	550	248	458	46
15	715	664	131	644	529	600	909	532	271	439	45
16	738	647	154	627	553	583	932	514	293	421	44
17	762	631	178	610	576	565	955	496	316	402	43
18	786	614	201	593	599	547	978	478	338	384	42
19	810	597	225	576	622	530	62001	460	361	366	41
20	57833	81580	59248	80558	60645	79512	024	78442	63383	77347	40
21	857	563	272	541	668	494	046	424	406	329	39
22	881	546	295	524	691	477	069	405	428	310	38
23	904	530	318	507	714	459	092	387	451	292	37
24	928	513	342	489	738	441	115	369	473	273	36
25	952	496	365	472	761	424	138	351	496	255	35
26	976	479	389	455	784	406	160	333	518	236	34
27	999	462	412	438	807	388	183	315	540	218	33
28	58023	445	436	420	830	371	206	297	563	199	32
29	047	428	459	403	853	353	229	279	585	181	31
30	070	81412	59482	80386	60876	79335	62251	78261	63608	77162	30
31	094	395	506	368	899	318	274	243	630	144	29
32	118	378	529	351	922	300	297	225	653	125	28
33	141	361	552	334	945	282	320	206	675	107	27
34	165	344	576	316	968	264	342	188	698	088	26
35	189	327	599	299	991	247	365	170	720	070	25
36	212	310	622	282	61015	229	388	152	742	051	24
37	236	293	646	264	038	211	411	134	765	033	23
38	260	276	669	247	061	193	433	116	787	014	22
39	283	259	693	230	084	176	456	098	810	76996	21
40	58307	81242	59716	80212	61107	79158	62479	78079	63832	977	20
41	330	225	739	195	130	140	502	061	854	959	19
42	354	208	763	178	153	122	524	043	877	940	18
43	378	191	786	160	176	105	547	025	899	921	17
44	401	174	809	143	199	087	570	007	922	903	16
45	425	157	832	125	222	069	592	77988	944	884	15
46	449	140	856	108	245	051	615	970	966	866	14
47	472	123	879	091	268	033	638	952	989	847	13
48	496	106	902	073	291	016	660	934	64011	828	12
49	519	089	926	056	314	78998	683	916	033	810	11
50	58543	81072	59949	80038	61337	980	62706	77897	056	76791	10
51	567	055	972	021	860	962	728	879	078	772	9
52	590	038	995	003	383	944	751	861	100	754	8
53	614	021	60019	79986	406	926	774	843	123	735	7
54	637	004	042	968	429	908	796	824	145	717	6
55	661	80987	065	951	451	891	819	806	167	698	5
56	684	970	089	934	474	873	842	788	190	679	4
57	708	953	112	916	497	855	864	769	212	661	3
58	731	936	135	899	520	837	887	751	234	642	2
59	755	919	158	881	543	819	909	733	256	623	1
60	779	902	182	864	566	801	932	715	279	604	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	54 Deg.		53 Deg.		52 Deg.		51 Deg.		50 Deg.		

	40 Deg.		41 Deg.		42 Deg.		43 Deg.		44 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	64279	76604	65606	75471	66913	74314	68200	73135	69466	71934	60
1	301	586	628	452	935	295	221	116	487	914	59
2	323	567	650	433	956	276	242	096	508	894	58
3	346	548	672	414	978	256	264	076	529	873	57
4	368	530	694	395	999	237	285	056	549	853	56
5	390	511	716	375	67021	217	306	036	570	833	55
6	412	492	738	356	043	198	327	016	591	813	54
7	435	473	759	337	064	178	349	72996	612	792	53
8	457	455	781	318	086	159	370	976	633	772	52
9	479	436	803	299	107	139	391	957	654	752	51
10	64501	76417	65825	75280	67129	74120	68412	72937	69675	71732	50
11	524	398	847	261	151	100	434	917	696	711	49
12	546	380	869	241	172	080	455	897	717	691	48
13	568	361	891	222	194	061	476	877	737	671	47
14	590	342	913	203	215	041	497	857	758	650	46
15	612	323	935	184	237	022	518	837	779	630	45
16	635	304	956	165	258	002	539	817	800	610	44
17	657	286	978	146	280	73983	561	797	821	590	43
18	679	267	66000	126	301	963	582	777	842	569	42
19	701	248	022	107	323	944	603	757	862	549	41
20	64723	76229	66044	75088	67344	73924	68624	72737	69883	71529	40
21	746	210	066	069	366	904	645	717	904	508	39
22	768	192	088	050	387	885	666	697	925	488	38
23	790	173	109	030	409	865	688	677	946	468	37
24	812	154	131	011	430	846	709	657	966	447	36
25	834	135	153	74992	452	826	730	637	987	427	35
26	856	116	175	973	473	806	751	617	70008	407	34
27	878	097	197	953	495	787	772	597	029	386	33
28	901	078	218	934	516	767	793	577	049	366	32
29	923	059	240	915	538	747	814	557	070	345	31
30	64945	76041	66262	74896	67559	73728	68835	72537	70091	71325	30
31	967	022	284	876	580	708	857	517	112	305	29
32	989	003	306	857	602	688	878	497	132	284	28
33	65011	75984	327	838	623	669	899	477	153	264	27
34	033	965	349	818	645	649	920	457	174	243	26
35	055	946	371	799	666	629	941	437	195	223	25
36	077	927	393	780	688	610	962	417	215	203	24
37	100	908	414	760	709	590	983	397	236	182	23
38	122	889	436	741	730	570	69004	377	257	162	22
39	144	870	458	722	752	551	025	357	277	141	21
40	65166	851	66480	74703	67773	73531	69046	72337	70298	71121	20
41	188	832	501	683	795	511	067	317	319	100	19
42	210	813	523	664	816	491	088	297	339	080	18
43	232	794	545	644	837	472	109	277	360	059	17
44	254	775	566	625	859	452	130	257	381	039	16
45	276	756	588	606	880	432	151	236	401	019	15
46	298	738	610	586	901	413	172	216	422	70998	14
47	320	719	632	567	923	393	193	196	443	978	13
48	342	700	653	548	944	373	214	176	463	957	12
49	364	680	675	528	965	353	235	156	484	937	11
50	65386	75661	66697	74509	67987	73333	69256	72136	70505	70916	10
51	408	642	718	489	68008	314	277	116	525	896	9
52	430	623	740	470	029	294	298	095	546	875	8
53	452	604	762	451	051	274	319	075	567	855	7
54	474	585	783	431	072	254	340	055	587	834	6
55	496	566	805	412	093	234	361	035	608	813	5
56	518	547	827	392	115	215	382	015	628	793	4
57	540	528	848	373	136	195	403	71995	649	772	3
58	562	509	870	353	157	175	424	974	670	752	2
59	584	490	891	334	179	155	445	954	690	731	1
60	606	471	913	314	200	135	466	934	711	711	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	49 Deg.		84 Deg.		47 Deg.		46 Deg.		45 Deg.		

	25 Deg.		26 Deg.		27 Deg.		28 Deg.		29 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	42262	90631	43837	89879	45399	89101	46947	88295	48481	87462	60
1	288	618	863	867	425	087	973	281	506	448	59
2	315	606	889	854	451	074	999	267	532	434	58
3	341	594	916	841	477	061	47024	254	557	420	57
4	367	582	942	828	503	048	050	240	583	406	56
5	394	569	968	816	529	035	076	226	608	391	55
6	420	557	994	803	554	021	101	213	634	377	54
7	446	545	44020	790	580	008	127	199	659	363	53
8	473	532	046	777	606	88995	153	185	684	349	52
9	499	520	072	764	632	981	178	172	710	335	51
10	42525	90507	44098	89752	45658	88968	47204	88158	48735	87321	50
11	552	495	124	739	684	955	229	144	761	306	49
12	578	483	151	726	710	942	255	130	786	292	48
13	604	470	177	713	736	928	281	117	811	278	47
14	631	458	203	700	762	915	306	103	837	264	46
15	657	446	229	687	787	902	332	089	862	250	45
16	683	433	255	674	813	888	358	075	888	235	44
17	709	421	281	662	839	875	383	062	913	221	43
18	736	408	307	649	865	862	409	048	938	207	42
19	762	396	333	636	891	848	434	034	964	193	41
20	42788	90383	44359	89623	45917	88835	47460	88020	48989	87178	40
21	815	371	385	610	942	822	486	006	49014	164	39
22	841	358	411	597	968	808	511	87993	040	150	38
23	867	346	437	584	994	795	537	979	065	136	37
24	894	334	464	571	46020	782	562	965	090	121	36
25	920	321	490	558	046	768	588	951	116	107	35
26	946	309	516	545	072	755	614	937	141	093	34
27	972	296	542	532	097	741	639	923	166	079	33
28	999	284	568	519	123	728	665	909	192	064	32
29	43025	271	594	506	149	715	690	896	217	050	31
30	051	90259	44620	89493	46175	88701	47716	87882	49242	87036	30
31	077	246	646	480	201	688	741	868	268	021	29
32	104	233	672	467	226	674	767	854	293	007	28
33	130	221	698	454	252	661	793	840	318	86993	27
34	156	208	724	441	278	647	818	826	344	978	26
35	182	196	750	428	304	634	844	812	369	964	25
36	209	183	776	415	330	620	869	798	394	949	24
37	235	171	802	402	355	607	895	784	419	935	23
38	261	158	828	389	381	593	920	770	445	921	22
39	287	146	854	376	407	580	946	756	470	906	21
40	43313	90133	44880	89363	46433	88566	47971	87743	49495	86892	20
41	340	120	906	350	458	553	997	729	521	878	19
42	366	108	932	337	484	539	48022	715	546	863	18
43	392	095	958	324	510	526	048	701	571	849	17
44	418	082	984	311	536	512	073	687	596	834	16
45	445	070	45010	298	561	499	099	673	622	820	15
46	471	057	036	285	587	485	124	659	647	805	14
47	497	045	062	272	613	472	150	645	672	791	13
48	523	032	088	259	639	458	175	631	697	777	12
49	549	019	114	245	664	445	201	617	723	762	11
50	43575	90007	45140	89232	46690	88431	48226	87603	49748	86748	10
51	602	89994	166	219	716	417	252	589	773	733	9
52	628	981	192	206	742	404	277	575	798	719	8
53	654	968	218	193	767	390	303	561	824	704	7
54	680	956	243	180	793	377	328	546	849	690	6
55	706	943	269	167	819	363	354	532	874	675	5
56	733	930	295	153	844	349	379	518	899	661	4
57	759	918	321	140	870	336	405	504	924	646	3
58	785	905	347	127	896	322	430	490	950	632	2
59	811	892	373	114	921	308	456	476	975	617	1
60	837	879	399	101	947	295	481	462	50000	603	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	64 Deg.		63 Deg.		62 Deg.		61 Deg.		60 Deg.		

	30 Deg.		31 Deg.		32 Deg.		33 Deg.		34 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	50000	86603	51504	85717	52992	84805	54464	83867	55919	82904	60
1	025	588	529	702	53017	789	488	851	943	887	59
2	050	573	554	687	041	774	513	835	968	871	58
3	076	559	579	672	066	759	537	819	992	855	57
4	101	544	604	657	091	743	561	804	56016	839	56
5	126	530	628	642	115	728	586	788	040	822	55
6	151	515	653	627	140	712	610	772	064	806	54
7	176	501	678	612	164	697	635	756	088	790	53
8	201	486	703	597	189	681	659	740	112	773	52
9	227	471	728	582	214	666	683	724	136	757	51
10	50252	86457	51753	85567	53238	84650	54708	83708	56160	82741	50
11	277	442	778	551	263	635	732	692	184	724	49
12	302	427	803	536	288	619	756	676	208	708	48
13	327	413	828	521	312	604	781	660	232	692	47
14	352	398	852	506	337	588	805	645	256	675	46
15	377	384	877	491	361	573	829	629	280	659	45
16	403	369	902	476	386	557	854	613	305	643	44
17	428	354	927	461	411	542	878	597	329	626	43
18	453	340	952	446	435	526	902	581	353	610	42
19	478	325	977	431	460	511	927	565	377	593	41
20	50503	86310	52002	85416	53484	84495	54951	83549	56401	82577	40
21	528	295	026	401	509	480	975	533	425	561	39
22	553	281	051	385	534	464	999	517	449	544	38
23	578	266	076	370	558	448	55024	501	473	528	37
24	603	251	101	355	583	433	048	485	497	511	36
25	628	237	126	340	607	417	072	469	521	495	35
26	654	222	151	325	632	402	097	453	545	478	34
27	679	207	175	310	656	386	121	437	569	462	33
28	704	192	200	294	681	370	145	421	593	446	32
29	729	178	225	279	705	355	169	405	617	429	31
30	50754	86163	52250	85264	53730	84339	55194	83389	56641	82413	30
31	779	148	275	249	754	324	218	373	665	396	29
32	804	133	299	234	779	308	242	356	689	380	28
33	829	119	324	218	804	292	266	340	713	363	27
34	854	104	349	203	828	277	291	324	736	347	26
35	879	089	374	188	853	261	315	308	760	330	25
36	904	074	399	173	877	245	339	292	784	314	24
37	929	059	423	157	902	230	363	276	808	297	23
38	954	045	448	142	926	214	388	260	832	281	22
39	979	030	473	127	951	198	412	244	856	264	21
40	51004	86015	52498	85112	53975	84182	55436	83228	56880	82248	20
41	029	000	522	096	54000	167	460	212	904	231	19
42	054	85985	547	081	024	151	484	195	928	214	18
43	079	970	572	066	049	135	509	179	952	198	17
44	104	956	597	051	073	120	533	163	976	181	16
45	129	941	621	035	097	104	557	147	57000	165	15
46	154	926	646	020	122	088	581	131	024	148	14
47	179	911	671	005	146	072	605	115	047	132	13
48	204	896	696	84989	171	057	630	098	071	115	12
49	229	881	720	974	195	041	654	082	095	098	11
50	51254	85866	52745	84959	54220	84025	55678	83066	57119	82082	10
51	279	851	770	943	244	009	702	050	143	065	9
52	304	836	794	928	269	83994	726	034	167	048	8
53	329	821	819	913	293	978	750	017	191	032	7
54	354	806	844	897	317	962	775	001	215	015	6
55	379	792	869	882	342	946	799	82985	238	81999	5
56	404	777	893	866	366	930	823	969	262	982	4
57	429	762	918	851	391	215	847	953	286	865	3
58	454	747	943	836	415	899	871	936	310	949	2
59	479	732	967	820	440	883	895	920	334	932	1
60	504	717	992	805	464	867	919	904	358	915	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	59 Deg.		58 Deg.		57 Deg.		56 Deg.		55 Deg.		

	35 Deg.		36 Deg.		37 Deg.		38 Deg.		39 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos	N.sine	N.cos.	N.sine	N.cos.	M
0	57358	81915	58779	80902	60182	79864	61566	78801	62932	77715	60
1	381	899	802	885	205	846	589	783	955	696	59
2	405	882	826	867	228	829	612	765	977	678	58
3	429	865	849	850	251	811	635	747	63000	660	57
4	453	848	873	833	274	793	658	729	022	641	56
5	477	832	896	816	298	776	681	711	045	623	55
6	501	815	920	799	321	758	704	694	068	605	54
7	524	798	943	782	344	741	726	676	090	586	53
8	548	782	967	765	367	723	749	658	113	568	52
9	572	765	990	748	390	706	772	640	135	550	51
10	57596	81748	59014	80730	60414	79688	61795	78622	63158	77531	50
11	619	731	037	713	437	671	818	604	180	513	49
12	643	714	061	696	460	653	841	586	203	494	48
13	667	698	084	679	483	635	864	568	225	476	47
14	691	681	108	662	506	618	887	550	248	458	46
15	715	664	131	644	529	600	909	532	271	439	45
16	738	647	154	627	553	583	932	514	293	421	44
17	762	631	178	610	576	565	955	496	316	402	43
18	786	614	201	593	599	547	978	478	338	384	42
19	810	597	225	576	622	530	62001	460	361	366	41
20	57833	81580	59248	80558	60645	79512	024	78442	63383	77347	40
21	857	563	272	541	668	494	046	424	406	329	39
22	881	546	295	524	691	477	069	405	428	310	38
23	904	530	318	507	714	459	092	387	451	292	37
24	928	513	342	489	738	441	115	369	473	273	36
25	952	496	365	472	761	424	138	351	496	255	35
26	976	479	389	455	784	406	160	333	518	236	34
27	999	462	412	438	807	388	183	315	540	218	33
28	58023	445	436	420	830	371	206	297	563	199	32
29	047	428	459	403	853	353	229	279	585	181	31
30	070	81412	59482	80386	60876	79335	62251	78261	63608	77162	30
31	094	395	506	368	899	318	274	243	630	144	29
32	118	378	529	351	922	300	297	225	653	125	28
33	141	361	552	334	945	282	320	206	675	107	27
34	165	344	576	316	968	264	342	188	698	088	26
35	189	327	599	299	991	247	365	170	720	070	25
36	212	310	622	282	61015	229	388	152	742	051	24
37	236	293	646	264	038	211	411	134	765	033	23
38	260	276	669	247	061	193	433	116	787	014	22
39	283	259	693	230	084	176	456	098	810	76996	21
40	58307	81242	59716	80212	61107	79158	62479	78079	63832	977	20
41	330	225	739	195	130	140	502	061	854	959	19
42	354	208	763	178	153	122	524	043	877	940	18
43	378	191	786	160	176	105	547	025	899	921	17
44	401	174	809	143	199	087	570	007	922	903	16
45	425	157	832	125	222	069	592	77988	944	884	15
46	449	140	856	108	245	051	615	970	966	866	14
47	472	123	879	091	268	033	638	952	989	847	13
48	496	106	902	073	291	016	660	934	64011	828	12
49	519	089	926	056	314	78998	683	916	033	810	11
50	58543	81072	59949	80038	61337	980	62706	77897	056	76791	10
51	567	055	972	021	360	962	728	879	078	772	9
52	590	038	995	003	383	944	751	861	100	754	8
53	614	021	60019	79986	406	926	774	843	123	735	7
54	637	004	042	968	429	908	796	824	145	717	6
55	661	80987	065	951	451	891	819	806	167	698	5
56	684	970	089	934	474	873	842	788	190	679	4
57	708	953	112	916	497	855	864	769	212	661	3
58	731	936	135	899	520	837	887	751	234	642	2
59	755	919	158	881	543	819	909	733	256	623	1
60	779	902	182	864	566	801	932	715	279	604	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	54 Deg.		53 Deg.		52 Deg.		51 Deg.		50 Deg.		

	40 Deg.		41 Deg.		42 Deg.		43 Deg.		44 Deg.		
M	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	M
0	64279	76604	65606	75471	66913	74314	68200	73135	69465	71934	60
1	301	586	628	452	935	295	221	116	487	914	59
2	323	567	650	433	956	276	242	096	508	894	58
3	346	548	672	414	978	256	264	076	529	873	57
4	368	530	694	395	999	237	285	056	549	853	56
5	390	511	716	375	67021	217	306	036	570	833	55
6	412	492	738	356	043	198	327	016	591	813	54
7	435	473	759	337	064	178	349	72996	612	792	53
8	457	455	781	318	086	159	370	976	633	772	52
9	479	436	803	299	107	139	391	957	654	752	51
10	64501	76417	65825	75280	67129	74120	68412	72937	69675	71732	50
11	524	398	847	261	151	100	434	917	696	711	49
12	546	380	869	241	172	080	455	897	717	691	48
13	568	361	891	222	194	061	476	877	737	671	47
14	590	342	913	203	215	041	497	857	758	650	46
15	612	323	935	184	237	022	518	837	779	630	45
16	635	304	956	165	258	002	539	817	800	610	44
17	657	286	978	146	280	73983	561	797	821	590	43
18	679	267	66000	126	301	963	582	777	842	569	42
19	701	248	022	107	323	944	603	757	862	549	41
20	64723	76229	66044	75088	67344	73924	68624	72737	69853	71529	40
21	746	210	066	069	366	904	645	717	904	508	39
22	768	192	088	050	387	885	666	697	925	488	38
23	790	173	109	030	409	865	688	677	946	468	37
24	812	154	131	011	430	846	709	657	966	447	36
25	834	135	153	74992	452	826	730	637	987	427	35
26	856	116	175	973	473	806	751	617	70008	407	34
27	878	097	197	953	495	787	772	597	029	386	33
28	901	078	218	934	516	767	793	577	049	366	32
29	923	059	240	915	538	747	814	557	070	345	31
30	64945	76041	66262	74896	67559	73728	68835	72537	70091	71325	30
31	967	022	284	876	580	708	857	517	112	305	29
32	989	003	306	857	602	688	878	497	132	284	28
33	65011	75984	327	838	623	669	899	477	153	264	27
34	033	965	349	818	645	649	920	457	174	243	26
35	055	946	371	799	666	629	941	437	195	223	25
36	077	927	393	780	688	610	962	417	215	203	24
37	100	908	414	760	709	590	983	397	236	182	23
38	122	889	436	741	730	570	69004	377	257	162	22
39	144	870	458	722	752	551	025	357	277	141	21
40	65166	851	66480	74703	67773	73531	69046	72337	70298	71121	20
41	188	832	501	683	795	511	067	317	319	100	19
42	210	813	523	664	816	491	088	297	339	080	18
43	232	794	545	644	837	472	109	277	360	059	17
44	254	775	566	625	859	452	130	257	381	039	16
45	276	756	588	606	880	432	151	236	401	019	15
46	298	738	610	586	901	413	172	216	422	70998	14
47	320	719	632	567	923	393	193	196	443	978	13
48	342	700	653	548	944	373	214	176	463	957	12
49	364	680	675	528	965	353	235	156	484	937	11
50	65386	75661	66697	74509	67987	73333	69256	72136	70505	70916	10
51	408	642	718	489	68008	314	277	116	525	896	9
52	430	623	740	470	029	294	298	095	546	875	8
53	452	604	762	451	051	274	319	075	567	855	7
54	474	585	783	431	072	254	340	055	587	834	6
55	496	566	805	412	093	234	361	035	608	813	5
56	518	547	827	392	115	215	382	015	628	793	4
57	540	528	848	373	136	195	403	71995	649	772	3
58	562	509	870	353	157	175	424	974	670	752	2
59	584	490	891	334	179	155	445	954	690	731	1
60	606	471	913	314	200	135	466	934	711	711	0
M	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	N.cos.	N.sine	M
	49 Deg.		84 Deg.		47 Deg.		46 Deg.		45 Deg.		

III. A Table of Logarithmical Sines and Tangents, to every degree and minute of the Quadrant; the natural radius being *Ten thousand millions*, and the logarithmical radius 10.

From which the secants and versed sines are readily found; for,

The cosine subtracted from 20, leaves the secant, and the sine subtracted from 20, leaves the co-secant (p. 131.)

Also,

If from double the sine of the *half* of any *given Arc*, you subtract the number 9·69897, the remainder will be the versed sine of *that arc* (p. 131.)

0 Degrees.

M	Sine	Co-sine	Tang.	Co-tang.	
0	0.00000	10.00000	0.00000	0.00000	60
1	6.46373	10.00000	6.46372	13.53627	59
2	6.76476	10.00000	6.76476	13.23524	58
3	6.94085	10.00000	6.94085	13.05915	57
4	7.06579	10.00000	7.06579	12.93421	56
5	7.16270	10.00000	7.16270	12.83730	55
6	7.24188	10.00000	7.24188	12.75812	54
7	7.30882	10.00000	7.30882	12.69118	53
8	7.36682	10.00000	7.36682	12.63318	52
9	7.41797	10.00000	7.41797	12.58203	51
10	7.46373	10.00000	7.46373	12.53627	50
11	7.50512	10.00000	7.50512	12.49488	49
12	7.54291	10.00000	7.54291	12.45709	48
13	7.57767	10.00000	7.57767	12.42233	47
14	7.60985	10.00000	7.60986	12.39014	46
15	7.63982	10.00000	7.63982	12.36018	45
16	7.66784	10.00000	7.66785	12.33215	44
17	7.69417	9.99999	7.69418	12.30582	43
18	7.71900	9.99999	7.71900	12.28100	42
19	7.74248	9.99999	7.74248	12.25752	41
20	7.76475	9.99999	7.76476	12.23524	40
21	7.78594	9.99999	7.78595	12.21405	39
22	7.80615	9.99999	7.80615	12.19385	38
23	7.82545	9.99999	7.82546	12.17454	37
24	7.84393	9.99999	7.84394	12.15606	36
25	7.86166	9.99999	7.86167	12.13833	35
26	7.87870	9.99999	7.87871	12.12129	34
27	7.89509	9.99999	7.89510	12.10490	33
28	7.91088	9.99999	7.91089	12.08911	32
29	7.92612	9.99999	7.92613	12.07387	31
M	Co-sine	Sine	Co-tan.	Tang.	M

89 Degrees.

0 Degrees.

M	Sine	Co-sine	Tang.	Co-tang	
30	7.94084	9.99998	7.94086	12.05914	
31	7.95508	9.99998	7.95510	12.04490	
32	7.96887	9.99998	7.96889	12.03111	
33	7.98223	9.99998	7.98225	12.01775	27
34	7.99520	9.99998	7.99522	12.00478	26
35	8.00779	9.99998	8.00781	11.99219	25
36	8.02002	9.99998	8.02004	11.97996	24
37	8.03192	9.99997	8.03194	11.96806	23
38	8.04350	9.99997	8.04353	11.95647	22
39	8.05478	9.99997	8.05481	11.94519	21
40	8.06578	9.99997	8.06581	11.93419	20
41	8.03650	9.99997	8.07653	11.92347	19
42	8.08696	9.99997	8.08700	11.91300	18
43	8.09718	9.99997	8.09722	11.90278	17
44	8.10717	9.99996	8.10720	11.89280	16
45	8.11693	9.99996	8.11696	11.88304	15
46	8.12647	9.99996	8.12651	11.87349	14
47	8.13581	9.99996	8.13585	11.86415	13
48	8.14495	9.99996	8.14500	11.85500	12
49	8.15391	9.99996	8.15395	11.84605	11
50	8.16268	9.99995	8.16273	11.83727	10
51	8.17128	9.99995	8.17133	11.82867	9
52	8.17971	9.99995	8.17976	11.82024	8
53	8.18798	9.99995	8.18804	11.81196	7
54	8.19610	9.99995	8.19616	11.80384	6
55	8.20407	9.99994	8.20413	11.79587	5
56	8.21189	9.99994	8.21195	11.78805	4
57	8.21958	9.99994	8.21964	11.78036	3
58	8.22713	9.99994	8.22720	11.77280	2
59	8.23456	9.99994	8.23462	11.76538	1
60	8.24186	9.99993	8.24192	11.75808	0
	Co-sine	Sine	Co-tang	Tang.	M

89 Degrees.

	9 Degrees.				10 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9·19433	9·99462	9·19971	10·80029	9·23967	9·99385	9·24632	10·75368	60
1	513	60	9·20053	10·79947	9·24039	33	706	294	59
2	592	58	134	864	110	31	779	221	58
3	672	56	216	784	181	28	853	147	57
4	751	54	297	703	253	26	926	074	56
5	830	52	378	622	324	24	9·25000	000	55
6	909	50	459	541	395	22	073	10·74927	54
7	988	48	540	460	466	19	146	854	53
8	9·20067	46	621	379	536	17	219	781	52
9	145	44	701	299	607	15	292	708	51
10	223	9·99442	9·20782	10·79218	9·24677	9·99313	9·25365	10·74635	50
11	302	40	862	138	748	10	437	563	49
12	380	38	942	058	818	08	510	490	48
13	458	36	9·21022	10·78978	888	06	582	418	47
14	535	34	102	898	958	04	655	345	46
15	613	32	182	818	9·25028	01	727	273	45
16	691	29	261	739	098	9·99299	799	201	44
17	768	27	341	659	168	97	871	129	43
18	845	25	420	580	237	94	943	057	42
19	922	23	499	501	307	92	9·26015	10·73985	41
20	999	9·99421	9·21578	10·78422	9·25376	9·99290	086	914	40
21	9·21076	19	657	343	445	88	158	842	39
22	153	17	736	264	514	85	229	771	38
23	229	15	814	186	583	83	301	699	37
24	306	13	893	107	652	81	372	628	36
25	382	11	971	029	721	78	443	557	35
26	458	09	9·22049	10·77951	790	76	514	486	34
27	534	07	127	873	858	74	585	415	33
28	610	04	205	795	927	71	655	345	32
29	685	02	283	717	995	69	726	274	31
30	9·21761	00	9·22361	10·77639	9·26063	9·99267	9·26797	10·73203	30
31	836	9·99398	438	562	131	64	867	133	29
32	912	96	516	484	199	62	937	063	28
33	987	94	593	407	267	60	9·27008	10·72992	27
34	9·22062	92	670	330	335	57	078	922	26
35	137	90	747	253	403	55	148	852	25
36	211	88	824	176	470	52	218	782	24
37	286	85	901	099	538	50	288	712	23
38	361	83	977	023	605	48	357	643	22
39	435	81	9·22054	10·76946	672	45	427	573	21
40	9·22509	9·99379	130	870	9·26739	9·99243	9·27496	10·72504	20
41	583	77	206	794	806	41	566	434	19
42	657	75	283	717	873	38	635	365	18
43	731	72	359	641	940	36	704	296	17
44	805	70	435	565	9·27007	33	773	227	16
45	878	68	510	490	073	31	842	158	15
46	952	66	586	414	140	29	911	089	14
47	9·23025	64	661	339	206	26	980	020	13
48	098	62	737	263	273	24	9·28049	10·71951	12
49	171	59	812	188	339	21	117	883	11
50	9·23244	9·99357	9·23887	10·76113	9·27405	9·99219	186	10·71814	10
51	317	55	962	038	471	17	254	746	9
52	390	53	9·24037	10·75963	537	14	323	677	8
53	462	51	112	888	602	12	391	609	7
54	535	48	186	814	668	09	459	541	6
55	607	46	261	739	734	07	527	473	5
56	679	44	335	665	799	04	595	405	4
57	752	42	410	590	864	02	662	338	3
58	823	40	484	516	930	00	730	270	2
59	895	37	558	442	995	9·99197	798	202	1
60	967	35	632	368	9·28060	95	865	135	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	80 Degrees.				79 Degrees.				

	7 Degrees.				8 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.08589	9.99675	9.08914	10.91086	9.14356	9.99575	9.14780	10.85220	60
1	692	74	9.09019	10.90981	445	73	872	128	59
2	795	72	123	877	535	72	963	037	58
3	897	70	227	773	624	70	9.15054	10.84946	57
4	999	69	330	676	714	68	145	855	56
5	9.09101	67	434	566	803	66	236	764	55
6	202	66	537	463	891	65	327	673	54
7	304	64	640	360	980	63	417	583	53
8	405	63	742	258	9.15069	61	508	492	52
9	506	61	845	155	157	59	598	402	51
10	9.09606	9.99659	947	053	9.15245	9.99557	9.15688	10.84312	50
11	707	58	9.10049	10.89951	333	56	777	223	49
12	807	56	150	850	421	54	867	133	48
13	907	55	252	748	508	52	956	044	47
14	9.10006	53	353	647	596	50	9.16046	10.83954	46
15	106	51	454	546	683	48	135	865	45
16	205	50	555	445	770	46	224	776	44
17	304	48	656	344	857	45	312	688	43
18	402	47	756	244	944	43	401	599	42
19	501	45	856	144	9.16030	41	489	511	41
20	9.10599	9.99643	956	044	116	9.99539	9.16577	10.83423	40
21	697	42	9.11056	10.88944	203	37	665	335	39
22	795	40	155	845	289	35	753	247	38
23	893	38	254	746	374	33	841	159	37
24	990	37	355	647	460	32	928	072	36
25	9.11087	35	452	548	545	30	9.17016	10.82984	35
26	184	33	551	449	631	28	103	897	34
27	281	32	649	351	716	26	190	810	33
28	377	30	747	253	801	24	277	723	32
29	474	29	845	155	886	22	363	637	31
30	9.11570	9.99627	943	057	970	9.99520	9.17450	16.82550	30
31	666	25	9.12040	10.87960	9.17055	18	536	464	29
32	761	24	138	862	139	17	622	378	28
33	857	22	235	765	223	15	708	292	27
34	952	20	332	668	307	13	794	206	26
35	9.12047	18	428	572	391	11	880	120	25
36	142	17	525	475	474	09	965	035	24
37	236	15	621	379	558	07	9.18051	10.81949	23
38	331	13	717	283	641	05	136	864	22
39	425	12	813	187	724	03	221	779	21
40	9.12519	9.99610	909	091	9.17807	01	9.18306	10.81694	20
41	612	08	9.13004	10.86996	890	9.99499	391	609	19
42	706	07	099	901	973	97	475	525	18
43	799	05	194	806	9.18055	95	560	440	17
44	892	03	289	711	137	94	644	356	16
45	985	01	384	616	220	92	728	272	15
46	9.13078	00	478	522	302	90	812	188	14
47	171	9.99598	573	427	383	88	896	104	13
48	263	96	667	333	465	86	979	021	12
49	355	95	761	239	547	84	9.19063	10.80937	11
50	9.13447	9.99593	9.13854	146	9.18628	9.99482	146	854	10
51	539	91	948	052	709	80	229	771	9
52	630	89	9.14041	10.85959	790	78	312	688	8
53	722	88	134	866	871	76	395	605	7
54	813	86	227	773	952	74	478	522	6
55	904	84	320	680	9.19033	72	561	439	5
56	994	82	412	588	113	70	643	357	4
57	9.14085	81	504	496	193	68	725	275	3
58	175	79	597	403	273	66	807	193	2
59	266	77	688	312	353	64	889	111	1
60	356	75	780	220	433	62	971	029	0
M	Co-sine	Sine	Co-tang.	Tangent.	Co-sine	Sine	Co-tang.	Tangent.	M
	82 Degrees.				81 Degrees.				

	9 Degrees.				10 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.19433	9.99462	9.19971	10.80029	9.23967	9.99335	9.24632	10.75368	60
1	513	60	9.20053	10.79947	9.24039	33	706	294	59
2	592	58	134	864	110	31	779	221	58
3	672	56	216	784	181	28	853	147	57
4	751	54	297	703	253	26	926	074	56
5	830	52	378	622	324	24	9.25000	000	55
6	909	50	459	541	395	22	073	10.74927	54
7	988	48	540	460	466	19	146	854	53
8	9.20067	46	621	379	536	17	219	781	52
9	145	44	701	299	607	15	292	708	51
10	223	9.99442	9.20782	10.79218	9.24677	9.99313	9.25365	10.74635	50
11	302	40	862	138	748	10	437	563	49
12	380	38	942	058	818	08	510	490	48
13	458	36	9.21022	10.78978	888	06	582	418	47
14	535	34	102	898	958	04	655	345	46
15	613	32	182	818	9.25028	01	727	273	45
16	691	29	261	739	098	9.99299	799	201	44
17	768	27	341	659	168	97	871	129	43
18	845	25	420	580	237	94	943	057	42
19	922	23	499	501	307	92	9.26015	10.73985	41
20	999	9.99421	9.21578	10.78422	9.25376	9.99290	086	914	40
21	9.21076	19	657	343	445	88	158	842	39
22	153	17	736	264	514	85	229	771	38
23	229	15	814	186	583	83	301	699	37
24	306	13	893	107	652	81	372	628	36
25	382	11	971	029	721	78	443	557	35
26	458	09	9.22049	10.77951	790	76	514	486	34
27	534	07	127	873	858	74	585	415	33
28	610	04	205	795	927	71	655	345	32
29	685	02	283	717	995	69	726	274	31
30	9.21761	00	9.22361	10.77639	9.26063	9.99267	9.26797	10.73203	30
31	836	9.99398	438	562	131	64	867	133	29
32	912	96	516	484	199	62	937	063	28
33	987	94	593	407	267	60	9.27008	10.72992	27
34	9.22062	92	670	330	335	57	078	922	26
35	137	90	747	253	403	55	148	852	25
36	211	88	824	176	470	52	218	782	24
37	286	85	901	099	538	50	288	712	23
38	361	83	977	023	605	48	357	643	22
39	435	81	9.22054	10.76946	672	45	427	573	21
40	9.22509	9.99379	130	870	9.26739	9.99243	9.27496	10.72504	20
41	583	77	206	794	806	41	566	434	19
42	657	75	283	717	873	38	635	365	18
43	731	72	359	641	940	36	704	296	17
44	805	70	435	565	9.27007	33	773	227	16
45	878	68	510	490	073	31	842	158	15
46	952	66	586	414	140	29	911	089	14
47	9.23025	64	661	339	206	26	980	020	13
48	098	62	737	263	273	24	9.28049	10.71951	12
49	171	59	812	188	339	21	117	883	11
50	9.23244	9.99357	9.23887	10.76113	9.27405	9.99219	186	10.71814	10
51	317	55	962	038	471	17	254	746	9
52	390	53	9.24037	10.75963	537	14	323	677	8
53	462	51	112	888	602	12	391	609	7
54	535	48	186	814	668	09	459	541	6
55	607	46	261	739	734	07	527	473	5
56	679	44	335	665	799	04	595	405	4
57	752	42	410	590	864	02	662	338	3
58	823	40	484	516	930	00	730	270	2
59	895	37	558	442	995	9.99197	798	202	1
60	967	35	632	368	9.28060	95	865	135	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	80 Degrees.				79 Degrees.				

	7 Degrees.				8 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.08589	9.99675	9.08914	10.91086	9.14356	9.99575	9.14780	10.85220	60
1	692	74	9.09019	10.90981	445	73	872	128	59
2	795	72	123	877	535	72	963	037	58
3	897	70	227	773	624	70	9.15054	10.84946	57
4	999	69	330	676	714	68	145	855	56
5	9.09101	67	434	566	803	66	236	764	55
6	202	66	537	463	891	65	327	673	54
7	304	64	640	360	980	63	417	583	53
8	405	63	742	258	9.15069	61	508	492	52
9	506	61	845	155	157	59	598	402	51
10	9.09606	9.99659	947	053	9.15245	9.99557	9.15688	10.84312	50
11	707	58	9.10049	10.89951	333	56	777	223	49
12	807	56	150	850	421	54	867	133	48
13	907	55	252	748	508	52	956	044	47
14	9.10006	53	353	647	596	50	9.16046	10.83954	46
15	106	51	454	546	683	48	135	865	45
16	205	50	555	445	770	46	224	776	44
17	304	48	656	344	857	45	312	688	43
18	402	47	756	244	944	43	401	599	42
19	501	45	856	144	9.16030	41	489	511	41
20	9.10599	9.99643	956	044	116	9.99539	9.16577	10.83423	40
21	697	42	9.11056	10.88944	203	37	665	335	39
22	795	40	155	845	289	35	753	247	38
23	893	38	254	746	374	33	841	159	37
24	990	37	355	647	460	32	928	072	36
25	9.11087	35	452	548	545	30	9.17016	10.82984	35
26	184	33	551	449	631	28	103	897	34
27	281	32	649	351	716	26	190	810	33
28	377	30	747	253	801	24	277	723	32
29	474	29	845	155	886	22	363	637	31
30	9.11570	9.99627	943	057	970	9.99520	9.17450	16.82550	30
31	666	25	9.12040	10.87960	9.17055	18	536	464	29
32	761	24	138	862	139	17	622	378	28
33	857	22	235	765	223	15	708	292	27
34	952	20	332	668	307	13	794	206	26
35	9.12047	18	428	572	391	11	880	120	25
36	142	17	525	475	474	09	965	035	24
37	236	15	621	379	558	07	9.18051	10.81949	23
38	331	13	717	283	641	05	136	864	22
39	425	12	813	187	724	03	221	779	21
40	9.12519	9.99610	909	091	9.17807	01	9.18306	10.81694	20
41	612	08	9.13004	10.86996	890	9.99499	391	609	19
42	706	07	099	901	973	97	475	525	18
43	799	05	194	806	9.18055	95	560	440	17
44	892	03	289	711	137	94	644	356	16
45	985	01	384	616	220	92	728	272	15
46	9.13078	00	478	522	302	90	812	188	14
47	171	9.99598	573	427	383	88	896	104	13
48	263	96	667	333	465	86	979	021	12
49	355	95	761	239	547	84	9.19063	10.80937	11
50	9.13447	9.99593	9.13854	146	9.18628	9.99482	146	854	10
51	539	91	948	052	709	80	229	771	9
52	630	89	9.14041	10.85959	790	78	312	688	8
53	722	88	134	866	871	76	395	605	7
54	813	86	227	773	952	74	478	522	6
55	904	84	320	680	9.19033	72	561	439	5
56	994	82	412	588	113	70	643	357	4
57	9.14085	81	504	496	193	68	725	275	3
58	175	79	597	403	273	66	807	193	2
59	266	77	688	312	353	64	889	111	1
60	356	75	780	220	433	62	971	029	0
M	Co-sine	Sine	Co-tang.	Tangent.	Co-sine	Sine	Co-tang.	Tangent.	M
	82 Degrees.				81 Degrees.				

	9 Degrees.				10 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.19433	9.99462	9.19971	10.80029	9.23967	9.99335	9.24632	10.75368	60
1	513	60	9.20053	10.79947	9.24039	33	706	294	59
2	592	58	134	864	110	31	779	221	58
3	672	56	216	784	181	28	853	147	57
4	751	54	297	703	253	26	926	074	56
5	830	52	378	622	324	24	9.25000	000	55
6	909	50	459	541	395	22	073	10.74927	54
7	988	48	540	460	466	19	146	854	53
8	9.20067	46	621	379	536	17	219	781	52
9	145	44	701	299	607	15	292	708	51
10	223	9.99442	9.20782	10.79218	9.24677	9.99313	9.25365	10.74635	50
11	302	40	862	138	748	10	437	563	49
12	380	38	942	058	818	08	510	490	48
13	458	36	9.21022	10.78978	888	06	582	418	47
14	535	34	102	898	958	04	655	345	46
15	613	32	182	818	9.25028	01	727	273	45
16	691	29	261	739	098	9.99299	799	201	44
17	768	27	341	659	168	97	871	129	43
18	845	25	420	580	237	94	943	057	42
19	922	23	499	501	307	92	9.26015	10.73985	41
20	999	9.99421	9.21578	10.78422	9.25376	9.99290	086	914	40
21	9.21076	19	657	343	445	88	158	842	39
22	153	17	736	264	514	85	229	771	38
23	229	15	814	186	583	83	301	699	37
24	306	13	893	107	652	81	372	628	36
25	382	11	971	029	721	78	443	557	35
26	458	09	9.22049	10.77951	790	76	514	486	34
27	534	07	127	873	858	74	585	415	33
28	610	04	205	795	927	71	655	345	32
29	685	02	283	717	995	69	726	274	31
30	9.21761	00	9.22361	10.77639	9.26063	9.99267	9.26797	10.73203	30
31	836	9.99398	438	562	131	64	867	133	29
32	912	96	516	484	199	62	937	063	28
33	987	94	593	407	267	60	9.27008	10.72992	27
34	9.22062	92	670	330	335	57	078	922	26
35	137	90	747	253	403	55	148	852	25
36	211	88	824	176	470	52	218	782	24
37	286	85	901	099	538	50	288	712	23
38	361	83	977	023	605	48	357	643	22
39	435	81	9.22054	10.76946	672	45	427	573	21
40	9.22509	9.99379	130	870	9.26739	9.99243	9.27496	10.72504	20
41	583	77	206	794	806	41	566	434	19
42	657	75	283	717	873	38	635	365	18
43	731	72	359	641	940	36	704	296	17
44	805	70	435	565	9.27007	33	773	227	16
45	878	68	510	490	073	31	842	158	15
46	952	66	586	414	140	29	911	089	14
47	9.23025	64	661	339	206	26	980	020	13
48	098	62	737	263	273	24	9.28049	10.71951	12
49	171	59	812	188	339	21	117	883	11
50	9.23244	9.99357	9.23887	10.76113	9.27405	9.99219	186	10.71814	10
51	317	55	962	038	471	17	254	746	9
52	390	53	9.24037	10.75963	537	14	323	677	8
53	462	51	112	888	602	12	391	609	7
54	535	48	186	814	668	09	459	541	6
55	607	46	261	739	734	07	527	473	5
56	679	44	335	665	799	04	595	405	4
57	752	42	410	590	864	02	662	338	3
58	823	40	484	516	930	00	730	270	2
59	895	37	558	442	995	9.99197	798	202	1
60	967	35	632	368	9.28060	95	865	135	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent.	M
	80 Degrees.				79 Degrees.				

	11 Degrees.				12 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.28060	9.99195	9.28865	10.71135	9.31788	9.99040	9.32747	10.67253	60
1	125	92	933	067	847	38	810	190	59
2	190	90	9.29000	000	907	35	872	128	58
3	254	87	067	10.70933	966	32	933	067	57
4	319	85	134	866	9.32025	30	995	005	56
5	384	82	201	799	084	27	9.33057	10.66943	55
6	448	80	268	732	143	24	119	881	54
7	512	77	335	665	202	22	180	820	53
8	577	75	402	598	261	19	242	758	52
9	641	72	468	532	319	16	303	697	51
10	9.28705	9.99170	9.29535	10.70465	9.32378	9.99013	9.33365	10.66635	50
11	769	67	601	399	437	11	426	574	49
12	833	65	668	332	495	08	487	513	48
13	896	62	734	266	553	05	548	452	47
14	960	60	800	200	612	02	609	391	46
15	9.29024	57	866	134	670	00	670	330	45
16	087	55	932	068	728	9.98997	731	269	44
17	150	52	998	002	786	94	792	208	43
18	214	50	9.30064	10.69936	844	91	853	147	42
19	277	47	130	870	902	89	913	087	41
20	9.29340	9.99145	195	805	960	9.98986	974	026	40
21	403	42	261	739	9.33018	83	9.34034	10.65966	39
22	466	40	326	674	075	80	095	905	38
23	529	37	391	609	133	78	155	845	37
24	591	35	457	543	190	75	215	785	36
25	654	32	522	478	248	72	276	724	35
26	716	30	587	413	305	69	336	664	34
27	779	27	652	348	362	67	396	604	33
28	841	24	717	283	420	64	456	544	32
29	903	22	782	218	477	61	516	484	31
30	966	9.99119	9.30846	10.69154	9.33534	9.98958	9.34576	10.65424	30
31	9.30028	17	911	089	591	55	635	365	29
32	090	14	975	025	647	53	695	305	28
33	151	12	9.31040	10.68960	704	50	755	245	27
34	213	09	104	896	761	47	814	186	26
35	275	06	168	832	818	44	874	126	25
36	336	04	233	767	874	41	933	067	24
37	398	01	297	703	931	38	992	008	23
38	459	9.99099	361	639	987	36	9.35051	10.64949	22
39	521	96	425	575	9.34043	33	111	889	21
40	9.30582	93	9.31489	10.68511	100	9.98930	170	830	20
41	643	91	552	448	156	27	229	771	19
42	704	88	616	384	212	24	288	712	18
43	765	86	679	321	268	21	347	653	17
44	826	83	743	257	324	19	405	595	16
45	887	80	806	194	380	16	464	536	15
46	947	78	870	130	436	13	523	477	14
47	9.31008	75	933	067	491	10	581	419	13
48	068	72	996	004	547	07	640	360	12
49	129	70	9.32059	10.67941	602	04	698	302	11
50	9.31189	9.99067	122	10.67878	9.34658	01	9.35757	10.64243	10
51	250	64	185	815	713	9.98898	815	185	9
52	310	62	248	752	769	96	873	127	8
53	370	59	311	689	824	93	931	069	7
54	430	56	373	627	879	90	989	011	6
55	490	54	436	564	934	87	9.36047	10.63953	5
56	549	51	498	502	989	84	105	895	4
57	609	48	561	439	9.35044	81	163	837	3
58	669	46	623	377	099	78	221	779	2
59	728	43	685	315	154	75	279	721	1
60	788	40	747	253	209	72	336	664	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	78 Degrees.				77 Degrees.				

	13 Degrees.				14 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine.	Co-sine	Tang.	Co-tang.	M
0	9.35209	9.98872	9.36336	10.63664	9.38368	9.95690	9.39677	10.60323	60
1	263	69	394	606	418	87	731	269	59
2	318	67	452	548	469	84	785	215	58
3	373	64	509	491	519	81	838	162	57
4	427	61	566	434	570	78	892	108	56
5	481	58	624	376	620	75	945	055	55
6	536	55	681	319	670	71	999	001	54
7	590	52	738	262	721	68	9.40052	10.59948	53
8	644	49	795	205	771	65	106	894	52
9	698	46	852	148	821	62	159	841	51
10	9.35752	9.98843	9.36909	10.63091	9.38871	9.98659	9.40212	10.59788	50
11	806	40	966	034	921	56	266	734	49
12	860	37	9.37023	10.62977	971	52	319	681	48
13	914	34	080	920	9.39021	49	372	628	47
14	968	31	137	863	071	46	425	575	46
15	9.36022	28	193	807	121	43	478	522	45
16	075	25	250	750	170	40	531	469	44
17	129	22	306	694	220	36	584	416	43
18	182	19	363	637	270	33	636	364	42
19	236	16	419	581	319	30	689	311	41
20	9.36289	9.98813	9.37476	10.62524	9.39369	9.98627	9.40742	10.59258	40
21	342	10	532	468	418	23	795	205	39
22	395	07	588	412	467	20	847	153	38
23	449	04	644	356	517	17	900	100	37
24	502	01	700	300	566	14	952	048	36
25	555	9.98798	756	244	615	10	9.41005	10.58995	35
26	607	95	812	188	664	07	057	943	34
27	660	92	868	132	713	04	109	891	33
28	713	89	924	076	762	01	161	839	32
29	766	86	980	020	811	9.98597	214	786	31
30	9.36819	9.98783	9.38035	10.61965	9.39860	94	9.41266	10.58734	30
31	871	80	091	909	909	91	318	682	29
32	924	77	147	853	958	88	370	630	28
33	976	74	202	798	9.40006	84	422	578	27
34	9.37028	71	257	743	055	81	474	526	26
35	081	68	313	687	103	78	526	474	25
36	133	65	368	632	152	74	578	422	24
37	185	62	423	577	200	71	629	371	23
38	237	59	479	521	249	68	681	319	22
39	289	56	534	466	297	65	733	267	21
40	9.37341	9.98753	9.38589	10.61411	9.40346	9.98561	9.41784	10.58216	20
41	393	50	644	356	394	58	836	164	19
42	445	46	699	301	442	55	887	113	18
43	497	43	754	246	490	51	939	061	17
44	549	40	808	192	538	48	990	010	16
45	600	37	863	137	586	45	9.42041	10.57959	15
46	652	34	918	082	634	41	093	907	14
47	703	31	972	028	682	38	144	856	13
48	755	28	9.39027	10.60973	730	35	195	805	12
49	806	25	082	918	778	31	246	754	11
50	9.37858	9.98722	136	864	9.40825	9.98528	9.42297	10.57703	10
51	909	19	190	810	873	25	348	652	9
52	960	15	245	755	921	21	399	601	8
53	9.38011	12	299	701	968	18	450	550	7
54	062	09	353	647	9.41016	15	501	499	6
55	113	06	407	593	063	11	552	448	5
56	164	03	461	539	111	08	603	397	4
57	215	00	515	485	158	05	653	347	3
58	266	9.98697	569	431	205	01	704	296	2
59	317	94	623	377	252	9.98498	755	245	1
60	368	90	677	323	300	94	805	195	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	76 Degrees.				75 Degrees.				

	15 Degrees.				16 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.41300	9.98494	9.42805	10.57195	9.44034	9.98284	9.45750	10.54250	60
1	347	91	856	144	078	81	797	203	59
2	394	88	906	094	122	77	845	155	58
3	441	84	957	043	166	73	892	108	57
4	488	81	9.43007	10.56993	210	70	940	060	56
5	535	77	057	943	253	66	987	013	55
6	582	74	108	892	297	62	9.46035	10.53965	54
7	628	71	158	842	341	59	082	918	53
8	675	67	208	792	385	55	130	870	52
9	722	64	258	742	428	51	177	823	51
10	9.41768	9.98460	9.43308	10.56692	9.44472	9.98248	9.46224	10.53776	50
11	815	57	358	642	516	44	271	729	49
12	861	53	408	592	559	40	319	681	48
13	908	50	458	542	602	37	366	634	47
14	954	47	508	492	646	33	413	587	46
15	9.42001	43	558	442	689	29	460	540	45
16	047	40	607	393	733	26	507	493	44
17	093	36	657	343	776	22	554	446	43
18	140	33	707	293	819	18	601	399	42
19	186	29	756	244	862	15	648	352	41
20	9.42232	9.98426	9.43806	10.56194	9.44905	9.98211	9.46694	10.53306	40
21	278	22	855	145	948	07	741	259	39
22	324	19	905	095	992	04	788	212	38
23	370	15	954	046	9.45035	00	835	165	37
24	416	12	9.44004	10.55996	077	9.98196	881	119	36
25	461	09	053	947	120	92	928	072	35
26	507	05	102	898	163	89	975	025	34
27	553	02	151	849	206	85	9.47021	10.52979	33
28	599	9.98398	201	799	249	81	068	932	32
29	644	95	250	750	292	77	114	886	31
30	9.42690	91	9.44299	10.55701	9.45334	9.98174	9.47160	10.52840	30
31	735	88	348	652	377	70	207	793	29
32	781	84	397	603	419	66	253	747	28
33	826	81	446	554	462	62	299	701	27
34	872	77	495	505	504	59	346	654	26
35	917	73	544	456	547	55	392	608	25
36	962	70	592	408	589	51	438	562	24
37	9.43007	66	641	359	632	47	484	516	23
38	053	63	690	310	674	44	530	470	22
39	098	59	738	262	716	40	576	424	21
40	9.43143	9.98356	9.44787	10.55213	9.45758	9.98136	9.47622	10.52378	20
41	188	52	836	164	801	32	668	332	19
42	233	49	884	116	843	29	714	286	18
43	278	45	933	067	885	25	760	240	17
44	323	42	981	019	927	21	806	194	16
45	367	38	9.45029	10.54971	969	17	852	148	15
46	412	34	078	922	9.46011	13	897	103	14
47	457	31	126	874	053	10	943	057	13
48	502	27	174	826	095	06	989	011	12
49	546	24	222	778	136	02	9.48035	10.51965	11
50	9.43591	9.98320	9.45271	10.54729	9.46178	9.98098	080	920	10
51	635	17	319	681	220	94	126	874	9
52	680	13	367	633	262	90	171	829	8
53	724	09	415	585	303	87	217	783	7
54	769	06	463	537	345	83	262	738	6
55	813	02	511	489	386	79	307	693	5
56	857	9.98299	559	441	428	75	353	647	4
57	901	95	606	394	469	71	398	602	3
58	946	91	654	346	511	67	443	557	2
59	990	88	702	298	552	63	489	511	1
60	9.44034	84	750	250	594	60	534	466	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	74 Degrees.				73 Degrees.				

	17 Degrees.				18 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.46594	9.98060	9.48534	10.51466	9.48998	9.97821	9.51178	10.48822	60
1	635	56	579	421	9.49037	817	221	779	59
2	676	52	624	376	076	812	264	736	58
3	717	48	669	331	115	808	306	694	57
4	758	44	714	286	153	804	349	651	56
5	800	40	759	241	192	800	392	608	55
6	841	36	804	196	231	796	435	565	54
7	882	32	849	151	269	792	478	522	53
8	923	29	894	106	308	788	520	480	52
9	964	25	939	061	347	784	563	437	51
10	9.47005	9.98021	9.48984	10.51016	9.49385	9.97779	9.51606	10.48394	50
11	045	17	9.49029	10.50971	424	75	648	352	49
12	086	13	073	927	462	71	691	309	48
13	127	09	118	882	500	67	734	266	47
14	168	05	163	837	539	63	776	224	46
15	209	01	207	793	577	59	819	181	45
16	249	9.97997	252	748	615	54	861	139	44
17	290	93	296	704	654	50	903	097	43
18	330	89	341	659	692	46	946	054	42
19	371	86	385	615	730	42	988	012	41
20	9.47411	9.97982	9.49430	10.50570	9.49768	9.97738	9.52031	10.47969	40
21	452	78	474	526	806	34	073	927	39
22	492	74	519	481	844	29	115	885	38
23	533	70	563	437	882	25	157	843	37
24	573	66	607	393	920	21	200	800	36
25	613	62	652	348	958	17	242	758	35
26	654	58	696	304	996	13	284	716	34
27	694	54	740	260	9.50034	08	326	674	33
28	734	50	784	216	072	04	368	632	32
29	774	46	828	172	110	00	410	590	31
30	9.47814	9.97942	9.49872	10.50128	148	9.97696	9.52452	10.47548	30
31	854	38	916	084	185	91	494	506	29
32	894	34	960	10.49040	223	87	536	464	28
33	934	30	9.50004	996	261	83	578	422	27
34	974	26	048	952	298	79	620	380	26
35	9.48014	22	092	908	336	74	661	339	25
36	054	18	136	864	374	70	703	297	24
37	094	14	180	820	411	66	745	255	23
38	133	10	223	777	449	62	787	213	22
39	173	06	267	733	486	57	829	171	21
40	9.48213	02	9.50311	10.49689	9.50523	9.97653	9.52870	10.47130	20
41	252	9.97898	355	645	561	49	912	088	19
42	292	94	398	602	598	45	953	047	18
43	332	90	442	558	635	40	995	005	17
44	371	86	485	515	673	36	9.53037	10.46963	16
45	411	82	529	471	710	32	078	922	15
46	450	78	572	428	747	27	120	880	14
47	490	74	616	384	784	23	161	839	13
48	529	70	659	341	821	19	202	798	12
49	568	66	703	297	858	15	244	756	11
50	9.48607	9.97861	9.50746	10.49254	9.50896	9.97610	9.53285	10.46715	10
51	647	57	789	211	933	06	327	673	9
52	686	53	833	167	970	02	368	632	8
53	725	49	876	124	9.51007	9.97597	409	591	7
54	764	45	919	081	043	93	450	550	6
55	803	41	962	038	080	89	492	508	5
56	842	37	9.51005	10.48995	117	84	533	467	4
57	881	33	048	952	154	80	574	426	3
58	920	29	092	908	191	76	615	385	2
59	959	25	135	865	227	71	656	344	1
60	998	21	178	822	264	67	697	303	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	T	
	72 Degrees.				71 Degrees.				

	19 Degrees.				20 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang	Co-tang.	M
0	9.51264	9.97567	9.53697	10.46303	9.53405	9.97299	9.56107	10.43893	60
1	301	63	738	262	440	94	146	854	59
2	337	58	779	221	475	89	185	815	58
3	374	54	820	180	509	85	224	776	57
4	411	50	861	139	544	80	264	736	56
5	447	45	902	098	578	76	303	697	55
6	484	41	943	057	613	71	342	658	54
7	520	36	984	016	647	66	381	619	53
8	557	32	9.54024	10.45975	682	62	420	580	52
9	593	28	065	935	716	57	459	541	51
10	9.51629	9.97523	9.54106	10.45894	9.53751	9.97252	9.56498	10.43502	50
11	666	519	147	853	785	48	537	463	49
12	702	515	187	813	819	43	576	424	48
13	738	510	228	772	854	38	615	385	47
14	774	506	269	731	888	34	654	346	46
15	811	501	309	691	922	29	693	307	45
16	847	497	350	650	957	24	732	268	44
17	883	492	390	610	991	20	771	229	43
18	919	488	431	569	9.54025	15	810	190	42
19	955	484	471	529	059	10	849	151	41
20	9.51991	9.97479	9.54512	10.45488	9.54093	9.97206	9.56887	10.43113	40
21	9.52027	75	552	448	127	01	926	074	39
22	063	70	593	407	161	9.97196	965	035	38
23	099	66	633	367	195	92	9.57004	10.42996	37
24	135	61	673	327	229	87	042	958	36
25	171	57	714	286	263	82	081	919	35
26	207	52	754	246	297	78	120	880	34
27	242	48	794	206	331	73	158	842	33
28	278	44	835	165	365	68	197	803	32
29	314	39	875	125	399	63	235	765	31
30	9.52350	9.97435	9.54915	10.45085	9.54433	9.97159	9.57274	10.42726	30
31	385	30	955	045	466	54	312	688	29
32	421	26	995	005	500	49	351	649	28
33	456	21	9.55035	10.44965	534	45	389	611	27
34	492	17	075	925	567	40	428	572	26
35	527	12	115	835	601	35	466	534	25
36	563	08	155	845	635	30	504	496	24
37	598	03	195	805	668	26	543	457	23
38	634	9.97399	235	765	702	21	581	419	22
39	669	94	275	725	735	16	619	381	21
40	9.52705	9.97390	9.55315	10.44685	9.54769	9.97111	9.57658	10.42342	20
41	740	85	355	645	802	07	696	304	19
42	775	81	395	605	836	02	734	266	18
43	811	76	434	566	869	9.97097	772	228	17
44	846	72	474	526	903	92	810	190	16
45	881	67	514	486	936	87	849	151	15
46	916	63	554	446	969	83	887	113	14
47	951	58	593	407	9.55003	78	925	075	13
48	986	53	633	367	036	73	963	037	12
49	9.53021	49	673	327	069	68	9.58001	10.41999	11
50	056	9.97344	9.55712	10.44288	9.55102	9.97063	039	961	10
51	092	40	752	248	136	59	077	923	9
52	126	35	791	209	169	54	115	885	8
53	161	31	831	169	202	49	153	847	7
54	196	26	870	130	235	44	191	809	6
55	231	22	910	090	268	39	229	771	5
56	266	17	949	051	301	35	267	733	4
57	301	12	989	011	334	30	304	696	3
58	336	08	9.56028	10.43972	367	25	342	658	2
59	370	03	067	933	400	20	380	620	1
60	405	9.97299	107	893	433	15	418	582	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	70 Degrees.				69 Degrees.				

	21 Degrees.				22 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.55433	9.97015	9.58418	10.41582	9.57358	9.96717	9.60641	10.39359	60
1	466	10	455	545	389	11	677	323	59
2	499	05	493	507	420	06	714	286	58
3	532	01	531	469	451	01	750	250	57
4	564	9.96996	569	431	482	9.96696	786	214	56
5	597	91	606	394	514	91	823	177	55
6	630	86	644	356	545	86	859	141	54
7	663	81	681	319	676	81	895	105	53
8	695	76	719	281	607	76	931	069	52
9	728	71	757	243	638	70	967	033	51
10	9.55761	9.96966	9.58794	10.41206	9.57669	9.96665	9.61004	10.38996	50
11	793	62	832	168	700	60	040	960	49
12	826	57	869	131	731	55	076	924	48
13	858	52	907	093	762	50	112	888	47
14	891	47	944	056	793	45	148	852	46
15	923	42	981	019	824	40	184	816	45
16	956	37	9.59019	10.40981	854	34	220	780	44
17	988	32	056	944	885	29	256	744	43
18	9.56021	27	094	906	916	24	292	708	42
19	053	22	131	869	947	19	328	672	41
20	085	9.96917	9.59168	10.40832	9.57978	9.96614	9.61364	10.38636	40
21	118	12	205	795	9.58008	08	400	600	39
22	150	07	243	757	039	03	436	564	38
23	182	03	280	720	070	9.96598	472	528	37
24	215	9.96898	317	683	101	93	508	492	36
25	247	93	354	646	131	88	544	456	35
26	279	88	391	609	162	82	579	421	34
27	311	83	429	571	192	77	615	385	33
28	343	78	466	534	223	72	651	349	32
29	375	73	503	497	253	67	687	313	31
30	9.56408	9.96868	9.59540	10.40460	9.58284	9.96562	9.61722	10.38278	30
31	440	63	577	423	314	56	758	242	29
32	472	58	614	386	345	51	794	206	28
33	504	53	651	349	375	46	830	170	27
34	536	48	688	312	406	41	865	135	26
35	568	43	725	275	436	35	901	099	25
36	599	38	762	238	467	30	936	064	24
37	631	33	799	201	497	25	972	028	23
38	663	28	835	165	527	20	9.62008	10.37992	22
39	695	23	872	128	557	14	043	957	21
40	9.56727	9.96818	9.59909	10.40091	9.58588	9.96509	079	921	20
41	759	13	946	054	618	04	114	886	19
42	790	08	983	017	648	9.96498	150	850	18
43	822	03	9.60019	10.39981	678	93	185	815	17
44	854	9.96798	056	944	709	88	221	779	16
45	886	93	093	907	739	83	256	744	15
46	917	88	130	870	769	77	292	708	14
47	949	83	166	834	799	72	327	673	13
48	980	78	203	797	829	67	362	638	12
49	9.57012	72	240	760	859	61	398	602	11
50	044	9.96767	9.60276	10.39724	9.58889	9.96456	9.62433	10.37567	10
51	075	62	313	687	919	51	468	532	9
52	107	57	349	651	949	45	504	496	8
53	138	52	386	614	979	40	539	461	7
54	169	47	422	578	9.59009	35	574	426	6
55	201	42	459	541	039	29	609	391	5
56	232	37	495	505	069	24	645	355	4
57	264	32	532	468	098	19	680	320	3
58	295	27	568	432	128	13	715	285	2
59	326	22	605	395	158	08	750	250	1
60	358	17	641	359	188	03	785	215	
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent.	
	68 Degrees.				67 Degrees.				

	23 Degrees.				24 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.59188	9.96403	9.62785	10.37215	9.60931	9.96073	9.64858	10.35142	60
1	218	9.96397	820	180	960	67	892	108	59
2	247	92	855	145	988	62	926	074	58
3	277	87	890	109	9.61016	56	960	040	57
4	307	81	926	074	045	50	994	006	56
5	336	76	961	039	073	45	9.65028	10.34972	55
6	366	70	996	004	101	39	062	938	54
7	396	65	9.63031	10.36969	129	34	096	904	53
8	425	60	066	934	158	28	130	870	52
9	455	54	101	899	186	22	164	836	51
10	9.59484	9.96349	9.63135	10.36865	9.61214	9.96017	9.65197	10.34803	50
11	514	43	170	830	242	11	231	769	49
12	543	38	205	795	270	05	265	735	48
13	573	33	240	760	298	9.96000	299	701	47
14	602	27	275	725	326	94	333	667	46
15	632	22	310	690	354	88	366	634	45
16	661	16	345	655	382	82	400	600	44
17	690	11	379	621	411	77	434	566	43
18	720	05	414	586	438	71	467	533	42
19	749	00	449	551	466	65	501	499	41
20	9.59778	9.96294	9.63484	10.36516	9.61494	9.95960	9.65535	10.34465	40
21	808	89	519	481	522	54	568	432	39
22	837	84	553	447	550	48	602	398	38
23	866	78	588	412	578	42	636	364	37
24	895	73	623	377	600	37	669	331	36
25	924	67	657	343	634	31	703	297	35
26	954	62	692	308	662	25	736	264	34
27	983	56	726	274	689	20	770	230	33
28	9.60012	51	761	239	717	14	803	197	32
29	041	45	796	204	745	08	837	163	31
30	070	9.96240	9.63830	10.36170	9.61773	02	9.65870	10.34130	30
31	099	34	865	135	800	9.95897	904	096	29
32	128	29	899	101	828	91	937	063	28
33	157	23	934	066	856	85	971	029	27
34	186	18	968	032	883	79	9.66004	10.33996	26
35	215	12	9.64003	10.35997	911	73	038	962	25
36	244	07	037	963	939	68	071	929	24
37	273	01	072	928	966	62	104	896	23
38	302	9.96196	106	894	994	56	138	862	22
39	331	90	140	860	9.62021	50	171	829	21
40	9.60359	85	9.64175	10.35825	049	9.95844	9.66204	10.33796	20
41	388	79	209	791	076	39	238	762	19
42	417	74	243	757	104	33	271	729	18
43	446	68	278	722	131	27	304	696	17
44	474	62	312	688	159	21	337	663	16
45	503	57	346	654	186	15	371	629	15
46	532	51	381	619	214	10	404	596	14
47	561	46	415	585	241	04	437	563	13
48	589	40	449	551	268	9.95798	470	530	12
49	618	35	483	517	296	92	503	497	11
50	9.60646	9.96129	9.64517	10.35483	9.62323	86	9.66537	10.33463	10
51	675	23	552	448	350	80	570	430	9
52	704	18	586	414	377	75	603	397	8
53	732	12	620	380	405	69	636	364	7
54	761	07	654	346	432	63	669	331	6
55	789	01	688	312	459	57	702	298	5
56	818	9.96095	722	278	486	51	735	265	4
57	846	90	756	244	513	45	768	232	3
58	875	84	790	210	541	39	801	199	2
59	903	79	824	176	568	33	834	166	1
60	931	73	858	142	595	28	867	133	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	66 Degrees.				65 Degrees.				

	25 Degrees.				26 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.62595	9.95728	9.66867	10.33133	9.64184	9.95366	9.68818	10.31182	60
1	622	722	900	100	210	360	850	150	59
2	649	716	933	067	236	354	882	118	58
3	676	710	966	034	262	348	914	086	57
4	703	704	999	001	288	341	946	054	56
5	730	698	9.67032	10.32968	313	335	978	022	55
6	757	692	065	935	339	329	9.69010	10.30990	54
7	784	686	098	902	365	323	042	958	53
8	811	680	131	869	391	317	074	926	52
9	838	674	163	837	417	310	106	894	51
10	9.62865	9.95668	9.67196	10.32804	9.64442	9.95304	9.69138	10.30862	50
11	892	663	229	771	468	298	170	830	49
12	918	657	262	738	494	292	202	798	48
13	945	651	295	705	519	286	234	766	47
14	972	645	327	673	545	279	266	734	46
15	999	639	360	640	571	273	298	703	45
16	9.63026	633	393	607	596	267	329	671	44
17	052	627	426	574	622	261	361	639	43
18	079	621	458	542	647	254	393	607	42
19	106	615	491	509	673	248	425	575	41
20	133	9.95609	9.67524	10.32476	9.64698	9.95242	9.69457	10.30543	40
21	159	603	556	444	724	236	488	512	39
22	186	597	589	411	749	229	520	480	38
23	213	591	622	378	775	223	552	448	37
24	239	585	654	346	800	217	584	416	36
25	266	579	687	313	826	211	615	385	35
26	292	573	719	281	851	204	647	353	34
27	319	567	752	248	877	198	679	321	33
28	345	561	785	215	902	192	710	290	32
29	372	555	817	183	927	185	742	258	31
30	9.63398	9.95549	9.67850	10.32150	953	9.95179	9.69774	10.30226	30
31	425	43	882	118	978	173	805	195	29
32	451	37	915	085	9.65003	167	837	163	28
33	478	31	947	053	029	160	868	132	27
34	504	25	980	020	054	154	900	100	26
35	531	19	9.68012	10.31988	079	148	932	068	25
36	557	13	044	956	104	141	963	037	24
37	583	07	077	923	130	135	995	005	23
38	610	00	109	891	155	129	9.70026	10.29974	22
39	636	9.95494	142	858	180	122	058	942	21
40	9.63662	9.95488	9.68174	10.31826	9.65205	9.95116	089	911	20
41	689	82	206	794	230	110	121	879	19
42	715	76	239	761	255	103	152	848	18
43	741	70	271	729	281	097	184	816	17
44	767	64	303	697	306	090	215	785	16
45	794	58	336	664	331	084	247	753	15
46	820	52	368	632	356	078	278	722	14
47	846	46	400	600	381	071	309	691	13
48	872	40	432	568	406	065	341	659	12
49	898	34	465	535	431	059	372	628	11
50	9.63924	9.95427	9.[illegible]8497	10.31503	9.65456	9.95052	9.70404	10.29596	10
51	950	21	529	471	481	046	435	565	9
52	976	15	561	439	506	039	466	534	8
53	9.64002	09	593	407	531	033	498	502	7
54	028	03	626	374	556	027	529	471	6
55	054	9.95397	658	342	580	020	560	440	5
56	080	91	690	310	605	014	592	408	
57	106	84	722	278	630	007	623	377	
58	132	78	754	246	655	001	654	346	
59	158	72	786	214	680	9.94995	685	315	
60	184	66	818	182	705	988	717	283	
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	C	ngent	
	64 Degrees.				63				

	27 Degrees.				28 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.65705	9.94988	9.70717	10.29283	9.67161	9.94593	9.72567	10.27433	60
1	729	82	748	252	185	87	598	402	59
2	754	75	779	221	208	80	628	372	58
3	779	69	810	190	232	73	659	341	57
4	804	62	841	159	256	67	689	311	56
5	828	56	873	127	280	60	720	280	55
6	853	49	904	096	303	53	750	250	54
7	878	43	935	065	327	46	780	220	53
8	902	36	966	034	350	40	811	189	52
9	927	30	997	003	374	33	841	159	51
10	9.65952	9.94923	9.71028	10.28972	9.67398	9.94526	9.72872	10.27128	50
11	976	17	059	941	421	19	902	098	49
12	9.66001	11	090	910	445	13	932	068	48
13	025	04	121	879	468	06	963	037	47
14	050	9.94898	153	847	492	9.94499	993	007	46
15	075	91	184	816	515	92	9.73023	10.26977	45
16	099	84	215	785	539	85	054	946	44
17	124	78	246	754	562	79	084	916	43
18	148	71	277	723	586	72	114	886	42
19	173	65	308	692	609	65	144	856	41
20	9.66197	9.94858	9.71339	10.28661	9.67633	9.94458	9.73175	10.26825	40
21	221	52	370	630	656	51	205	795	39
22	246	45	401	599	680	45	235	765	38
23	270	39	431	569	703	38	265	735	37
24	295	32	462	538	726	31	295	705	36
25	319	26	493	507	750	24	326	674	35
26	343	19	524	476	773	17	356	744	34
27	368	13	555	445	796	10	386	614	33
28	392	06	586	414	820	04	416	584	32
29	416	9.94799	617	383	843	9.94397	446	554	31
30	9.66441	93	9.71648	10.28352	9.67866	90	9.73476	10.26524	30
31	465	86	679	321	890	83	507	493	29
32	489	80	709	291	913	76	537	463	28
33	513	73	740	260	936	69	567	433	27
34	537	67	771	229	959	62	597	403	26
35	562	60	802	198	982	55	627	373	25
36	586	53	833	167	9.68006	49	657	343	24
37	610	47	863	137	029	42	687	313	23
38	634	40	894	106	052	35	717	283	22
39	658	34	925	075	075	28	747	253	21
40	9.66682	9.94727	955	045	9.68098	9.94321	9.73777	10.26223	20
41	706	20	986	014	121	14	807	193	19
42	731	14	9.72017	10.27983	144	07	837	163	18
43	755	07	048	952	167	00	867	133	17
44	779	00	078	922	190	9.94293	897	103	16
45	803	9.94694	109	891	213	86	927	073	15
46	827	87	140	860	237	79	957	043	14
47	851	80	170	830	260	73	987	013	13
48	875	74	201	799	283	66	9.74017	10.25983	12
49	899	67	231	769	305	59	047	953	11
50	9.66922	9.94660	9.72262	10.27738	9.68328	9.94252	077	923	10
51	946	54	293	707	351	45	107	893	9
52	970	47	323	677	374	38	137	863	8
53	994	40	354	646	397	31	166	834	7
54	9.67018	34	384	616	420	24	196	804	6
55	042	27	415	585	443	17	226	774	5
56	066	20	445	555	466	10	256	744	4
57	090	14	476	524	489	03	286	714	3
58	113	07	506	494	512	9.94196	316	684	2
59	137	00	537	463	534	89	345	655	1
60	161	9.94593	567	433	557	82	375	625	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	62 Degrees.				61 Degrees.				

	33 Degrees.				34 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.73611	9.92359	9.81252	10.18748	9.74756	9.91857	9.82899	10.17101	60
1	630	51	279	721	775	49	926	074	59
2	650	43	307	693	794	40	953	047	58
3	669	34	335	665	812	32	980	020	57
4	689	26	362	638	831	23	9.83008	10.16992	56
5	708	18	390	610	850	15	035	965	55
6	727	10	418	582	868	06	062	938	54
7	747	02	445	555	887	9.91798	089	911	53
8	766	9.92293	473	527	906	89	117	883	52
9	785	85	500	500	924	81	144	856	51
10	9.73805	77	9.81528	10.18472	9.74943	72	9.83171	10.16829	50
11	824	69	556	444	961	63	198	802	49
12	843	60	583	417	980	55	225	775	48
13	863	52	611	389	999	46	252	748	47
14	882	44	638	362	9.75017	38	280	720	46
15	901	35	666	334	036	29	307	693	45
16	921	27	693	307	054	20	334	666	44
17	940	19	721	279	073	12	361	639	43
18	959	11	748	252	091	03	388	612	42
19	978	02	776	224	110	9.91695	415	585	41
20	997	9.92194	9.81803	10.18197	9.75128	86	9.83442	10.16558	40
21	9.74017	86	831	169	147	77	470	530	39
22	036	77	858	142	165	69	497	503	38
23	055	69	886	114	184	60	524	476	37
24	074	61	913	087	202	51	551	449	36
25	093	52	941	059	221	43	578	422	35
26	113	44	968	032	239	34	605	395	34
27	132	36	996	004	258	25	632	368	33
28	151	27	9.82023	10.17977	276	17	659	341	32
29	170	19	051	949	294	08	686	314	31
30	9.74189	11	078	922	9.75313	9.91599	9.83713	10.16287	30
31	208	02	106	894	331	91	740	260	29
32	227	9.92094	133	867	350	82	768	232	28
33	246	86	161	839	368	73	795	205	27
34	265	77	188	812	386	65	822	178	26
35	284	69	215	785	405	56	849	151	25
36	303	60	243	757	423	47	876	124	24
37	322	52	270	730	441	38	903	097	23
38	341	44	298	702	459	30	930	070	22
39	360	35	325	675	478	21	957	043	21
40	9.74379	9.92027	9.82352	10.17648	9.75496	12	984	016	20
41	398	18	380	620	514	04	9.84011	10.15989	19
42	417	10	407	593	533	9.91495	038	962	18
43	436	02	435	565	551	86	065	935	17
44	455	9.91993	462	538	569	77	092	908	16
45	474	85	489	511	587	69	119	881	15
46	493	76	517	483	605	60	146	854	14
47	512	68	544	456	624	51	173	827	13
48	531	59	571	429	642	42	200	800	12
49	549	51	599	401	660	33	227	773	11
50	9.74568	9.91942	9.82626	10.17374	9.75678	9.91425	9.84254	10.15746	10
51	587	34	653	347	696	16	280	720	9
52	606	25	681	319	714	07	307	693	8
53	625	17	708	292	733	9.91398	334	666	7
54	644	08	735	265	751	89	361	639	6
55	662	00	762	238	769	81	388	612	5
56	681	9.91891	790	210	787	72	415	585	4
57	700	83	817	183	805	63	442	558	3
58	719	74	844	156	823	54	469	531	2
59	737	66	871	129	841	45	496	504	1
60	756	57	899	101	859	36	523	477	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	56 Degrees.				55 Degrees.				

	31 Degrees.				32 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.71184	9.93307	9.77877	10.22123	9.72421	9.92842	9.79579	10.20421	60
1	205	9.93299	906	094	441	34	607	393	59
2	226	91	935	065	461	26	635	365	58
3	247	84	963	037	482	18	663	337	57
4	268	76	992	008	502	10	691	309	56
5	289	69	9.78020	10.21980	522	03	719	281	55
6	310	61	049	951	542	9.92795	747	253	54
7	331	53	077	923	562	87	776	224	53
8	252	46	106	894	582	79	804	196	52
9	373	38	135	865	602	71	832	168	51
10	9.71395	9.93230	9.78163	10.21837	9.72622	9.92763	9.79860	10.20140	50
11	414	23	192	808	643	55	888	112	49
12	435	15	220	780	663	47	916	084	48
13	456	07	249	751	683	39	944	056	47
14	477	00	277	723	703	31	972	028	46
15	498	9.93192	306	694	723	23	9.80000	000	45
16	519	84	334	666	743	15	028	10.19972	44
17	539	77	363	637	763	07	056	944	43
18	560	69	391	609	783	9.92699	084	916	42
19	581	61	419	581	803	91	112	888	41
20	9.71602	9.93154	9.78448	10.21552	9.72823	83	9.80140	10.19860	40
21	622	46	476	524	843	75	168	832	39
22	643	38	505	495	863	67	195	805	38
23	664	31	533	467	883	59	223	777	37
24	685	23	562	438	902	51	251	749	36
25	705	15	590	410	922	43	279	721	35
26	726	08	618	382	942	35	307	693	34
27	747	00	647	353	962	27	335	665	33
28	767	9.93092	675	325	982	19	363	637	32
29	788	84	704	296	9.73002	11	391	609	31
30	9.71809	77	9.78732	10.21268	022	03	9.80419	10.19581	30
31	829	69	760	240	041	9.92595	447	553	29
32	850	61	789	211	061	87	474	526	28
33	870	53	817	183	081	79	502	498	27
34	891	46	845	155	101	71	530	470	26
35	911	38	874	126	121	63	558	442	25
36	932	30	902	098	140	55	586	414	24
37	952	22	930	070	160	46	614	386	23
38	973	14	959	041	180	38	642	358	22
39	994	07	987	013	200	30	669	331	21
40	9.72014	9.92999	9.79015	10.20985	9.73219	9.92522	9.80697	10.19303	20
41	034	91	043	957	239	14	725	275	19
42	055	83	072	928	259	06	753	247	18
43	075	76	100	900	278	9.92498	781	219	17
44	096	68	128	872	298	90	808	192	16
45	116	60	156	844	318	82	836	164	15
46	137	52	185	815	337	73	864	136	14
47	157	44	213	787	357	65	892	108	13
48	177	36	241	759	377	57	919	081	12
49	198	29	269	731	396	49	947	053	11
50	9.72218	9.92921	9.79297	10.20703	9.73416	9.92441	975	025	10
51	238	13	326	674	435	33	9.81003	10.18997	9
52	259	05	354	646	455	25	030	970	8
53	279	9.92897	382	618	474	16	058	942	7
54	299	89	410	590	494	08	086	914	6
55	320	81	438	562	513	00	113	887	5
56	340	74	466	534	533	9.92392	141	859	4
57	360	66	495	505	552	84	169	831	3
58	381	58	523	477	572	76	196	801	2
59	401	50	551	449	591	67	224	776	1
60	421	42	579	421	611	59	252	748	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	58 Degrees.				57 Degrees.				

	37 Degrees.				38 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.77946	9.90235	9.87711	10.12289	9.78934	9.89653	9.89281	10.10719	60
1	963	25	738	262	950	43	307	693	59
2	980	16	764	236	967	33	333	667	58
3	997	06	790	210	983	24	359	641	57
4	9.78013	9.90197	817	183	999	14	385	615	56
5	030	87	843	157	9.79015	04	411	589	55
6	047	78	869	131	031	9.89594	437	563	54
7	063	68	895	105	047	84	463	537	53
8	080	59	922	078	063	74	489	511	52
9	097	49	948	052	079	64	515	485	51
10	9.78113	9.90139	974	026	9.79095	9.89554	9.89541	10.10459	50
11	130	30	9.88000	000	111	44	567	433	49
12	147	20	027	10.11973	128	34	593	407	48
13	163	11	053	947	144	24	619	381	47
14	180	01	079	921	160	14	645	355	46
15	197	9.90091	105	895	176	04	671	329	45
16	213	82	131	869	192	9.89495	697	303	44
17	230	72	158	842	208	85	723	277	43
18	246	63	184	816	224	75	749	251	42
19	263	53	210	790	240	65	775	225	41
20	9.78280	9.90043	9.88236	10.11764	9.79256	9.89455	9.89801	10.10199	40
21	296	34	262	738	272	45	827	173	39
22	313	24	289	711	288	35	853	147	38
23	329	14	315	685	304	25	879	121	37
24	346	05	341	659	319	15	905	095	36
25	362	9.89995	367	633	335	05	931	069	35
26	379	85	393	607	351	9.89395	957	043	34
27	395	76	420	580	367	85	983	017	33
28	412	66	446	554	383	75	9.90009	10.09991	32
29	428	56	472	528	399	64	035	965	31
30	9.78445	9.89947	9.88498	10.11502	9.79415	9.89354	061	939	30
31	461	37	524	476	431	44	086	914	29
32	478	27	550	450	447	34	112	888	28
33	494	18	576	424	463	24	138	862	27
34	510	08	603	397	478	14	164	836	26
35	527	9.89898	629	371	494	04	190	810	25
36	543	88	655	345	510	9.89294	216	784	24
37	560	79	681	319	526	84	242	758	23
38	576	69	707	293	542	74	268	732	22
39	592	59	733	267	558	64	294	706	21
40	9.78609	9.89849	9.88759	10.11241	9.79573	9.89254	9.90320	10.09680	20
41	625	40	786	214	589	44	346	654	19
42	642	30	812	188	605	33	371	629	18
43	658	20	838	162	621	23	397	603	17
44	674	10	864	136	636	13	423	577	16
45	691	01	890	110	652	03	449	551	15
46	707	9.89791	916	084	668	9.89193	475	525	14
47	723	81	942	058	684	83	501	499	13
48	739	71	968	032	699	73	527	473	12
49	756	61	994	006	715	62	553	447	11
50	9.78772	9.89752	9.89020	10.10980	9.79731	9.89152	9.90578	10.09422	10
51	788	42	046	954	746	42	604	396	9
52	805	32	073	927	762	32	630	370	8
53	821	22	099	901	778	22	656	344	7
54	837	12	125	875	793	12	682	318	6
55	853	02	151	849	809	01	708	292	5
56	869	9.89693	177	823	825	9.89091	734	266	4
57	886	83	203	797	840	81	759	241	3
58	902	73	229	771	856	71	785	215	2
59	918	63	255	745	872	60	811	189	1
60	934	53	281	719	887	50	837	163	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	52 Degrees.				51 Degrees.				

	35 Degrees.				36 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.75859	9.91336	9.84523	10.15477	9.76922	9.90796	9.86126	10.13874	60
1	877	328	550	450	939	87	153	847	59
2	895	319	576	424	957	77	179	821	58
3	913	310	603	397	974	68	206	794	57
4	931	301	630	370	991	59	232	768	56
5	949	292	657	343	9.77009	50	259	741	55
6	967	283	684	316	026	41	285	715	54
7	985	274	711	289	043	31	312	688	53
8	9.76003	266	738	262	061	22	338	662	52
9	021	257	764	236	078	13	365	635	51
10	039	9.91248	9.84791	10.15209	9.77095	04	9.86392	10.13608	50
11	057	239	818	182	112	9.90694	418	582	49
12	075	230	845	155	130	85	445	555	48
13	093	221	872	128	147	76	471	529	47
14	111	212	899	101	164	67	498	502	46
15	129	203	925	075	181	57	524	476	45
16	146	194	952	048	199	48	551	449	44
17	164	185	979	021	216	39	577	423	43
18	182	176	9.85006	10.14994	233	30	603	397	42
19	200	167	033	967	250	20	630	370	41
20	9.76218	9.91158	059	941	9.77268	11	9.86656	10.13344	40
21	236	149	086	914	285	02	683	317	39
22	253	141	113	887	302	9.90592	709	291	38
23	271	132	140	860	319	83	736	264	37
24	289	123	166	834	336	74	762	238	36
25	307	114	193	807	353	65	789	211	35
26	324	105	220	780	370	55	815	185	34
27	342	096	247	753	387	46	842	158	33
28	360	087	273	727	405	37	868	132	32
29	378	078	300	700	422	27	894	106	31
30	9.76395	9.91069	9.85327	10.14673	9.77439	18	9.86921	10.13079	30
31	413	060	354	646	456	09	947	053	29
32	431	051	380	620	473	9.90499	974	026	28
33	448	042	407	593	490	90	9.87000	000	27
34	466	033	434	566	507	80	027	10.12973	26
35	484	023	460	540	524	71	053	947	25
36	501	014	487	513	541	62	079	921	24
37	519	005	514	486	558	52	106	894	23
38	537	9.90996	540	460	575	43	132	868	22
39	554	987	567	433	592	34	158	842	21
40	9.76572	978	9.85594	10.14406	9.77609	9.90424	9.87185	10.12815	20
41	590	969	620	380	626	15	211	789	19
42	607	960	647	353	643	05	238	762	18
43	625	951	674	326	660	9.90396	264	736	17
44	642	942	700	300	677	86	290	710	16
45	660	933	727	273	694	77	317	683	15
46	677	924	754	246	711	68	343	657	14
47	695	915	780	220	727	58	369	631	13
48	712	905	807	193	744	49	396	604	12
49	730	896	834	166	761	39	422	578	11
50	9.76747	9.90887	9.85860	10.14140	9.77778	9.90330	9.87448	10.12552	10
51	765	878	887	113	795	20	475	525	9
52	782	869	913	087	812	11	501	499	8
53	800	860	940	060	829	01	527	473	7
54	817	851	967	033	846	9.90292	554	446	6
55	835	842	993	007	862	82	580	420	5
56	852	832	9.86020	10.13980	879	73	606	394	4
57	870	823	046	954	896	63	633	367	3
58	887	814	073	927	913	54	659	341	2
59	904	805	100	900	930	44	685	315	1
60	922	796	126	874	946	35	711	289	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	54 Degrees.				53 Degrees.				

	37 Degrees.				38 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.77946	9.90235	9.87711	10.12289	9.78934	9.89653	9.89281	10.10719	60
1	963	25	738	262	950	43	307	693	59
2	980	16	764	236	967	33	333	667	58
3	997	06	790	210	983	24	359	641	57
4	9.78013	9.90197	817	183	999	14	385	615	56
5	030	87	843	157	9.79015	04	411	589	55
6	047	78	869	131	031	9.89594	437	563	54
7	063	68	895	105	047	84	463	537	53
8	080	59	922	078	063	74	489	511	52
9	097	49	948	052	079	64	515	485	51
10	9.78113	9.90139	974	026	9.79095	9.89554	9.89541	10.10459	50
11	130	30	9.88000	000	111	44	567	433	49
12	147	20	027	10.11973	128	34	593	407	48
13	163	11	053	947	144	24	619	381	47
14	180	01	079	921	160	14	645	355	46
15	197	9.90091	105	895	176	04	671	329	45
16	213	82	131	869	192	9.89495	697	303	44
17	230	72	158	842	208	85	723	277	43
18	246	63	184	816	224	75	749	251	42
19	263	53	210	790	240	65	775	225	41
20	9.78280	9.90043	9.88236	10.11764	9.79256	9.89455	9.89801	10.10199	40
21	296	34	262	738	272	45	827	173	39
22	313	24	289	711	288	35	853	147	38
23	329	14	315	685	304	25	879	121	37
24	346	05	341	659	319	15	905	095	36
25	362	9.89995	367	633	335	05	931	069	35
26	379	85	393	607	351	9.89395	957	043	34
27	395	76	420	580	367	85	983	017	33
28	412	66	446	554	383	75	9.90009	10.09991	32
29	428	56	472	528	399	64	035	965	31
30	9.78445	9.89947	9.88498	10.11502	9.79415	9.89354	061	939	30
31	461	37	524	476	431	44	086	914	29
32	478	27	550	450	447	34	112	888	28
33	494	18	576	424	463	24	138	862	27
34	510	08	603	397	478	14	164	836	26
35	527	9.89898	629	371	494	04	190	810	25
36	543	88	655	345	510	9.89294	216	784	24
37	560	79	681	319	526	84	242	758	23
38	576	69	707	293	542	74	268	732	22
39	592	59	733	267	558	64	294	706	21
40	9.78609	9.89849	9.88759	10.11241	9.79573	9.89254	9.90320	10.09680	20
41	625	40	786	214	589	44	346	654	19
42	642	30	812	188	605	33	371	629	18
43	658	20	838	162	621	23	397	603	17
44	674	10	864	136	636	13	423	577	16
45	691	01	890	110	652	03	449	551	15
46	707	9.89791	916	084	668	9.89193	475	525	14
47	723	81	942	058	684	83	501	499	13
48	739	71	968	032	699	73	527	473	12
49	756	61	994	006	715	62	553	447	11
50	9.78772	9.89752	9.89020	10.10980	9.79731	9.89152	9.90578	10.09422	10
51	788	42	046	954	746	42	604	396	9
52	805	32	073	927	762	32	630	370	8
53	821	22	099	901	778	22	656	344	7
54	837	12	125	875	793	12	682	318	6
55	853	02	151	849	809	01	708	292	5
56	869	9.89693	177	823	825	9.89091	734	266	4
57	886	83	203	797	840	81	759	241	3
58	902	73	229	771	856	71	785	215	2
59	918	63	255	745	872	60	811	189	1
60	934	53	281	719	887	50	837	163	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	52 Degrees.				51 Degrees.				

	39 Degrees.				40 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.79887	9.89050	9.90837	10.09168	9.80807	9.88425	8.92381	10.07619	60
1	903	40	868	137	822	15	407	593	59
2	918	30	889	111	837	04	433	567	58
3	934	20	914	086	852	9.88394	458	542	57
4	950	09	940	060	867	83	484	516	56
5	965	9.88999	966	034	882	72	510	490	55
6	981	89	992	008	897	62	535	465	54
7	996	78	9.91018	10.08982	912	51	561	439	53
8	9.80012	68	043	957	927	40	587	413	52
9	027	58	069	931	942	30	612	388	51
10	043	9.88948	095	905	9.80957	19	9.92638	10.07362	50
11	058	37	121	879	972	08	663	337	49
12	074	27	147	853	987	9.88298	689	311	48
13	089	17	172	828	9.81002	87	715	285	47
14	105	06	198	802	017	76	740	260	46
15	120	9.88896	224	776	032	66	766	234	45
16	136	86	250	750	047	55	792	208	44
17	151	75	276	724	061	44	817	183	43
18	166	65	301	699	076	34	843	157	42
19	182	55	327	678	091	23	868	132	41
20	9.80197	9.88844	9.91353	10.08647	9.81106	12	9.92894	10.07106	40
21	213	34	379	621	121	01	920	080	39
22	228	24	404	596	136	9.88191	945	055	38
23	244	13	430	570	151	80	971	029	37
24	259	03	456	544	166	69	996	004	36
25	274	9.88793	482	518	180	58	9.93022	10.06978	35
26	290	82	507	493	195	48	048	952	34
27	305	72	533	467	210	37	073	927	33
28	320	61	559	441	225	26	099	901	32
29	336	51	585	415	240	15	124	876	31
30	9.80351	9.88741	9.91610	10.08390	9.81254	05	9.93150	10.06850	30
31	366	30	636	364	269	9.88094	175	825	29
32	382	20	662	338	284	83	201	799	28
33	397	09	688	312	299	72	227	773	27
34	412	9.88699	713	287	314	61	252	748	26
35	428	88	739	261	328	51	278	722	25
36	443	78	765	235	343	40	303	697	24
37	458	68	791	209	358	29	329	671	23
38	473	57	816	184	372	18	354	646	22
39	489	47	842	158	387	07	380	620	21
40	9.80504	9.88636	9.91868	10.08132	9.81402	9.87996	9.93406	10.06594	20
41	519	26	893	107	417	85	431	569	19
42	534	15	919	081	431	75	457	543	18
43	550	05	945	055	446	64	482	518	17
44	565	9.88594	971	029	461	53	508	492	16
45	580	84	996	004	475	42	534	467	15
46	595	73	9.92022	10.07978	490	31	559	441	14
47	610	63	048	952	505	20	584	416	13
48	625	52	073	927	519	09	610	390	12
49	641	42	099	901	534	9.87898	636	364	11
50	9.80656	9.88531	9.92125	10.07875	9.81549	87	9.93661	10.06339	10
51	671	21	150	850	563	77	687	313	9
52	686	10	176	824	578	66	712	288	8
53	701	9.88499	202	798	592	55	738	262	7
54	716	89	227	773	607	44	763	237	6
55	731	78	253	747	622	33	789	211	5
56	746	68	279	721	636	22	814	186	4
57	762	57	304	696	651	11	840	160	3
58	777	47	330	670	665	00	865	135	2
59	792	36	356	644	680	9.87789	891	109	1
60	807	25	381	619	694	78	916	084	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	50 Degrees.				49 Degrees.				

	41 Degrees.				42 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang.	Co-tang.	M
0	9.81694	9.87778	9.93916	10.06084	9.82551	9.87107	9.95444	10.04556	60
1	9.81709	67	42	058	565	096	469	531	59
2	23	56	67	033	579	085	495	505	58
3	38	45	93	007	593	073	520	480	57
4	52	34	9.94018	10.05982	607	062	545	455	56
5	67	23	44	956	621	050	571	429	55
6	81	12	69	931	635	039	596	404	54
7	96	01	95	905	649	027	622	378	53
8	9.81810	9.87690	9.94120	880	663	016	647	353	52
9	25	79	46	854	677	004	672	328	51
10	9.81839	68	71	10.05829	9.82691	9.86993	9.95698	10.04302	50
11	54	57	97	803	705	982	723	277	49
12	68	46	9.94222	778	719	970	748	252	48
13	82	35	48	752	733	959	774	226	47
14	97	24	73	727	747	947	799	201	46
15	9.81911	13	99	701	761	936	825	175	45
16	26	01	9.94324	676	775	924	850	150	44
17	40	9.87590	50	650	788	913	875	125	43
18	54	79	75	625	802	902	901	099	42
19	69	68	9.94401	599	816	890	926	074	41
20	9.81983	57	426	10.05574	9.82830	9.86879	9.95952	10.04048	40
21	98	46	452	548	844	867	977	023	39
22	9.82012	35	477	523	858	855	9.96002	10.03998	38
23	26	24	503	497	872	844	028	972	37
24	41	13	528	472	885	832	053	947	36
25	55	01	554	446	899	821	078	922	35
26	69	9.87490	579	421	913	809	104	896	34
27	84	79	604	396	927	798	129	871	33
28	98	68	630	370	941	786	155	845	32
29	9.82112	57	655	345	955	775	180	820	31
30	26	46	9.94681	10.05319	9.82968	9.86763	9.96205	10.03795	30
31	41	34	706	294	982	752	231	769	29
32	55	23	732	268	996	740	256	744	28
33	69	12	757	243	9.83010	728	281	719	27
34	84	01	783	217	023	717	307	693	26
35	98	9.87390	808	192	037	705	332	668	25
36	9.82212	78	834	166	051	694	357	643	24
37	26	67	859	141	065	682	383	617	23
38	40	56	884	116	078	670	408	592	22
39	55	45	910	090	092	659	433	567	21
40	9.82269	9.87334	9.94935	10.05065	9.83106	9.86647	9.96459	10.03541	20
41	83	22	961	039	120	635	484	516	19
42	97	11	986	014	133	624	510	490	18
43	9.82311	00	9.95012	10.04988	147	612	535	465	17
44	26	9.87288	037	963	161	600	560	440	16
45	40	77	062	938	174	589	586	414	15
46	54	66	088	912	188	577	611	389	14
47	68	55	113	887	202	565	636	364	13
48	82	43	139	861	215	554	662	338	12
49	96	32	164	836	229	542	687	313	11
50	9.82410	9.87221	9.95190	10.04810	9.83242	9.86530	9.96712	10.03288	10
51	24	09	215	785	256	518	738	262	9
52	39	9.87198	240	760	270	507	763	237	8
53	53	87	266	734	283	495	788	212	7
54	67	75	291	709	297	483	814	186	6
55	81	64	317	683	310	472	839	161	5
56	95	53	342	658	324	460	864	136	4
57	9.82509	41	368	632	338	448	890	110	3
58	23	30	393	607	351	436	915	085	2
59	37	19	418	582	365	425	940	060	1
60	51	07	444	556	378	413	966	034	0
M	Co-sine	Sine	Co-tang.	Tangent.	Co-sine	Sine	Co-tan		M
	48 Degrees.				47 Degre				

	43 Degrees.				44 Degrees.				
M	Sine	Co-sine	Tang.	Co-tang.	Sine	Co-sine	Tang	Co-tang.	M
0	9.83378	9.86413	9.96966	10.03034	9.84177	9.85693	9.98484	10.01516	60
1	392	401	991	009	190	681	509	491	59
2	405	389	9.97016	10.02984	203	669	534	466	58
3	419	377	042	958	216	657	560	440	57
4	432	366	067	933	229	645	585	415	56
5	446	354	092	908	242	632	610	390	55
6	459	342	118	882	255	620	635	365	54
7	473	330	143	857	269	608	661	339	53
8	486	318	168	832	282	596	686	314	52
9	500	306	193	807	295	583	711	289	51
10	9.83513	9.86295	9.97219	10.02781	9.84308	9.85571	9.98737	10.01263	50
11	527	283	244	756	321	559	762	238	49
12	540	271	269	731	334	547	787	213	48
13	554	259	295	705	347	534	812	188	47
14	567	247	320	680	360	522	838	162	46
15	581	235	345	655	373	510	863	137	45
16	594	223	371	629	385	497	888	112	44
17	608	211	396	604	398	485	913	087	43
18	621	200	421	579	411	473	939	061	42
19	634	188	447	553	424	460	964	036	41
20	9.83648	9.86176	9.97472	10.02528	9.84437	9.85448	989	011	40
21	661	164	497	503	450	436	9.99015	10.00985	39
22	674	152	523	477	463	423	040	960	38
23	688	140	548	452	476	411	065	935	37
24	701	128	573	427	489	399	090	910	36
25	715	116	598	402	502	386	116	884	35
26	728	104	624	376	515	374	141	859	34
27	741	092	649	351	528	361	166	834	33
28	755	080	674	326	540	349	191	809	32
29	768	068	700	300	553	337	217	783	31
30	9.83781	9.86056	9.97725	10.02275	9.84566	9.85324	9.99242	10.00758	30
31	795	044	750	250	579	312	267	733	29
32	808	032	776	224	592	299	293	707	28
33	821	020	801	199	605	287	318	682	27
34	834	008	826	174	618	274	343	657	26
35	848	9.85996	851	149	630	262	368	632	25
36	861	984	877	123	643	250	394	606	24
37	874	972	902	098	656	237	419	581	23
38	887	960	927	073	669	225	444	556	22
39	901	948	953	047	682	212	469	531	21
40	9.83914	9.85936	978	022	9.84694	9.85200	9.99495	10.00505	20
41	927	924	9.98003	10.01997	707	187	520	480	19
42	940	912	029	971	720	175	545	455	18
43	954	900	054	946	733	162	570	430	17
44	967	888	079	921	745	150	596	404	16
45	980	876	104	896	758	137	621	379	15
46	993	864	130	870	771	125	646	354	14
47	9.84006	851	155	845	784	112	672	328	13
48	020	839	180	820	796	100	697	303	12
49	033	827	206	794	809	087	722	278	11
50	9.84046	9.85815	9.98231	10.01769	9.84822	9.85074	9.99747	10.00253	10
51	059	803	256	744	835	062	773	227	9
52	072	791	281	719	847	049	798	202	8
53	085	779	307	693	860	037	823	177	7
54	098	766	332	668	873	024	848	152	6
55	112	754	357	643	885	012	874	126	5
56	125	742	383	617	898	9.84999	899	101	4
57	138	730	408	592	911	986	924	076	3
58	151	718	433	567	923	974	949	051	2
59	164	706	458	542	936	961	975	025	1
60	177	693	484	516	949	949	10.00000	10.00000	0
M	Co-sine	Sine	Co-tang.	Tangent	Co-sine	Sine	Co-tang.	Tangent	M
	46 Degrees.				45 Degrees.				

TABLE IV.

Refraction in altitude, of the heavenly bodies.

Alt.	Refr.	Alt.	Refr.	Alt.	Refr.	Alt.	Refr.	Alt.	Ref
D.M.	M.S.	D. M	M. S.	D. M.	M. S.	D.	M. S.	D.	M. S.
0. 0	33. 0	4. 0	11.51	2.20	4.16	30	1.38	60	0.33
0. 5	32.10	4.10	11.29	2.40	4. 9	31	1.35	61	0.32
0.10	31.22	4.20	11. 8	3. 0	4. 3	32	1.31	62	0.30
0.15	30.35	4.30	10.48	13.20	3.57	33	1.28	63	0.29
0.20	29.50	4.40	10.29	13.40	3.51	34	1.24	64	0.28
0.25	29. 6	4.50	10.11	14. 0	3.45	35	1.21	65	0.26
0.30	28.22	5. 0	9.54	14.20	3.40	36	1.18	66	0.25
0.35	27.41	5.10	9.38	14.40	3.35	37	1.16	67	0.24
0.40	27. 0	5.20	9.23	15. 0	3.30	38	1.13	68	0.23
0.45	26.20	5.30	9. 8	15.30	3.24	39	1.10	69	0.22
0.50	25.42	5.40	8.54	16. 0	3.17	40	1. 8	70	0.21
0.55	25. 5	5.50	8.41	16.30	3.10	41	1. 5	71	0.19
1. 0	24.29	6. 0	8.28	17. 0	3. 4	42	1. 3	72	0.18
1. 5	23.54	6.10	8.15	17.30	2.59	43	1. 1	73	0.17
1.10	23.20	6.20	8. 3	18. 0	2.54	44	0.59	74	0.16
1.15	22.47	6.30	7.51	18.30	2.49	45	0.57	75	0.15
1.20	22.15	6.40	7.40	19. 0	2.44	46	0.55	76	0.14
1.25	21.44	6.50	7.30	19.30	2.39	47	0.53	77	0.13
1.30	21.15	7. 0	7.20	20. 0	2.35	48	0.51	78	0.12
1.35	20.46	7.10	7.11	20.30	2.31	49	0.49	79	0.11
1.40	20.18	7.20	7. 2	21. 0	2.27	50	0.48	80	0.10
1.45	19.51	7.30	6.53	21.30	2.24	51	0.46	81	0. 9
1.50	19.25	7.40	6.45	22. 0	2.20	52	0.44	82	0. 8
1.55	19. 0	7.50	6.37	23. 0	2.14	53	0.43	83	0. 7
2. 0	18.35	8. 0	6.29	24. 0	2. 7	54	0.41	84	0. 6
2. 5	18.11	8.10	6.22	25. 0	2. 2	55	0.40	85	0. 5
2.10	17.48	8.20	6.15	26. 0	1.56	56	0.38	86	0. 4
2.15	17.26	8.30	6. 8	27. 0	1.51	57	0.37	87	0. 3
2.20	17. 4	8.40	6. 1	28. 0	1.47	58	0.35	88	0. 2
2.25	16.44	8.50	5.55	29. 0	1.42	59	0.34	89	0. 1
2.30	16.24	9. 0	5.48						
2.35	16. 4	9.10	5.42						
2.40	15.45	9.20	5.36						
2.45	15.27	9.30	5.31						
2.50	15. 9	9.40	5.25						
2.55	14.52	9.50	5.20						
3. 0	14.36	10. 0	5.15						
3. 5	14.20	10.15	5. 7						
3.10	14. 4	10.30	5. 0						
3.15	13.49	10.45	4.53						
3.20	13.34	11. 0	4.47						
3.25	13.20	11.15	4.40						
3.30	13. 6	11.30	4.34						
3.40	12.40	11.45	4.29						
3.50	12.15	12. 0	4.23						

TABLE V.

Depression of the horizon, at sea.

Height of the eye.	Dip.
Feet.	M. S.
1	0.57
2	1.21
3	1.39
4	1.55
5	2. 8
6	2.20
7	2.31
8	2.42
9	2.52
10	3. 1
11	3.10
12	3.18
13	3.26
14	3.34
15	3.42
16	3.49
17	3.56
18	4. 3
19	4.10
20	4.16
21	4.22
22	4.28
23	4.34
24	4.40
26	4.52
28	5. 3
30	5.14
35	5.39
40	6. 2

TABLE VI.

The Sun's parallax in altitude.

Alt.	Paral.	Alt.	Paral.
D.	S.	D.	S.
0	9	60	4
10	9	65	4
20	8	70	3
30	8	75	2
40	7	80	2
50	6	85	1
55	5	90	0

TABLE VII.

Augmentation of the moon's semidiameter.

Alt.	Aug.	Alt.	Aug.
D.	S.	D.	S.
0	0	40	10
5	1	45	11
10	3	50	12
15	4	55	13
20	6	60	14
25	7	70	15
30	8	80	16
35	9	—	—

TABLE VIII.

The mean right ascensions and declinations of 36 principal fixed stars, from the Nautical Almanac (1818), corrected to the beginning of the year 1822. The stars marked with an asterisk * are used to find the longitude at sea. (See pages 337 and 338.)

Names of the stars.	Mag.	Mean right ascension in time.	Annual variation.	Declination.	Annual variation.
		h ′ ″	″	° ′ ″	″
γ Pegasi	2	0· 4· 4·69	3·08	14·11·40·80 N.	+20·20
α Arietis *	2·3	1·57· 9·42	3·35	22·37· 0.09 N.	+17·40
α Ceti	2	2·52·58·78	3·12	3·23· 2·00 N.	+14·75
Aldebaran *	1	4·25·42·88	3·43	16· 8·36·29 N.	+ 7·95
Capella	1	5· 3·33·09	4·41	45·48·20·67 N.	+ 4·57
Rigel	1	5· 5·59·01	2·88	8·24·49·64 S.	− 4·92
β Tauri	2	5·15· 2·66	3·78	28·26·50·80 N.	+ 3·83
α Orion	1	5·45·32·11	3·25	7·21·56·68 N.	+ 1·37
Sirius	1	6·37·17·98	2·64	16·28·39·64 S.	+ 4·36
Castor	2	7·23·13·50	3·85	32·16· 9·72 N.	− 7·06
Procyon	1·2	7·29·58·67	3·15	5·40·28·88 N.	− 8·54
Pollux *	2	7·34·24·54	3·69	28·26·51·58 N.	− 8·00
α Hydræ	2	9·18·50·21	2·95	7·58·27·53 S.	+15·19
Regulus *	1	9·58·52·86	3·21	12·50· 1·41 N.	−17·33
β Leonis	1·2	11·39·58·28	3·07	15·34· 1·99 N.	−20·04
β Virginis	3	11·41·25·07	3·12	2·47· 9·00 N.	−20·22
Spica Virginis *	1	13·15·49·42	3·14	10·13·41·55 S.	+18·95
Arcturus	1	14· 7·32·59	2·73	20· 6·50·09 N.	−18·99
1 α Libræ	6	14·40·51·31	3·29	15·14·55·54 S.	+15·20
2 α Libræ	2	14·41· 2·74	3·29	15·17·39·40 S.	+15·20
α Cor. Borealis.	2·3	15·27· 9·09	2·53	27·19·12·15 N.	−12·49
α Serpentis	2	15·35·30·24	2·94	7·59·35·20 N.	−11·73
Antares *	1	16·18·30·24	3·66	26· 1·33·40 S.	+ 8·62
α Herculis	2	17· 6·31·98	2·73	14·36· 5.76 N.	− 4·48
α Ophiuchi	2	17·26·40·38	2·77	12·41·53·22 N.	− 3·10
α Lyræ	1	18·30·54·64	2·03	38·37·26·52 N.	+ 3·00
γ Aquilæ *	3	19·37·47·69	2·85	10·11·14·82 N.	+ 8·38
α* Aquilæ *	1·2	19·42· 5·70	2·93	8.24·22·24 N.	+ 9·06
β Aquilæ *	3·4	19·46·33·99	2·95	5·58·13·03 N.	+ 8·57
1 α Capricorni	3	20· 7·46·24	3·33	13· 3.58·20 S.	−10·80
2 α Capricorni	3	20· 8·10·12	3·33	13· 5·15·10 S.	−10·80
α Cygni	1·2	20·35·21·81	2·04	44·38·56·69 N.	+12·63
α Aquarii	3	21·56·38·14	3·09	1·10·45·29 S.	−17·37
Fomalhaut *	1·2	22·47·47·28	3·34	30·33·47·60 S.	−19·10
α Pegasi *	2	22·55·53·91	2·98	14·15· 3·24 N.	+19·43
α Andromedæ	2	23·59·12·22	3·08	28· 6·29·75 N.	+19·99

THE END.

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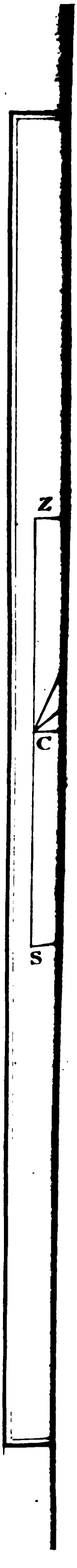
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PLATE IV

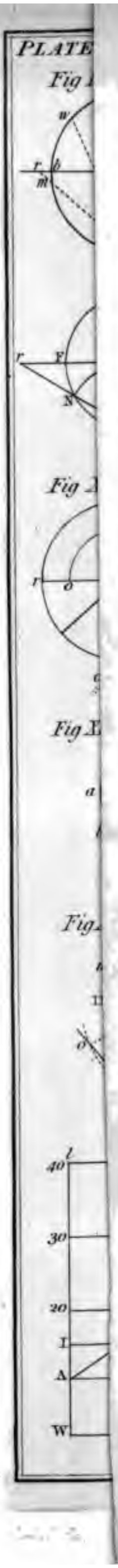
PLATE
Fig I
w
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Fig X
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Fig X
a
Fig
40
30
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I
A
W